PDE midterm (Full score = 105. Points above 100 will be truncated).

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1. **1D wave equation with no damping (25 points).** In this question, you are asked to solve the 1-dimensional wave equation $\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$ at $0 < x < L$ where $y = y(x,t)$ with the following boundary conditions: $y(0,t) = 0 = y(L,t)$. We further assume the following initial conditions,

$$y(x,0) = f(x) = \begin{cases} \sin\left(\frac{2\pi}{L}x\right), & 0 < x < \frac{L}{2} \\ 0, & \frac{L}{2} \le x < L, \end{cases}$$

and $y_t(x,0) = 0, \forall x \in (0,L)$.

(a) [*10 pts*] Assume that $f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right)$. Calculate a_1, a_2, a_3, and a_4.

(b) [*5 pts*] For any arbitrary initial displacement with zero initial velocity, show that the general solution can be written as $y(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right)\cos\omega_n t$. Express the angular frequencies ω_n in terms of n and other parameters.

(c) [*10 pts*] Plot $y(x,t)$ at $t = \frac{L}{4c}, \frac{L}{2c}$, and $\frac{L}{c}$. (*Hint*: D'Alembert decomposition and periodic extension).

2. **2D wave equation with damping (25 points)**. Denote the vertical displacement over a membrane with tension as $\xi = \xi(x, y, t)$ and assume that, within the square region bounded by $0 < x < a$ and $0 < y < a$, ξ satisfies the following equation:

$$\xi_{xx} + \xi_{yy} = \frac{1}{c^2}(\xi_{tt} + b\xi_t). \qquad (1)$$

Further, assume that $\xi(x, y, t) = 0$ at all the four sides of the square; that is, $\xi(x, 0, t) = \xi(x, a, t) = \xi(0, y, t) = \xi(a, y, t) = 0$.

(a) [*10 pts*] To start solving this equation, assume that $\tilde{\xi}(x, y, t) = X(x)Y(y)T(t)$ satisfies Eq. (1). Show that we must have $\frac{X''}{X} + \frac{Y''}{Y} = -\kappa^2$, and find out all the possible κ's so that the boundary conditions can be satisfied. (Hint: It should be natural to index the set of κ's by two positive integers).

(b) [*10 pts*] Denote the general solution as
$\xi(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{a} T_{mn}(t)$.
What ordinary differential equation (ODE) should $T_{mn}(t)$ satisfy? Write an expression of $T_{mn}(t)$ by solving the ODE. How large should b be for some vibrational mode to be *over-damped* (that is, for some $T_{mn}(t)$ to never change sign)?

(c) [*5 pts*] Continuing from (b), assume that the initial velocity $\xi_t(x, y, 0) = 0$ for all (x, y). Describe how to find the coefficients a_{mn} given the initial displacement $\xi(x, y, 0) = f(x, y)$.

3. **Diffusion with different boundary conditions (25 points)**. Let us consider a long narrow tube of length L filled uniformly with a special gas that does not exist naturally in the atmosphere. Assume that it has the following initial concentration: $u(x, 0) = D_0$ for $0 < x < L$.

(a) [15 pts] Assume that, at $t = 0$, we open one end of the tube so the gas starts diffusing outward. Let us model this situation as $u(L, t) = 0$ and $u_x(0, t) = 0$ for $t > 0$. Assume that the variation of u follows the "heat equation": $u_t = \alpha^2 u_{xx}$ so nothing dramatic such as explosion would happen. Just simply diffusion. Solve $u(x, t)$ for $0 < x < L$ and $t > 0$.

(b) [5 pts] Estimate roughly what's the amount of time it will take in terms of α and other parameters for half of the gas molecules to leak out.

(c) [5 pts] Roughly speaking, how much faster would it take for half of the gas molecules to leak out if we also open the other end of the tube at $t = 0$?

註：(b)-(c) 部分只是測試大家粗略估算的能力，不需要精確，但要描述推理過程；請儘量說明推理過程中，有那些項次是忽略的

4. Laplace equation between two circles (15 points). Assume that $u = u(r, \theta)$ satisfies the Laplace equation $\nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$ in a region between two circles $S = \{(r, \theta): a < r < b\}$. Also assume that we have the following boundary conditions: $u(a, \theta) = C_1$ and $u(b, \theta) = C_2$, where C_1 and C_2 are constants.

(a) [*10 pts*] Solve $u = u(r, \theta)$.

(b) [*5 pts*] Make a 3D sketch of your solution.

5. **Laplace equation inside a circle (15 points).** We have shown that $\phi_n(r, \theta) = r^n \sin n\theta$ satisfies the Laplace equation. In this question, you are asked to explore the shape of this function a little bit.

(a) [*5 pts*] In particular, $\phi_1(r, \theta) = r \sin \theta$. Show that $\nabla \phi_1$ is a constant vector. [Definition of the symbol is as we introduced in class: $\nabla = \left(\frac{\partial}{\partial_x}, \frac{\partial}{\partial_y}\right)$].

(b) [*5 pts*] Continuing from (a), what is the geometric interpretation?

(c) [*5 pts*] Make a 3D sketch of the function $\phi_2(r, \theta)$.