Midterm exam (04/12/2022)

Note: The important thing is to obtain the correct and accurate answer. Every problem is designed to be independent. In other words, no need to get correct answer of one problem to find the answer of another problem.

1.(20%) In Sec 11.11 of Kreyszig's book, you can find solution of Laplace equation for electrostatic potential in spherical coordinates.

Find the solution for the interior problem

For the interior problem of potential (r<1) is given by a series

$$u(r,\phi)=\sum_{n=0}^{\infty}A_n r^n P_n(cos\phi)$$

If the potential on the spherical surface (r=1) is given by

$$u(r=1,\phi)=cos^3\phi$$

Find the potential for the interior problem (r<1) and exterior problem (r>1) (20%)

You can also use the following Legendre polynomial.

n	$P_n(x)$
0	1
1	x
2	$\frac{1}{2}\left(3x^2-1\right)$
3	$\frac{1}{2}\left(5x^3-3x\right)$
4	$\frac{1}{8}\left(35x^4-30x^2+3\right)$
5	$\frac{1}{8}\left(63x^5-70x^3+15x\right)$

Note: for the exterior problem (r>1), the solution is given by another series

$$u(r,\phi)=\sum_{n=0}^{\infty}B_n\frac{1}{r^{n+1}}P_n(cos\phi)$$

Answer:

You can find the linear combination of costheta ^4

$$cos^3\theta=\frac{2}{5}P_3(\cos\theta)+\frac{3}{5}P_1(\cos\theta)$$

r<1

$$u=\frac{2}{5}r^3P_3(\cos\theta)+\frac{3}{5}r^1P_1(\cos\theta)$$

r>1

$$u=\frac{2}{5}\frac{1}{r^4}P_3(\cos\theta)+\frac{3}{5}\frac{1}{r^2}P_1(\cos\theta)$$

2. (20%) Solve this problem by Laplace transform to find the solution

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial t^2} + 8\frac{\partial \ \ w}{\partial t} + 16w$$

$$w(x,0) = 0 \ if \ x \geq 0$$

$$w_t(x,0) = 0 \ if \ x \geq 0$$

$$w(0,t) = \exp{(-t)} \ if \ t \geq 0$$

(To have a reasonable and physical solution, you can assume w(x,t) ->0 when x goes to infinity.)

The final solution is given by

Take the Laplace transform of both side with respect to t

$$\frac{\partial^2 W(x,s)}{\partial x^2} = sW(x,s) + 8sW(x,s) + 16W(x,s)$$

$$W(x,s) = B(s)e^{-(s+4)x} + C(s)e^{s+4}$$

We can reject C(s) because it blows up the solution.

$$W(0,s) = B(s) \quad = \mathcal{L}\{w(0,t)\} = F(s)$$

$$W(x,s) = F(s)e^{-(s+3)x}$$

We can use the shift theorem to find the inverse transform

$f(t-a)u(t-a) = \mathcal{L}^{-1}\{e^{-as}F(s)\}$

$$w(x,t) = e^{-4x}\exp{[-(\mathrm{t}-\mathrm{x})]}u(t-x)$$

You can see that the extra term w_t and w introduces a damping term e^{-4x} in space.

3. (10%) For the quantum mechanical simple harmonic oscillator in two dimension,

$$[-\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{1}{2}(x^2 + 4y^2))\phi \quad (x,y,z) = E\phi(x,y,z)$$

We can use the operator method to find the lowest eigen function with the procedure given below. The ground state satisfies

$$a_x a_y \phi_0 = 0$$

where ax, ay are annihilation operation for x and y coordinate, respectively.

Solve this partial differential equation to find ground state

$$ax \equiv (x + \frac{\partial}{\partial x})$$

$$ay \equiv (2y + \frac{\partial}{\partial y})$$

Solution:

$$a_y \phi_0(x,y,z) = 0$$

$$\phi_0(x,y,z) = C(x)\exp{[-y\text{^}2]}$$

since we can interchange ax and ay

$$a_x \phi_0(x, y, z) = 0$$

$$a_x C(x) \exp[-y^2] \quad = 0$$

$$C(x) = D\exp[-\frac{x^{\wedge}2}{2}]$$

D is an arbitrary constant, the final answer is

$$\phi_0(x, y, z) = D\exp[-\frac{(x^2 + 4y^2)}{2}]$$

C is a constant for normalization. (You do not need to normalize the function)

4.(10%) From the asymptotic approximation formula, find the third zero for $J_3(x)$.

Hint: The exact value of first zero J3 is 6.38.

Note that the first zero of J3(x) is NOT x=0 from the usual convention.

$$J_n(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)$$

Answer:

$$J_3(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{3\pi}{2} - \frac{\pi}{4}\right) \sim$$

$$x = \frac{9\pi}{4}, \frac{13\pi}{4}, \frac{17\pi}{4} \quad \frac{21\pi}{4} \ldots$$

x= 7.0686, 10.2102, 13.3518, 16.4934

We can see x= 7.0686 corresponds to first zero.

Therefore, x= 17*pi/4 corresponds to third zero.

5. (10% each) For 2D wave equation for a circular membrane with $c^2=1$,

$$u_{tt} = (u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta})$$

again, the boundary condition is u(r, θ) =0 on the edge of the membrane.

Find the lowest eigenfunction of a membrane of Octant (八分圓) of a circle of radius R= 1 ， defined by r<1 and 0<θ< π/4

Hint: separation of variable with u= G(r)F(r,theta) gives an ODE and PDE

$$\ddot{G} + \lambda^2 G = 0$$

$$F_{rr} + \frac{1}{r} F_r + k^2 F = 0$$

F=W(r)Q(theta)

$$\ddot{Q} + n^2 Q = 0$$

$$r^2W" + rW' + (k^2r^2 \; - n^2)W = 0$$

This ODE is actually Bessel function of order n.

solution :

$J_4(\alpha_{4,1}r)\sin(4\theta)$ $\alpha_{2,1}$ is the first zero of Bessel function of second order J_2

You can easily verify sin(2*(pi/2))=0

6. (20%) For a 2D Laplace equation in polar coordinate, $\nabla^2 u(r,\theta) = 0$, we have worked out the solution using separation of variable. The solution in the disk r<R, (assuming R=1, for simplicity) is given by

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} a_n r^n \cos(n\theta) + b_n r^n \sin n\theta$$

if the potential is specified at R=1 as

$$u(r=1,\theta) = 1 \;\; if \; 0 < \theta < \pi$$

$$u(r=1,\theta) = -1 \;\; if -\pi < \theta < 0$$

Find the solution $u(r,\theta)$ to satisfy this boundary value problem.

You may use Fourier expansion formula for a periodic function of period 2pi

$$f(x) = \; c_0 + \sum_{n=1}^{\infty} c_n \cos(nx) + d_n \sin nx$$

the Fourier coeffients are given by

$$d_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$cn = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$c0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

solution : Fourier series expansion let f(x) be a period function of period 2pi

$$f(x) = \; u(r=1,x) = a_0 + \sum_{n=0}^{\infty} a_n \cos(nx) + b_n \sin nx$$

$$f(x) = a0 + \sum_{n=1}^{\infty}(an\; cosnx + bn\; sinnx)$$

Note that this pontial is an odd function so that an =0 and a0 =0. We only need to calculate bn

$$bn = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{2}{n\pi}(1 - \cos n\pi)$$

cos npi = -1 for odd n

cos npi = 1 for even n

thefore we can conclude for even n, bn=0

for odd n

$$bn = \frac{4}{n\pi}$$

Final solution

$$u(r,\theta) = \sum_{n=0}^{\infty} b_n r^n \sin n\theta = \sum_{n=1,3,5,\dots}^{\infty} \frac{4}{n\pi} r^n \sin n\theta$$

7. (10%) Sturm-Liouville theorem states that for ODE of the following form

$$(p(x)y')' + [q(x) + \lambda w(x)]y = 0$$

Suppose $y_m(x)$ and $y_n(x)$ are solutions satisfying certain boundary condition to this ODE with corresponding eigen value λ_m and λ_n. (You can think of m and n are indices to label eigenvalues.)

Find the solution to the ODE

$$(y')' + \lambda y = 0$$

with the boundary condition y'(0) =0 and y'(a)=0 and their corresponding eigen values lamda .

Solution

$$y(x) = \cos(kx)$$

$$y'(x) = -\mathrm{k}\sin(kx)$$

$$y'(a) = -\mathrm{k}\sin(ka) = 0$$

k = npi/a , n= 1,2,3,4…..

y(x) = cos (n pi /a)   lamda = (n*pi/a)^2