Partial Differential Equations and Complex Variables
2020 Fall
Homework 3

19.2.6. If there are some damping of the vibration, the modified wave equation becomes

$$c^2 y_{xx} = y_{tt} + a y_t, \tag{6.1}$$

where a is a known constant. Solve (6.1) by separation of variables, subject to the conditions.

$$y(0,t) = 0, \quad y(L,t) = 0,$$

$$y(x,0) = f(x), \quad y_t(x,0) = 0,$$

for definiteness, suppose that $0 < a < 2\pi c/L$.

sol.

Separating the variables by letting $y(x,t) = X(x)T(t)$, we have

$$\frac{X''}{X} = \frac{T'' + aT'}{c^2 T} = -\kappa^2,$$

which leads to

$$\begin{cases} X'' + \kappa^2 X = 0 \\ T'' + aT' + \kappa^2 c^2 T = 0 \end{cases}$$

Hence we have

$$X(x) = \begin{cases} A\cos\kappa x + B\sin\kappa x, & \kappa \neq 0 \\ C + Dx, & \kappa = 0 \end{cases}$$

and

$$T(t) = \begin{cases} Ee^{\lambda_1 t} + Fe^{\lambda_2 t}, & \kappa \neq 0 \\ G + He^{-at}, & \kappa = 0 \end{cases}$$

where $\lambda = (-a \pm \sqrt{a^2 - 4\kappa^2 c^2})/2$. Now, since B.C. is separable, which means $y(0,t) = y(L,t) = 0$ implies $X(0) = X(L) = 0$ (otherwise $T(t) = 0$ leads to a trivial solution), we may apply B.C. To begin with, if $X(0) = 0 \Rightarrow A = C = 0 \Rightarrow X(x) = Dx$ or $B\sin\kappa x$. Besides, since $X(L) = 0 \Rightarrow D = 0, \quad B\sin\kappa L = 0 \Rightarrow \kappa L = n\pi$. Hence eigenvalues $\kappa = n\pi/L$, $n \in \mathbb{N}$. Then, using the given inequality,

$$\begin{aligned} a^2 + 4\kappa^2 c^2 &= a^2 - \frac{4n^2\pi^2 c^2}{L^2} \\ &= a^2 - \left(\frac{2\pi c}{L}\right)^2 \cdot n \\ &< 0 \end{aligned} \tag{1}$$

Let

$$\omega_n = \sqrt{\left(\frac{n\pi c}{L}\right)^2 - \left(\frac{a}{2}\right)^2},$$

then (show your details)

$$T(t) = \begin{cases} e^{-\frac{at}{2}}(E\cos\omega_n t + F\sin\omega_n t), & \kappa \neq 0 \\ G + He^{-at}, & \kappa = 0 \end{cases}$$

So we have

$$X(x)T(t) = \sin\frac{n\pi x}{L}(e^{-\frac{at}{2}}(E\cos\omega_n t + F\sin\omega_n t))$$

for the basis, and hence

$$y(x,t) = e^{-\frac{at}{2}}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}(E_n\cos\omega_n t + F_n\sin\omega_n t)$$

Applying I.C. to solve for coefficient. However this contribute to differentiating a Fourier series. In fact, by Theorem 17.5.2, it reasonable to employ term-wise differentiation on a Fourier series (and the proof is supplied in advanced calculus text). Write

$$y(x,0) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{L}$$

$$\Rightarrow E_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx.$$

On top of that, because

$$y_t(x,0) = \sum_{n=1}^{\infty}(\omega_n F_n - \frac{a}{2}E_n)\sin\frac{n\pi x}{L} = 0,$$

$$(\omega_n F_n - \frac{a}{2}E_n) = 0 \Rightarrow F_n = \frac{aE_n}{2\omega_n}.$$

Finally, the total solution is given

$$y(x,t) = e^{-\frac{at}{2}} \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{L}(\cos\omega_n t + \frac{a}{2\omega_n}\sin\omega_n t)$$

$$E_n = \frac{L}{2}\int_0^L f(x)\sin\frac{n\pi x}{L}dx.$$

$$\omega_n = \sqrt{\left(\frac{n\pi c}{L}\right)^2 - \left(\frac{a}{2}\right)^2}$$

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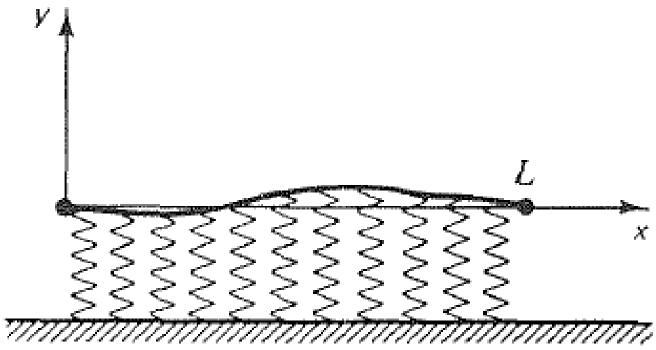
19.2.7. If, as shown in the figure,



a lateral distributed spring is included, then the modified equation of motion of motion for the vibrating string is

$$\tau y_{xx} - ky = \sigma_{tt},$$

where k is the spring stiffness per unit length (newtons per meter per meter) or

$$c^2 y_{xx} - by = y_{tt} \quad \left(c^2 = \frac{\tau}{\sigma}, b = \frac{k}{\sigma}\right) \tag{7.1}$$

Solve (7.1) by separation of variables, subject to the conditions

$$y(0,t) = 0, \quad y(L,t) = 0,$$

$$y(x,0) = f(x), \quad y_t(x,0) = 0,$$

Summarize, in words, the effect(s) of the spring term by in (7.1).

Hint: This equation is homogeneous and all BC are zero, so SOV applied.

sol. Let $y = X(x)T(t)$, we have $c^2X''(x)T(t) - bX(x)T(t) = X(x)T''(t)$

$$\begin{aligned}
&\Rightarrow c^2\frac{X''}{X} - b = \frac{T''}{T} \\
&\Rightarrow \frac{X''}{X} = \frac{T''}{c^2T} + \frac{b}{c^2} \equiv -\kappa^2 \\
&\Rightarrow \begin{cases} X'' + \kappa^2 X = 0 \\ T'' + (b^2 + c^2\kappa^2)T = 0 \end{cases} \\
&\Rightarrow X = \begin{cases} A + Bx, & \kappa = 0 \\ D\cos\kappa x + E\sin\kappa x, & \kappa \neq 0 \end{cases} \\
&\Rightarrow T = \begin{cases} H\cos(bt) + I\sin(bt) & \kappa = 0 \\ J\cos(\sqrt{b^2+c^2\kappa^2}t) + K\sin(\sqrt{b^2+c^2\kappa^2}t), & \kappa \neq 0 \end{cases} \\
&\Rightarrow y(x,t) = (A+Bx)(H\cos(bt) + I\sin(bt)) + (D\cos\kappa x + E\sin\kappa x)(J\cos(\sqrt{b^2+c^2\kappa^2}t) + K\sin(\sqrt{b^2+c^2\kappa^2}t)),
\end{aligned} \tag{2}$$

for some constant A, B, C, E, H, I, J, K. Now applying BC's, $\because y(0,t) = 0$,

$$\therefore A(H\cos(bt) + I\sin(bt)) + D(J\cos(\sqrt{b^2+c^2\kappa^2}t) + K\sin(\sqrt{b^2+c^2\kappa^2}t)) = 0,$$

$\Rightarrow A, D = 0$. We may merge B into H, I and E into J, K. then

$$y(x,t) = (H\cos(bt) + I\sin(bt))x + (J\cos(\sqrt{b^2 + c^2\kappa^2}t) + K\sin(\sqrt{b^2 + c^2\kappa^2}t))\sin\kappa x.$$

Again, $\because y(L,t) = 0$

$$\therefore (H\cos(bt) + I\sin(bt))L + (J\cos(\sqrt{b^2 + c^2\kappa^2}t) + K\sin(\sqrt{b^2 + c^2\kappa^2}t))\sin\kappa L = 0,$$

we have $H = I = 0$ and $\kappa L = n\pi$. Hence the eigenvalues $\kappa_n = n\pi/L$.

Now the fundamental solution forms like

$$y = \phi_n(x,t) = \sin\frac{n\pi x}{L}\left(J\cos\left(\sqrt{b^2 + (\frac{n\pi c}{L})^2}t\right) + K\sin\left(\sqrt{b^2 + (\frac{n\pi c}{L})^2}t\right)\right)$$

Let

$$y = \sum_{n=0}^{\infty}\sin\frac{n\pi x}{L}\left(J_n\cos\left(\sqrt{b^2 + (\frac{n\pi c}{L})^2}t\right) + K_n\sin\left(\sqrt{b^2 + (\frac{n\pi c}{L})^2}t\right)\right).$$

Applying I.C., since $y(x,0) = f(x)$,

$$y = \sum_{n=0}^{\infty} J_n\sin\frac{n\pi x}{L} = f(x).$$

Hence

$$J_n = \int_0^1 f(x)\sin\frac{n\pi x}{L}dx. \tag{7.3}$$

Also, since $y_t(x,0) = 0$, It is clear that $K_n = 0$. Hence we have

$$y = \sum_{n=0}^{\infty} J_n\sin\frac{n\pi x}{L}\cos\sqrt{b^2 + (\frac{n\pi c}{L})^2}t.$$

where J_n is given by (7.3). For word interpretation, you may argue that whether the harmonics appears in the same frequency as a vibrating string . ■

19.2.9. (Non-constant forcing function) In Exercise 8 we included a forcing term that was a constant. The suggested solution technique would have worked even if the forcing term were a non-constant function of x. But in this exercise we allow for t dependence as well. Thus, consider the problem

$$\begin{aligned} c^2 y_{xx} &= y_{tt} + F(x,t) \\ y(0,t) = 0, &\quad y(L,t) = 0, \\ y(x,0) = f(x), &\quad y_t(x,0) = 0. \end{aligned} \tag{9.1}$$

To solve, we can use essentially the same eigenvector expansion method.

(a) Accordingly, solve (9.1) by seeking

$$y(x,t) = \sum_{n=1}^{\infty} h_n(t)\sin\frac{n\pi x}{L}$$

and expanding

$$F(x,t) = \sum_{n=1}^{\infty} F_n(t)\sin\frac{n\pi x}{L}$$

and

$$f(x) = \sum_{n=1}^{\infty} f_n(t)\sin\frac{n\pi x}{L}$$

where the coefficients

$$F_n(t) = \int_0^L F(x,t)\sin\frac{n\pi x}{L}$$

and

$$f_n = \int_0^L f(x)\sin\frac{n\pi x}{L}$$

are considered as known [i.e., compute-able from $F(x,t)$ and $f(x)$]. With $\omega_n = n\pi c/L$, show that

$$y(x,t) = \sum_{n=1}^{\infty}\left[f_n\cos\omega_n t + \frac{1}{\omega_n}\int_0^t F_n(\tau)\sin\omega_n(\tau - t)d\tau\right]\sin\frac{n\pi x}{L}$$

(b) With the help of the Leibniz rule formally verify that (9.6) satisfies (9.1).

(c) Work out the solution (9.6) for the case where $F(x,t) = F_0\sin\Omega t$ and $f(x) = 0$. assuming that the driving frequency does not equal any of the natural frequencies, say ω_k

(d) Same as (c), but where Ω equals one of the natural frequencycies, say ω_k.

sol. (a) Since $c^2 y_{xx} = y_{tt} + F(x,t)$, by termwise differentiation,

$$-c^2 \sum_{n=1}^{\infty} h_n(t)\left(\frac{n\pi}{L}\right)^2 \sin\frac{n\pi x}{L} = \sum_{n=1}^{\infty} h_n''(t)\sin\frac{n\pi x}{L} + \sum_{n=1}^{\infty} F_n(t)\sin\frac{n\pi x}{L}.$$

Hence $\forall n \in \mathbb{N}$,

$$-c^2\left(\frac{n\pi}{L}\right)^2 h_n(t) = h_n''(t) + F_n(t).$$

and I.C.'s of this ODE are $h_n(0) = f_n$, $h_n'(0) = 0$. By Laplace transform,

$$s^2\hat{h}_n(s) - sf_n\left(\frac{n\pi c}{L}\right)^2\hat{h}_n(s) = -\hat{F}_n(s).$$

After doing some algebra, we have

$$\begin{aligned}
\hat{h}(s) &= \frac{s}{s^2 + (n\pi c/L)^2} f_n - \hat{F}_n(s)\frac{1}{s^2 + (n\pi c/L)^2} \\
\Rightarrow h_n(t) &= f_n \cos\frac{n\pi ct}{L} - F_n(t) * \frac{L}{n\pi c}\sin\frac{n\pi ct}{L} \qquad (3) \\
&= f_n\cos\frac{n\pi ct}{L} - \frac{L}{n\pi c}\int_0^t F_n(\tau)\sin\frac{n\pi c}{L}(t-\tau)d\tau
\end{aligned}$$

so

$$y(x,t) = \sum_{n=1}^{\infty}\left(f_n\cos\omega_n t + \frac{1}{\omega_n}\int_0^t F_n(\tau)\sin\omega_n(t-\tau)d\tau\right)\sin\frac{n\pi x}{L},$$

where

$$\omega_n = \frac{n\pi c}{L}.$$

(b) Let

$$f(t) = \int_0^t g(t,\tau)h(\tau)d\tau,$$

Leibniz rule gives that

$$f'(t) = g(t,t)h(t) + \int_0^t\left(\frac{\partial}{\partial t}g(t,\tau)\right)h(\tau)d\tau.$$

The BC's and the IC $y(x,0) = 0$is obvious, we only have to check whether $y_t(x,0) = 0$

$$y_t(x,0) = \sum_{n=1}^{\infty}\left(-f_n\omega_n\sin\omega_n t\Big|_{t=0} + \frac{1}{\omega_n}\int_0^t F_n(\tau)\sin\omega_n(\tau - t)d\tau\Big|_{t=0}\right)\sin\frac{n\pi x}{L} = 0$$

(c) Compute F_n by the formula of Fourier series,

$$\begin{aligned}
F_n(t) &= \frac{2}{L}\int_0^L\left(F_0\sin\Omega t + \sin\frac{n\pi x}{L}\right)dx \\
&= \begin{cases} \frac{4F_0}{n\pi}\sin\Omega t & \text{odd } n, \\ 0 & \text{even } n. \end{cases} \qquad (4)
\end{aligned}$$

Since $f(x) = 0$, we must have $f_n = 0$. Then

$$y(x,t) = \frac{4F_0}{n\pi}\sum_{n=1,3,5,\ldots}\frac{1}{n\omega_n}\frac{\omega_n\sin\Omega t - \Omega\sin\omega_n t}{\Omega^2 - \omega_n^2}\sin\frac{n\pi x}{L}$$

(d) If $\Omega = \omega_k$ for some odd k, then one of the term in the series would have a zero denominator. This could be avoided by taking limit for that term.

$$y(x,t) = \frac{4F_0}{n\pi}\sum_{n=1,3,5,\ldots,n\neq k}\frac{1}{n\omega_n}\frac{\omega_n\sin\Omega t - \Omega\sin\omega_n t}{\Omega^2-\omega_n^2}\sin\frac{n\pi x}{L} + \lim_{\Omega=\omega_k}\frac{1}{k\omega_k}\frac{\omega_k\sin\Omega t - \Omega\sin\omega_k t}{\Omega^2 - \omega_k^2}\sin\frac{k\pi x}{L}$$

By L'Hospital rule,

$$\lim_{\Omega=\omega_k}\frac{\omega_k\sin\Omega t - \Omega\sin\omega_k t}{\Omega^2-\omega_k^2}\sin\frac{k\pi x}{L} = \frac{\omega_k t\cos\omega_k t - \sin\omega_k t}{2\omega_k}$$

Putting it back,

$$y(x,t) = \frac{4F_0}{n\pi}\sum_{n=1,3,5,\ldots,n\neq k}\frac{1}{n\omega_n}\frac{\omega_n\sin\Omega t - \Omega\sin\omega_n t}{\Omega^2-\omega_n^2}\sin\frac{n\pi x}{L} + \frac{\omega_k t\cos\omega_k t - \sin\omega_k t}{2k\omega_k^2}$$

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Supplemented problem. Derive wave equation by Maxwell's equation and find the speed of light.

sol. Taking curl on both side of

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

we have

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = \nabla \times (-\frac{\partial \mathbf{B}}{\partial t}).$$

Since $\nabla \cdot \mathbf{E} = 0$, we have

$$\nabla \times (-\frac{\partial \mathbf{B}}{\partial t}) = -\nabla^2 \mathbf{E}.$$

So,

$$\begin{aligned}
\nabla^2 \mathbf{E} &= \nabla \times \frac{\partial \mathbf{B}}{\partial t} \\
&= \frac{\partial}{\partial t}(\nabla \times \mathbf{B}) \\
&= \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \\
&= \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}
\end{aligned} \tag{5}$$

So we have speed of light $c = 1/\sqrt{\mu_0 \epsilon_0}$

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Remark. So far, we have known some techniques of solving PDEs:

- **Separation of variables.** Sometimes we need to divide a problem it to 2 or more problems.
- **Eigenfunction expansion.** An application of Fourier series

and, as a EE student, you should at least know the following techniques to evaluate an integral:

- **Change of variables.** Such as u-substitution or triangular substitution.
- **Integration by parts.**
- **Integral transform.** Such as Fourier transform and Laplace transform. These could be useful in communication system and signal and system.
- **Construct an ordinary differential equation.** Usually accompany with Leibniz rule.
- **Complex integration** We'll meet it soon. It's powerful in digital signal processing and communication system.