Final exam (06/14/2022) You need to turn on you video camera and show your face to TA. TA will monitor on-line.

Note: The important thing is to obtain the correct and accurate answer.(數字全對才給分) Every problem is designed to be independent. In other words, no need to get correct answer of one problem to find the answer of another problem.

Total score is 120 % (20% bonus) but you can only get up to 100% maximal points even if your score exceed 100%.

code: kyoto

please write this code for identification

1. (20%)Use the conform mapping to map a non-coaxial geometry to a coaxial geometry

$f(z) = \frac{z-b}{bz-1}$

where b is a real number. In other words, we wish map $C1: |z| = 1$ into $C_1^*: |z| = 1$ (a unit disk to a unit disk) and map C2 into C2* with

$C2: \left|z - \frac{5}{2*13}\right| = \frac{5}{2*13}$ into $C_2^*: |z| = r_0$

Find b and r0 （10% each）

(Hint: check example 1 in Chapter 18.2)

Solution: similar to example 1 in Kreyszig 18.2 , we wish to map z=0 into $-r_0$ and z=1/sq2 to z=0

We will arrive two equation for b and r0

$r_0 = b$

$$-cb^2 + 2b - c = 0$$

$$c = 5/13$$

solving the quadratic equation , we obtain

b= 1/5 , 5 , we need to reject 5 >1

so

b= 1/5

r0 = 1/5

2.(20%) Compute the following integral with residue integration method

$$I = \oint_C (\frac{df}{dz})^2 dz$$

where C: $|z| = 1$ counterclockwise

$$f(z) = e^{i2\pi/3}\left(z + \frac{9}{z}\right) - i\ln z$$

Answer: the integral is in Blasius theorem for gamma =1 U= $e^{ipi/3}$

$$I = 2\pi i\left(-\frac{ie^{i2\pi/3}}{1}\right) =$$

or you can do the differentiation

$$f'(z) = e^{i2\pi/3}\left(1 - \frac{9}{z^2}\right) - \frac{i}{z}$$

$$I = \oint_C [e^{i2\pi/3}\left(1 - \frac{3^2}{z^2}\right) - \frac{i}{z}]^2 dz$$

It seems complicated at first glance but you only need a term proportional to ~ 1/z .

According to residue integration method, other terms amount to zero integral.

You can do explicit expansion of $(f'(z))^2$ and identify 1/z. The end result is

$$[e^{i2\pi/3}\left(1 - \frac{9}{z^2}\right) - \frac{i}{z}]^2 = [e^{i2\pi/3}\left(1 - \frac{9}{z^2}\right)]^2 + [-\frac{i}{z}]^2 + 2 * e^{i2\pi/3}(1 - \frac{3^2}{z^2})(-\frac{i}{z})$$

$=+2 * e^{\frac{i2\pi}{3}}\left(-\frac{i}{z}\right) + other\ terms$ Only $2 * e^{2i\pi/3}\left(-\frac{i}{z}\right)$ is important and is the residue so

$I = 2\pi i * (-2\frac{e^{\frac{i2\pi}{3}} i}{1}) = 4\pi e^{i2\pi/3}$

3. (20%) Use Newton's method with 2 iterations to solve $x^5 = 2$ and calculate $2^{1/5}$ and use initial value x0=1. In other words, find x1 and x 2 to 3 decimal digits(小數點以下三位). (each 10%)

Ans:

$f(x) = x^5 - c = 0$ $f'(x) = 5x^4$

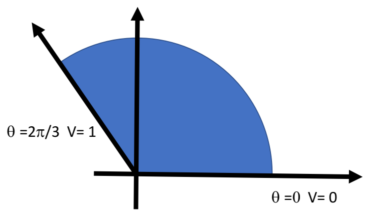
$x_{n+1} = x_n - (x_n^5 - c)/(5x^4) = (4/5)x_n + (c/5x_n^4)$

c=2

iteration x1 = 1.200 x2= 1.153

the exact value is 1.149 (3 digits)

4. (20%) Find the complex potential for the electrostatics problem in the sector $0 <= \theta <= \pi/3$ between the boundary theta θ=0 kept at 0 V and θ= 2π/3 kept at 1 V



solution

f(z) = a*i*ln z +b a, b real

$$\ln(z) = \ln(r) + i(\theta)$$

The real part of the complex potential is Re f(z) = $\Phi(r,\theta) = -a * \theta + b$

θ=0 kept at 0 V

$\Phi(r,\theta) = -a * 0 + b = 0$ b=0

θ= 2π/3 kept at 1 V

$$\Phi(r,\theta) = -a * 2\pi/3 + b = 1$$

b= 0

a= -3/pi

$$f(z) = -\left(\frac{3}{2\pi}\right) i\ln(z)$$

5. (20%) Evaluate the integral

$$\int_c \frac{dz}{z^2+4}$$

for a counter clockwide circuit center at z=0 and radius 3

$$|z-i| = 2$$

The singularities are +2i and -2 i

The circle enclose both singularities.

Method 1: using residue theorem

$$\int_c \frac{1}{z^2+4} dz = 2\pi i (\frac{1}{2z}) \quad {}_{z=2i} = \pi/2$$

Method 2: using Cauchy integral theorem

$$\int_c \frac{dz}{z^2+4} = 2\pi i \left(\frac{1}{z+2i_{z=+2i}}\right) = \pi/2$$

6. (20%) Evaluate the integral using residue theorem

$$I = \int_0^{2\pi} \frac{d\theta}{13+5cos\theta}$$

solution:

We can do the contour integral along a unit circle with counter clockwide direction.

$$\cos\theta = \left(\frac{1}{2}\right)\left(z+\frac{1}{z}\right) \quad z = e^z \; dz = izd\theta$$

$$I = \oint_C \frac{dz/iz}{13 + (\frac{5}{2})(z + \frac{1}{z})} = \oint_C f(z)dz$$

$$f(z) = \frac{2}{i}\frac{1}{26z + (5z^2 + 5)}$$

The denominator $5z^2+26z+5=0$ two solution $z_+= -1/5$ $z_-=-5$ There are two simple poles, z_+ and z_-. Only z_+ is enclosed by the unit circle. $z_-=5$ is outside the unit cicle. We can evaluate the residue at z_+

$$Resf(z) = \frac{2}{i}\frac{1}{10z + 26}_{z=-1/5} = \frac{2}{i}\frac{1}{24} = \frac{1}{12i}$$

$$I = 2\pi i \frac{1}{12i} = \frac{\pi}{6}$$