PDE midterm (Full score = 105. Points above 100 will be truncated).

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1. **1D wave equation with no damping (25 points).** In this question, you are asked to solve the 1-dimensional wave equation $\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$ at $0 < x < L$ where $y = y(x,t)$ with the following boundary conditions: $y(0,t) = 0 = y(L,t)$. We further assume the following initial conditions,

$$y(x,0) = f(x) = \begin{cases} \sin\left(\frac{2\pi}{L}x\right), & 0 < x < \frac{L}{2} \\ 0, & \frac{L}{2} \le x < L, \end{cases}$$

and $y_t(x,0) = 0, \forall x \in (0,L)$.

(a) [*10 pts*] Assume that $f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right)$. Calculate a_1, a_2, a_3, and a_4.

(b) [*5 pts*] For any arbitrary initial displacement with zero initial velocity, show that the general solution can be written as $y(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right)\cos\omega_n t$. Express the angular frequencies ω_n in terms of n and other parameters.

(c) [*10 pts*] Plot $y(x,t)$ at $t = \frac{L}{4c}, \frac{L}{2c}$, and $\frac{L}{c}$. (*Hint*: D'Alembert decomposition and periodic extension).

1

a. $\int_0^{\frac{L}{2}} \sin\left(\frac{2\pi}{L}x\right)\cdot\sin\left(\frac{n\pi}{L}x\right)dx$

$=\int_0^{\frac{L}{2}} \frac{1}{2}\left[-\cos\left(\frac{n+2}{L}\pi x\right)+\cos\left(\frac{n-2}{2}\pi x\right)dx\right.$

$=\frac{1}{2}\left[-\frac{L}{(n+2)\pi}\sin\left(\frac{n+2}{L}\pi x\right)+\frac{L}{(n-2)\pi}\sin\left(\frac{n-2}{L}\pi x\right)\right]_0^{\frac{L}{2}}$

$=\frac{1}{2}\left[-\frac{L}{(n+2)\pi}\sin\left(\frac{n+2}{2}\pi\right)+\frac{L}{(n-2)\pi}\sin\left(\frac{n-2}{2}\pi\right)\right]\ (n\neq 2)$

$a_n=\frac{\cancel{2}}{L}\cdot\frac{1}{\cancel{2}}\left[-\frac{\cancel{L}}{(n+2)\pi}\sin\left(\frac{n+2}{2}\pi\right)+\frac{\cancel{L}}{(n-2)\pi}\sin\left(\frac{n-2}{2}\pi\right)\right.$

$$\begin{cases} a_1=\frac{1}{L}\cdot\left[-\frac{L}{3\pi}\cdot(-1)+\frac{1}{-\pi}\cdot(-1)\right]=\frac{5}{3\pi} \\ a_3=\frac{1}{L}\left[-\frac{L}{5\pi}\cdot(1)+\frac{1}{\pi}\cdot 1\right]=\frac{4}{5\pi} \\ a_4=0 \end{cases}$$

For $n=2$

$\int_0^{\frac{L}{2}}\sin^2\left(\frac{2\pi}{L}x\right)dx=\int_0^{\frac{L}{2}}\frac{1-\cos\left(\frac{4\pi}{L}x\right)}{2}dx=\left(\frac{1}{2}x-\frac{L}{4\pi}\sin\left(\frac{4\pi}{L}x\right)\right)_0^{\frac{L}{2}}=\frac{L}{4}$

$a_2=\frac{2}{L}\cdot\frac{L}{4}=\frac{1}{2}$ #

(b)

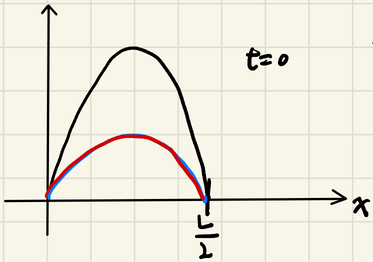
$\omega_n=cK_n$ #

"Most students attain correct answer!!"

$y(x,t)=\sum_n A_n\cdot\sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi}{L}ct\right)$ #

c.

Solution is $y(x,t)=\sum_n A_n \sin(k_n x)\cos(ck_n t)$



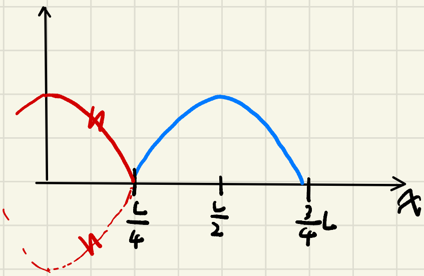
$t=0$

$y=\frac{1}{2}(y_L+y_R)$

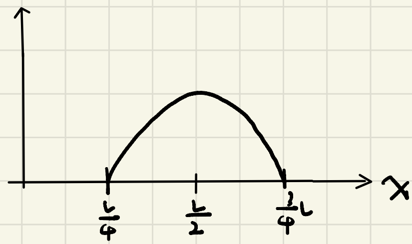
$c=\sqrt{\frac{\tau}{\rho}}$ // constant

#固定端反射

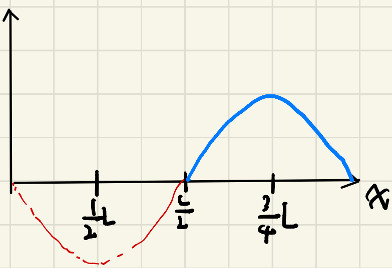
$t=\frac{L}{4c} \Rightarrow \Delta x=\frac{L}{4}$



$\Rightarrow$

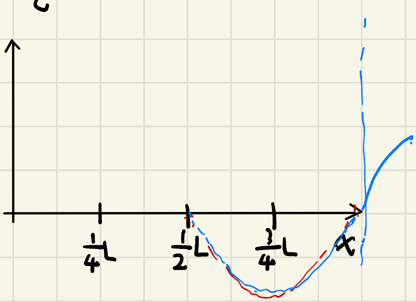


$t=\frac{L}{2c} \Rightarrow \Delta x=\frac{L}{2}$

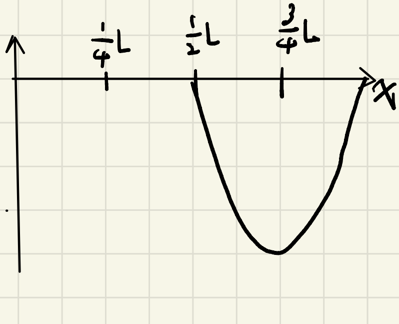


$\Rightarrow$

$t=\frac{L}{c}$ $\Delta x=L$



$\Rightarrow$



2. **2D wave equation with damping (25 points)**. Denote the vertical displacement over a membrane with tension as $\xi = \xi(x, y, t)$ and assume that, within the square region bounded by $0 < x < a$ and $0 < y < a$, ξ satisfies the following equation:

$$\xi_{xx} + \xi_{yy} = \frac{1}{c^2}(\xi_{tt} + b\xi_t). \qquad (1)$$

Further, assume that $\xi(x, y, t) = 0$ at all the four sides of the square; that is, $\xi(x, 0, t) = \xi(x, a, t) = \xi(0, y, t) = \xi(a, y, t) = 0$.

(a) [*10 pts*] To start solving this equation, assume that $\tilde{\xi}(x, y, t) = X(x)Y(y)T(t)$ satisfies Eq. (1). Show that we must have $\frac{X''}{X} + \frac{Y''}{Y} = -\kappa^2$, and find out all the possible κ's so that the boundary conditions can be satisfied. (Hint: It should be natural to index the set of κ's by two positive integers).

(b) [*10 pts*] Denote the general solution as

$\xi(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{a} T_{mn}(t)$.

What ordinary differential equation (ODE) should $T_{mn}(t)$ satisfy? Write an expression of $T_{mn}(t)$ by solving the ODE. How large should b be for some vibrational mode to be *over-damped* (that is, for some $T_{mn}(t)$ to never change sign)?

(c) [*5 pts*] Continuing from (b), assume that the initial velocity $\xi_t(x, y, 0) = 0$ for all (x, y). Describe how to find the coefficients a_{mn} given the initial displacement $\xi(x, y, 0) = f(x, y)$.

3. **Diffusion with different boundary conditions (25 points)**. Let us consider a long narrow tube of length L filled uniformly with a special gas that does not exist naturally in the atmosphere. Assume that it has the following initial concentration: $u(x, 0) = D_0$ for $0 < x < L$.

(a) [15 pts] Assume that, at $t = 0$, we open one end of the tube so the gas starts diffusing outward. Let us model this situation as $u(L, t) = 0$ and $u_x(0, t) = 0$ for $t > 0$. Assume that the variation of u follows the "heat equation": $u_t = \alpha^2 u_{xx}$ so nothing dramatic such as explosion would happen. Just simply diffusion. Solve $u(x, t)$ for $0 < x < L$ and $t > 0$.

2. $\xi_{xx}+\xi_{yy}=\frac{1}{c^2}\left(\xi_{tt}+b\,\xi_t\right)$, $\xi = X(x)\,Y(y)\,T(t)$.

$\Rightarrow X''YT+XY''T=\frac{1}{c^2}\left(XYT''+bXYT'\right) \Rightarrow \frac{X''}{X}+\frac{Y''}{Y}=\frac{1}{c^2}\left(\frac{T''+bT'}{T}\right)=-k^2$

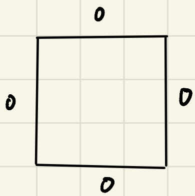
$\begin{cases} \frac{X''}{X}+\frac{Y''}{Y}=-k^2 \\ T''+bT''=-c^2k^2T \end{cases}$

① $\frac{X''}{X}=-\frac{Y''}{Y}-k^2=-k_x^2$

$\Rightarrow \begin{cases} X''+k_x^2=0 \\ Y''+\underbrace{(k^2-k_x^2)}_{k_y^2}=0 \end{cases} \Rightarrow \begin{cases} X''+k_x^2X=0 \\ Y''+k_y^2Y=0 \end{cases}$

$\Rightarrow \begin{cases} X(x)=A\cos k_x x+B\sin k_x x \\ Y(y)=C\cos k_y y+D\sin k_y y \end{cases}$

(a)



x-axis :

$A=0$, $B\sin k_x x=0$; $k_x=\frac{n\pi}{a}$

y-axis :

$C=0$, $D\sin k_y x=0$; $k_y=\frac{n\pi}{a}$

(b). denoted $k_x=\frac{m\pi}{a}$, $k_y=\frac{n\pi}{a}$

$\cancel{\sum_m}\cancel{\sum_n}\cancel{a_{mn}}(-k_x^2)\sin \cancel{k_x x}\,\cancel{\sin k_y y}+\cancel{a_{mn}}\,\sin \cancel{k_x x}(-k_y^2)\sin \cancel{k_y y}$ //

$=\frac{1}{c^2}\left(\cancel{\sum_m}\cancel{\sum_n}\cancel{a_{mn}}\,\cancel{\sin k_x x}\,\cancel{\sin k_y y}\cdot T''_{mn}+\cancel{a_{mn}}\,\cancel{\sin k_x x}\,\cancel{\sin k_y y}\cdot b\,T'_{mn}\right)$

$\Rightarrow -k_x^2-k_y^2=\frac{1}{c^2}\left[T''_{mn}+bT'_{mn}\right] \Rightarrow T''+bT'+c^2\left[k_x^2+k_y^2\right]T$

over-damped : $D>0 \rightarrow b^2-4\cdot c^2\left[k_x^2+k_y^2\right]>0 \Rightarrow b^2>4c^2\left[k_x^2+k_y^2\right]$

$\Rightarrow b>2c\left[k_x+k_y\right]^{\frac{1}{2}} \vee b<-2c\left[k_x+k_y\right]^{\frac{1}{2}}$

#

c.

$$\xi_e(x, y, 0) = X(x)Y(y)T'(0) = f(x, y).$$

$$\Rightarrow T'(0) = \left[\left(-\frac{b}{2} + \cancel{\frac{\sqrt{b^2-4k^2c^2}}{2}}\right)\cancelto{1}{\exp(0)} + \left(-\frac{b}{2} - \cancel{\frac{\sqrt{b^2-4k^2c^2}}{2}}\right)\cdot \cancelto{1}{\exp(0)}\right]$$

$$= -b.$$

$$f(x, y) = \sum_m \sum_n a_{mn} \cdot T'(0) \cdot \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{a}y\right)$$

$$\# \quad [a_{mn} \cdot \cancelto{-b}{T'(0)}] = \frac{a^2}{4} \cdot \int_0^a \int_0^a f(x, y) \cdot \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{a}y\right)\, dy\, dx$$

$$\Rightarrow a_{mn} = \frac{1}{-b} \frac{a^2}{4} \int_0^a \int_0^a f(x, y) \cdot \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{a}y\right)\, dy\, dx.$$

3. (a)

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = k^2 \quad \begin{cases} X'' - k^2 X = 0 \\ T' - \alpha^2 k^2 T = 0 \end{cases}$$

$A\sinh kx + B\cosh kx$

$A=0$ $(u_x(0,t)=0)$ $B=0$ $(u(L,t)=0)$

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = 0 \quad \Big(X = Ax + B \quad A=0, \quad B=0.$$

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = -k^2 \quad \begin{cases} X'' + k^2 X = 0 \rightarrow A\cos kx + B\sin kx \\ T' + k^2\alpha^2 T = 0 \rightarrow \Lambda \cdot e^{-k^2\alpha^2 t} \end{cases}$$

$$u_x(0,t) = -kA\sin kx + k\cancel{B}^{\,0}\cos kx = 0$$

$$u(L,t) = A\cdot\cos kL = 0 \quad kL = \left(\tfrac{1}{2}+n\right)\pi \Rightarrow \left(\tfrac{1}{2}+n\right)\frac{\pi}{L}$$

$u(x,t) = (\Pi_n)\cos\left(\left(\tfrac{1}{2}+n\right)\frac{\pi}{L}x\right)\cdot\exp(-\alpha^2 k^2 t)$

$$\Pi_n = \int_0^L D_0 \cdot \cos\left(\left(\tfrac{1}{2}+n\right)\frac{\pi}{L}x\right)dx$$

$$= D_0 \frac{L}{\left(\frac{1}{2}+n\right)\pi}\sin\left(\frac{\left(\frac{1}{2}+n\right)}{\cancel{L}}\pi \cancel{L}\right) = \frac{D_0 L}{\left(\frac{1}{2}+n\right)\pi}(-1)^n, \quad n \geq 0.$$

(b) attain $t=\tau$ (half time)

$$\int_0^L u(x,\tau)\,dx = \sum_n \Pi_n \int_0^L \cos\left(\left(\tfrac{1}{2}+n\right)\frac{\pi}{L}x\right)dx\cdot\exp(-\alpha^2 k^2 \tau) = \frac{D_0}{2}L$$

$n\uparrow \Rightarrow$ exp. term n^2 decade

$n=0$ mode ; $\Pi_0 = \frac{2L}{\pi}$: 1

$n=1$ mode ; $\Pi_1 = \frac{2L}{3\pi}$: $\frac{1}{3}$

$n=2$ mode ; $\Pi_2 = \frac{2L}{5\pi}$: $\frac{1}{5}$

$\simeq 15:5:3.$ // total: 21. 10.5

* mode 0 dominates

$$\exp\left(-\alpha^2\frac{\pi^2}{4L^2}\cdot\tau\right) \simeq \frac{1}{2}$$

$$\tau \simeq \frac{4\cdot\ln 2}{\alpha^2\cdot\left(\frac{\pi}{L}\right)^2}$$ #

c.

① If your statement affirms $\tau' = \frac{1}{2}\tau$ // given physics hunch

$$\tau' = \frac{2 \cdot \ln 2}{\alpha^2 \left(\frac{z}{L}\right)^2} \quad \#$$ // my version!!

②

If your statement affirms $K' = 2K$; thus the τ would be $\frac{1}{4}$.

$$\tau' = \frac{1}{4}\tau = \frac{\ln 2}{\alpha^2 \left(\frac{z}{L}\right)^2} \quad \#$$

* The grading of this question mainly be judged by the clarity of your statement!!

(b) [5 pts] Estimate roughly what's the amount of time it will take in terms of α and other parameters for half of the gas molecules to leak out.

(c) [5 pts] Roughly speaking, how much faster would it take for half of the gas molecules to leak out if we also open the other end of the tube at $t = 0$?

註：(b)-(c) 部分只是測試大家粗略估算的能力，不需要精確，但要描述推理過程；請儘量說明推理過程中，有那些項次是忽略的

4. Laplace equation between two circles (15 points). Assume that $u = u(r, \theta)$ satisfies the Laplace equation $\nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$ in a region between two circles $S = \{(r, \theta): a < r < b\}$. Also assume that we have the following boundary conditions: $u(a, \theta) = C_1$ and $u(b, \theta) = C_2$, where C_1 and C_2 are constants.

(a) [*10 pts*] Solve $u = u(r, \theta)$.

(b) [*5 pts*] Make a 3D sketch of your solution.

5. **Laplace equation inside a circle (15 points)**. We have shown that $\phi_n(r, \theta) = r^n \sin n\theta$ satisfies the Laplace equation. In this question, you are asked to explore the shape of this function a little bit.

(a) [*5 pts*] In particular, $\phi_1(r, \theta) = r \sin \theta$. Show that $\nabla \phi_1$ is a constant vector. [Definition of the symbol is as we introduced in class: $\nabla = \left(\frac{\partial}{\partial_x}, \frac{\partial}{\partial_y}\right)$].

(b) [*5 pts*] Continuing from (a), what is the geometric interpretation?

(c) [*5 pts*] Make a 3D sketch of the function $\phi_2(r, \theta)$.

4.

By textbook.

$$\begin{cases} r^2R''+rR'-n^2R=0 \longrightarrow \begin{cases} A+B\ln r & n=0 \\ C\cdot r^n+D r^{-n}, & n\in\mathbb{N} \end{cases} \\ Д\cdot\cos n\theta+П\cdot\sin n\theta \end{cases}$$

$(A+B\ln r)+(C\cdot r^n+D\cdot r^{-n})\cdot(Д\cdot\cos n\theta+П_n\sin\theta)$.

B.C. $u(a,\theta)=C_1$; $u(b,\theta)=C_2$.

$$\begin{cases} A+B\ln a=\frac{1}{2\pi}\int_0^{2\pi}C_1\,d\theta=C_1 \\ A+B\ln b=\frac{1}{2\pi}\int_0^{2\pi}C_2\,d\theta=C_2 \end{cases}$$

$$A=\frac{\begin{vmatrix} C_1 & \ln a \\ C_2 & \ln b \end{vmatrix}}{\begin{vmatrix} 1 & \ln a \\ 1 & \ln b \end{vmatrix}}=\frac{C_1\ln b-C_2\ln a}{\ln b-\ln a} \checkmark$$

$$B=\frac{\begin{vmatrix} 1 & C_1 \\ 1 & C_2 \end{vmatrix}}{\begin{vmatrix} 1 & \ln a \\ 1 & \ln b \end{vmatrix}}=\frac{C_2-C_1}{\ln b-\ln a} \checkmark$$

$(C\cdot a^n+Da^{-n})[Д_n\cdot\cos n\theta+П_n\cdot\sin n\theta)$

$(C\cdot b^n+Db^{-n})[Д_n\cdot\cos n\theta+П_n\cdot\sin n\theta)$

$$\begin{cases} P=CД, & Y=DП \\ Ш=CД, & Я=DП. \end{cases}$$

$$\Rightarrow\begin{cases} (P_na^n+Y_na^{-n})\cdot\cos n\theta+(Ш_na^n+Я_na^{-n})\sin n\theta. \\ (P_nb^n+Y_nb^{-n})\cdot\cos n\theta+(Ш_nb^n+Я_nb^{-n})\sin n\theta \end{cases}$$

Eq. 1.

$$a^n\cdot P_n+a^{-n}Y_n=\frac{1}{\pi}\int_0^{2\pi}C_1\cdot\cos(n\theta)\,d\theta$$

$$b^n\cdot P_n+b^{-n}Y_n=\frac{1}{\pi}\int_0^{2\pi}C_2\cdot\cos(n\theta)\,d\theta.$$

Eq. 2.

$$a^nШ_n+a^{-n}Я_n=\frac{1}{\pi}\int_0^{2\pi}C_1\cdot\sin(n\theta)\,d\theta$$

$$b^nШ_n+b^{-n}Я_n=\frac{1}{\pi}\int_0^{2\pi}C_2\cdot\sin(n\theta)\,d\theta$$

$$\int_0^{2\pi} \cos n\theta \, d\theta = \frac{1}{n}\sin n\theta \Big|_0^{2\pi} = 0 \qquad \int_0^{2\pi} \sin n\theta \, d\theta = \frac{1}{n}(-\cos n\theta)\Big|_0^{2\pi} = \frac{1}{n}\left[(-1)-(-1)\right] = 0.$$

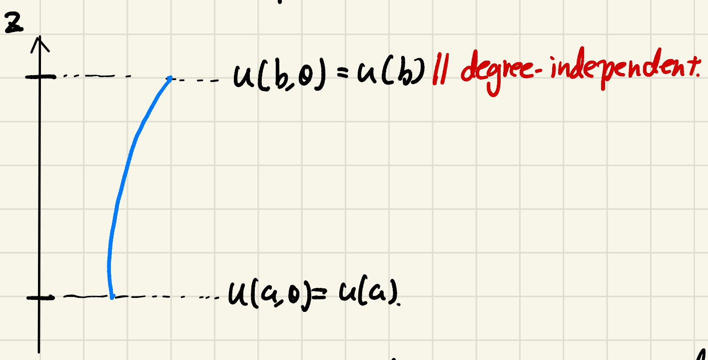
All $P_n, \gamma_n, \text{Ш}_n, q_n = 0$ #

Total sol.

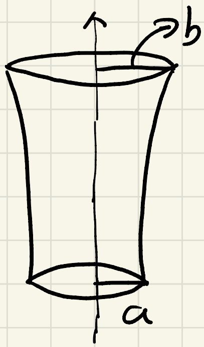
$$\Rightarrow u(r,\theta) = \frac{C_1 \ln b - C_2 \ln a}{\ln b - \ln a} + \frac{C_1 - C_2}{\ln b - \ln a} \ln r \quad \#$$

(b)

dimension decomposition

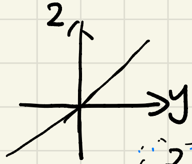


$\because$ θ is independent $\Rightarrow$ invariant value regards to degree.

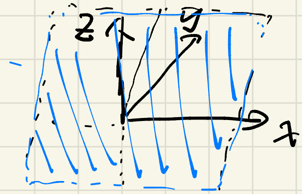


5.

a. $r\sin\theta \Rightarrow y$ (coordinate transform).



$\nabla\phi_1 = (0, 1)$ #

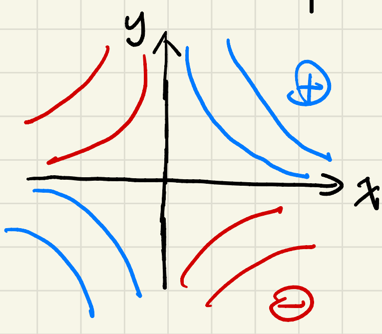


b. For every point, both magnitude and direction of changing trend are invariant.

c. $r^2 \cdot \sin^2\theta$

$= r \cdot (2\cdot\sin\theta\cdot\cos\theta) = r^2 \cdot 2 \cdot \left(\frac{y}{r}\cdot\frac{x}{r}\right) = 2xy.$ #

dimension decomposition



$\forall\ x=y \quad z = 2xy = 2x^2$

$x=-y \quad z = -2x^2$