NTHU EE2020: PDE and Functions of a Complex Variable

Final Exam

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請注意：為簡化作答，任何課本上有的定理或上課證明過的結果，都可以直接使用。

1. (16 points). Let $u = u(r,\theta)$, where (r,θ) denotes the polar coordinate, be a function that satisfies the Laplace equation:

$\nabla^2 u = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) u = 0$. By separation of variables, set $\tilde{u} = R(r)\Theta(\theta)$.

We have

$$\begin{cases} r^2 R'' + rR' - \kappa^2 R = 0 \\ \Theta'' + \kappa^2 \Theta = 0 \end{cases}$$

Assume that we are given the boundary condition $u(b,\theta) = f(\theta)$, and would like to solve Laplace equation in the open disk $r < b$. We know that the solution has the following expression:

$u(r,\theta) = I + \sum_{n=1}^{\infty} r^n (p_n \cos n\theta + q_n \sin n\theta)$.

(a) [8%] Express p_n and q_n in terms of $f(\theta)$.

(b) [3%] Show that $I = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, d\theta$.

(c) [5%] Continuing from (b), briefly explain how this result leads to the *maximum principle* --- let D be any simply connected open region in $\mathbb{R}^2$ then any $u(x,y)$ satisfying $\nabla^2 u = 0$ in D cannot attain its maximum in the interior of D (unless u is a constant everywhere).

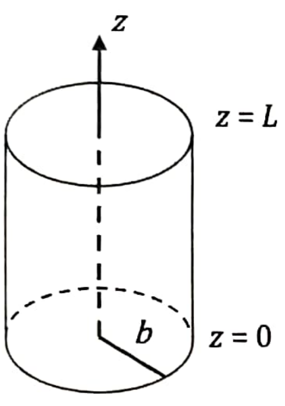
2. (12 points). Continuing from Problem 1, let $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$. Assume that we have the special case that $q_n = 0$ for all $n \in N$.

(a) [5%] Show that $u(r,\theta) = I + \sum_{n=1}^{\infty} p_n r^n \cos n\theta$ can be written as the real-part of a power series of z. (Hint: $r^n \cos n\theta = Re[r^n e^{in\theta}]$).

(b) [7%] Denote that power series as $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Argue that its radius of convergence is greater than or equal to b.

3. (10 points). Sturm-Liouville Theory. Assume that we are solving the Laplace equation $\nabla^2 u = 0$ inside a cylinder shown below. In the cylindrical coordinate, assume that a solution can be written as $u(r,\theta,z) = R(r)Z(z)$. Assume that we have the following boundary condition: $u(b,\theta,z) = 0$. Then, we can derive the following two equations:

$$\begin{cases} R'' + \frac{1}{r}R' + \kappa^2 R = 0, & \text{(Eq. 1)} \\ Z'' - \kappa^2 Z = 0. & \text{(Eq. 2)} \end{cases}$$

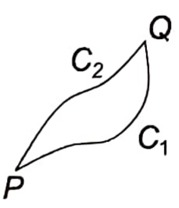


(a) Explain how to find values of κ such that (Eq. 1) has non-trivial solutions. Draw a picture if necessary. (Hint: Bessel function)

(b) Continuing from above, let $\{\kappa_1, \kappa_2, \dots, \kappa_n, \dots\}$ denote the set of all such κ's. Then, the eigenfunction $R_n(r)$ associated with eigenvalue κ_n^2 satisfies the following equation: $(rR_n')' + \kappa_n^2 r R_n = 0$. Under what definition are $R_n(r)$ and $R_m(r)$ orthogonal (if $n \neq m$)?

以下皆為複變題，均假設$z = x + iy$, 並且$f(z) = u(x,y) + iv(x,y)$ 其中u與v 分別為實部與虛部。

4. (12 points). Basic properties of complex functions.



(a) As shown on the left, let $P, Q \in \mathbb{C}$, and let C_1 and C_2 be two paths connecting from P to Q. What is a sufficient condition such that $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$?

(b) Show an example of $f(z)$ that is continuous everywhere in $\mathbb{C}$ but is not differentiable with respect to z anywhere.

5. **(10 points). Cauchy-Riemann relations.** Let $f(z) = (x^3 - 3xy^2) + iv(x, y)$. Find $v(x, y)$ such that $f(z)$ is analytic over the entire z plane.

6. **(10 points).** Show that the range of $f(z) = \cos z$ is the entire complex plane.

7. **(20 points).** Let $f(z) = \sin z / z^4$, $g(z) = \frac{1}{f(z)} = \frac{z^4}{\sin z}$, and let C be the counterclockwise circle $|z| = 1$.

(a) [4%] What is the order of pole of $f(z)$ at $z = 0$?

(b) [6%] Calculate $\oint_C f(z)dz$.

(c) [4%] Find all the poles of $g(z)$.

(d) [6%] Continuing from above, evaluate $\oint_C g(z)dz$.

(Hint: $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$)

8. **(10 points).** Calculate $\int_0^{2\pi} \frac{1}{(3-2\cos\theta)^2} d\theta$.