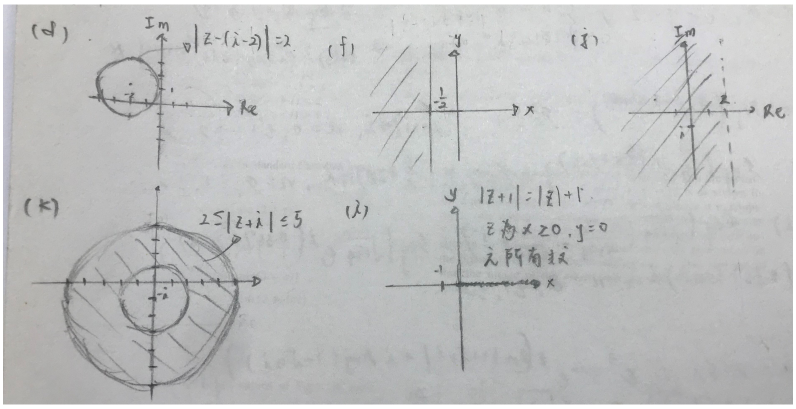
Partial Differential Equations and Complex Variables
Homework 5

21.3.1 With a labeled sketch, show the points sets defined by the following.

sol.



■

21.3.3 Show whether or not $|e^z| = e^{|z|}$. More generally, is $|w(z)| = w(|z|)$?

sol. Both answer are no. For example, if $z = i$, then $|e^z| = 1$ but $e^{|z|} = e$. ■

21.3.4 Show whether or not $\overline{e^z} = e^{\bar{z}}$. More generally, is $\overline{w(z)} = w(\bar{z})$?

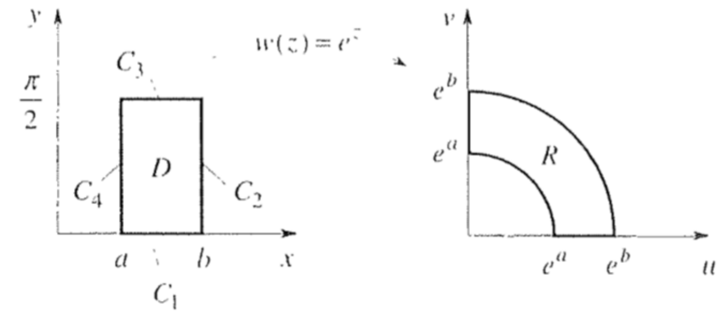
sol. Let x, y be real numbers and $z = x + yi$, then we have $\overline{e^z} = e^x(\cos y - i\sin y) = e^{x-iy} = e^{\bar{z}}$. However in general, $\overline{w(z)} \neq w(\bar{z})$. For example, if $w(z) = \text{Im}\{z\}$, then $\overline{w(z)} = \bar{y} = y$ but $w(\bar{z}) = w(x - yi) = -y$. ■

21.3.9 Evaluate each of the following in standard Cartesian form (d) $\sin(3 + \pi i)$ (e) $\cos(-2 + 3\pi i)$ (h) $\tan(-\frac{3\pi}{4}i)$ (i) $\cot(\frac{\pi i}{4})$ (k) $\cosh(1 - \pi i)$ (l) $\tanh(2 + 4\pi i)$

sol. (d) $\sin 3\cosh\pi + i\sinh\pi\cos 3$ (e) $\cos 2\cosh 3\pi + i\sin 2\sinh 3\pi$ (h) $-i\tanh(\frac{3\pi}{4})$ (i) $-i\coth(\frac{\pi}{4})$ (k) $-\cosh 1$ (l) $\tanh 2$ ■

21.3.12 Find the image of $R = \{x + iy : a < x < b, 0 < y < \pi/2\}$ under the function e^z

sol. Write e^z as $e^x(\cos y + i\sin y)$, where x, y are real numbers. If we coincide $\mathbb{C}$ with $\mathbb{R}^2$, then we have a parameterized region $(e^x\cos y, e^x\sin y)$. Since e^x ranges in (e^a, e^b) and y ranges in $(0, \pi/2)$, the region shall looks like:



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21.4.6 Obtain, in Cartesian form, all values of $\log z$ for each given z.

(a) -2

(d) $-5i$

(g) $13 - 5i$

sol. Use definition.

(a) $\log(-2) = \log(2e^{(\pi+2n\pi)i}) = \ln 2 + (2n+1)\pi i$

(d) $\log(-5i) = \log(5e^{(-\pi/2+2n\pi)i}) = \ln 5 + (4n-1)\pi i/2$

(f) $\log(13 - 5i) = \log(\sqrt{194}e^{(\theta+2n\pi)i}) = \frac{1}{2}\ln 194 + (\theta + 2n\pi i)$, where $\theta = \arctan(-12/5)$.

$\forall n \in \mathbb{Z}$ ■

21.4.10 Let c be any real or complex number other than 0. Then for each integer value of k

$$e^z = e^{z\log c} = e^{z(\log|c|+i(\arg c+2k\pi))}$$

defines a distinct single-valued function of z. Suppose that we set $k = 0$, because then if c is real and equal to e, c^z reduces to the familiar exponential function e^z . Thus let us define

$$e^z = e^{z(\ln|c|+i\arg c)}$$

We call it the generalized exponential function because it allows for any value of $c(\neq 0)$ and reduces to the exponential function e^z for the case where $c = e$. We now state the problem. With e^z defined by (10.2), evaluate c^z for $c = 1 + \sqrt{i}$ and $z = 2 - 5i$.

sol. $(1+\sqrt{i})^{(2-5i)} = e^{(2-5i)(\ln 2+i\pi/3)} = e^{(2\ln 2+5\pi/3)}e^{i(2\pi/3-5\ln 2)} = 4e^{5\pi/3}(\cos(\frac{2\pi}{3} - 5\ln 2) + i\sin(\frac{2\pi}{3} - 5\ln 2))$ ■

21.4.13 We define the inverse of the sine function

$$w(z) = \sin^{-1} z$$

such that $z = \sin w$

(a) Writing the latter as

$$z = \frac{e^{iw} - e^{-iw}}{2i}.$$

show that $e^{iw} = iz + (1 - z^2)^{1/2}$. and hence that

$$\sin^{-1} z = -\log\left[iz + \sqrt{1-z^2}\right]$$

(b) Observe that $\sin^{-1} z$ is multi-valued because of the $(1 - z^2)^{1/2}$ and also because of the $\log[\cdot]$. Specifically, for each value of $z(\neq \pm 1)$. the $(1 - z^2)^{1/2}$ gives two values. To illustrate this point. show that

$$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6} + 2k\pi \quad \text{or} \quad \frac{5\pi}{6} + 2k\pi$$

for $k = 0, \pm 1, \pm 2, ...$

(c) Determine all possible value of $\sin^{-1} 2$

(d) Determine all possible value of $\sin^{-1}(2i)$

sol.

(a) $z = \sin w = (e^{iw} - e^{-iw})/2i$ gives $(e^{iw})^2 - 2iz(e^{iw}) - 1 = 0$, so

$$e^{iw} = \frac{2iz \pm \sqrt{-4z^2+4}}{2} = iz + \sqrt{1-z^2}.$$

We may drop the $\pm$ symbol since the square root of a complex number always gives the $\pm$.

$$\begin{aligned} iw &= \log(iz + \sqrt{1-z^2}) \\ \Rightarrow w &= -i\log(iz + \sqrt{1-z^2}) \\ \Rightarrow \sin^{-1} z &= -i\log(iz + \sqrt{1-z^2}) \end{aligned} \tag{1}$$

(b) Similar to (a),

$$\begin{aligned} \sin^{-1}\left(\frac{1}{2}\right) &= -i\log(\frac{1}{2}i \pm \frac{\sqrt{3}}{2}) \\ &= -i\log(\frac{\pm\sqrt{3}}{2} + \frac{i}{2}) \\ &= -i\log(1e^{i(\frac{\pi}{6}+2k\pi)}) = \frac{\pi}{6} + 2k\pi, \text{ or} \\ &= -i\log(1e^{i(\frac{5\pi}{6}+2k\pi)}) = \frac{5\pi}{6} + 2k\pi, \forall k \in \mathbb{Z} \end{aligned} \tag{2}$$

(c) DIY.

$$\sin^{-1} 2 = (\frac{\pi}{2} + 2k\pi) - i\ln(2 \pm \sqrt{3}), \quad \forall k \in \mathbb{Z}$$

(d) DIY.

$$\sin^{-1}(2i) = \begin{cases} 2k\pi - i\ln(\sqrt{5}-2) \\ (2k+1)\pi - i\ln(\sqrt{5}+2) \end{cases} \quad \forall k \in \mathbb{Z}$$

■

sp.1 判斷21.3的第1題(a)到(l)的所有集合爲open還是closed?connected還是disconnected?請使用上課所教的定義判斷，並直接寫下答案，不需要推論過程。

sol.

open: (b)(c)(e)(g)(h)(j)(l)

closed: (a)(d)(f)(i)(k)

connected: (a)(b)(c)(e)(f)(g)(h)(i)(j)(k)(l)

disconnected: (d)

■

sp.2 令z爲複數，問:sin(z)之值域爲何?

sol. Let $W = \{\sin(z) | z \in \mathbb{C}\}$, we claim that $W = \mathbb{C}$.

" $\subseteq$ " : This is obvious.

" $\supseteq$ " : It means that for every $w \in \mathbb{C}$, we have to find $z \in \mathbb{C}$ such that $w = \sin(z)$. This could be done by exercise 21.4.13, where $z = \sin^{-1} w = -i\log(iw + \sqrt{1-w^2})$ since $i\log(iw + \sqrt{1-w^2})$ always exists for $w \in \mathbb{C}$.

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Remark. 爲何這一題不能令$z = x + iy$後代入去解其值域呢?

思考題 (1) 有沒有既open又closed的集合?

(2) 有沒有不open也不closed的集合?

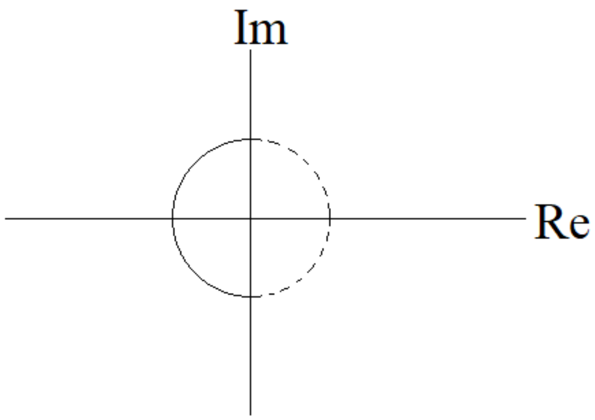
(3) 有沒有既connected又disconnected的集合?

(4) 有沒有不connected也不disconnected的集合?

sol.

(a) $\mathbb{C}$ and empty set $\varnothing$ is both open and closed in $\mathbb{C}$.

(b) $\{z : |z| < 1 \text{ and } \mathrm{Re}\{z\} > 0\} \cup \{z : |z| \leqslant 1 \text{ and } \mathrm{Re}\{z\} < 0\}$



(c) No, since a set which is not connected must be disconnected.

(d) No.

■