Partial Differential Equations and Complex Variables
Final Exam

1. (a) You may write down the result directly.

$$p_n = \frac{1}{b^n \pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta d\theta$$

$$q_n = \frac{1}{b^n \pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta d\theta$$

■

(b) This follows directly from the mean value property of harmonic functions and that

$$u(0, \theta) = I$$

.

(c) For any point $z \in D$, by the mean property of harmonic functions, $\because u|_p$ is bounded by maximum and minimum value on $B_r(p)$, for any $r > 0$ such that $B_r(p) \subset D$, $\therefore u$ cannot reach its extreme value in D. This implies either u =constant or u attains its maximum/minimum in $\partial B_r(p)$. ■

2. (a) Consider

$$\begin{aligned} f(z) &= u(r,\theta) + i \sum_{n=0}^{\infty} p_n r^n \sin n\theta \\ &= \sum_{n=0}^{\infty} p_n r^n \cos n\theta + i \sum_{0}^{\infty} p_n r^n \sin n\theta \\ &= \sum_{n=0}^{\infty} p_n r^n (\cos n\theta + i \sin n\theta) \\ &= \sum_{n=0}^{\infty} a_n z^n \end{aligned} \tag{1}$$

(b) By lemma 24.2.1, since the power series $f(z)$ converges at 0, it converges everywhere in the open disk $|z| < |b|$. So the radius of convergence is greater or equal to b. ■

3. (a) By solving the two ODEs, we obtain

$$R = \begin{cases} A + B \ln r & \text{if } \kappa = 0 \\ C J_0(\kappa r) + D Y_0(\kappa r) & \text{if } \kappa \neq 0 \end{cases}$$

$$Z = \begin{cases} E + Fz & \text{if } \kappa = 0 \\ G \cosh(\kappa z) + H \sinh(\kappa z) & \text{if } \kappa \neq 0 \end{cases}$$

for some constant A, B, C, D, E, F, G, H. Then

$$u(r,z) = (A + B \ln r)(E + Fz) + (C J_0(\kappa r) + D Y_0(\kappa r))(G \cosh(\kappa z) + H \sinh(\kappa z)).$$

If we require u to be bounded, $B = 0$ and $D = 0$, merge A into E and F and D into G and H, then

$$u(b,z) = E + Fz + J_0(\kappa r)(G \cosh(\kappa z) + H \sinh(\kappa z)).$$

Applying $u(b, \theta, z) = 0$,

$$u(b,z) = E + Fz + J_0(\kappa b)(G \cosh(\kappa z) + H \sinh(\kappa z)).$$

So $E = F = 0$, and $\kappa b = z_n$ the zeros of J_0 from small to large. We define $\kappa_n = z_n/b$. Hence we have found at least one nontrivial solution $J_0(z_n r/b)(G \cosh(z_n z/b) + H \sinh(z_n z/b))$, $n = 0, 1, 2, ...$

(b) This equation satisfies the form of Sturm-Liouville theorem with weight function $w(r) = r$. The inner product should be defined as

$$< R_n(r), R_m(r) >= \int_0^b R_n(r) R_m(r) r dr$$

■

4. (a) $f(z)$ is analytic in the region that contains P and Q.

(b) Claim: $f(z) = \bar{z}$ is continuous everywhere in $\mathbb{C}$. However it is nowhere analytic in $\mathbb{C}$. Proof of claim: On the direction parallel to x-axis,

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{x + \Delta x - x}{\Delta x} = 1.$$

However, on the direction parallel to y-axis

$$\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}=\lim_{\Delta y\to 0}\frac{-iy-i\Delta y+iy}{i\Delta y}=-1.$$

Hence $f(z)=\bar{z}$ is nowhere analytic in $\mathbb{C}$. ■

5. By Cauchy-Riemann relation, $u_x=v_y=3x^2-3y^2$, so $v=3x^2y-y^3+A(x)$. Also $u_y=-v_x=-6xy+A'(x)$, so $A'(x)=0$ and $A(x)=$ constant. Hence $v(x,y)=3x^2y-y^3+C$, for any constant C. ■

6. Claim: $\cos^{-1}z=-i\ln(z+\sqrt{z^2-1})$

Proof of claim: Let $w=\cos^{-1}z$ or $z=\cos w=(e^{iw}-e^{-iw})/2$ gives $(e^{iw})^2-2z(e^{iw})-1=0$, so

$$e^{iw}=\frac{2z\pm\sqrt{4z^2-4}}{2}=iz+\sqrt{1-z^2}.$$

We may drop the $\pm$ symbol since the square root of a complex number always gives the $\pm$. So $w=\cos^{-1}z=-i\ln(z+\sqrt{z^2-1})$.

Let $W=\{\cos(z)|z\in\mathbb{C}\}$, we claim that $W=\mathbb{C}$.

" $\subseteq$ " : This is obvious.

" $\supseteq$ " : Since $i\log(iw+\sqrt{1-w^2})$ always exists for $w\in\mathbb{C}$, done.

■

7. (a) Expand $f(z)$ as Laurent series.

$$f(z)=\frac{\sin z}{z^4}=z^{-4}\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-...\right)=\frac{1}{z^3}-\frac{1}{3!z}+\frac{z}{5!}-...$$

So it is a 3rd-order pole.

(b) Observe that the residue at $z=0$ is -1/6, so by Residue theorem,

$$\oint_C f(z)dz=2\pi i\times\frac{-1}{6}=-\frac{\pi i}{3}.$$

(c) $z=\pm\pi,\pm2\pi,\pm3\pi,...$ are poles. For $z=0$, $\sin z$ has a first-order zero, and z^4 has a fourth-order zero. By theorem 24.4.1, since $1<4$, $z^4/\sin z$ is analytic at $z=0$.

(d) All poles are outside C, so the integral equals to 0. ■

8. Let $z=e^i\theta$, $\cos\theta=\frac{z^2+1}{2z}$, and C be the contour $|z|=1$, then

$$\begin{aligned}\int_0^{2\pi}\frac{1}{(3-2\cos\theta)^2}d\theta&=\oint_C\frac{z}{z(3-\frac{z^2+1}{z})^2}\frac{dz}{iz}\\&=\oint_C\frac{-iz}{(3z-(z^2+1))^2}dz\\&=\oint_C\frac{-iz}{((z-z_0)(z-z_1))^2}dz,\end{aligned}\tag{2}$$

where

$$z_0,z_1=\frac{3\pm\sqrt{5}}{2}.$$

However only z_1 (who takes the minus sign) is in the inside of C. So by residue theorem,

$$\begin{aligned}\oint_C\frac{-iz}{((z-z_0)(z-z_1))^2}dz&=-i\times2\pi i\mathrm{Res}_{z=z1}\frac{-iz}{((z-z_0)(z-z_1))^2}\\&=2\pi\Big(\lim_{z\to z_1}\frac{d}{dz}\frac{z}{(z-z_0)^2}\Big)\\&=2\pi\left(\frac{(z-z_0)^2-2z(z-z_0)}{(z-z_0)^4}\right)\\&=2\pi\frac{-3}{(-\sqrt{5})^3}\\&=\frac{6\pi}{5\sqrt{5}}\end{aligned}\tag{3}$$

We conclude that

$$\int_0^{2\pi}\frac{1}{(3-2\cos\theta)^2}d\theta=\frac{6\pi}{5\sqrt{5}}.$$

■