Partial Differential Equations and Complex Variables
Homework 7

23.5.3 We'll prove **fundamental theorem of algebra** in this problem.

(a) Show from

$$\oint_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i}{n!}f^{(n)}(a).$$

that if C is a circle of radius ρ with center at z, $f(z)$ is analytic inside and on C, and M is the maximum value of $|f(z)|$ on C, then

$$|f^{(n)}(z)| \leqslant \frac{n!M}{\rho^n}.$$

(b) Prove **Liouville's theorem**: If f is entire (i.e. analytic for all finite z) and bounded for all z, then f is a constant.

(c) Since $f(z) = \sin z$ is entire and not a constant, it must not be bounded (according to Liouville's theorem). Demonstrate that, in fact, it is not bounded.

(d) Prove **fundamental theorem of algebra**: if $P(z)$ is a polynomial function of z, of degree 1 or greater;

$$P(z) = a_n z^n + a_{n-1}z^{n-1} + ... + a_0 \quad (a_n \neq 0)$$

then $P(z) = 0$ has at least one root.

HINT: Suppose that $P(z)$ is nonzero everywhere. Then $f(z) = 1/P(z)$ is analytic everywhere and is bounded.

sol. (a) ML bound gives that

$$\left|\frac{2\pi i}{n!}f^{(n)(z)}\right| = \left|\oint_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}d\zeta\right| \leqslant \frac{M}{\rho^{n+1}} \cdot 2\pi\rho$$

so

$$|f^{(n)}(z)| \leqslant \frac{n!M}{\rho^n}.$$

(b) Claim: Let f be a holomorphic function on an open connected domain $\Omega \in \mathbf{C}$. Suppose $f' = 0$, Then f is a constant function.

proof of claim: May assume Ω is path connected, arbitrarily choose a curve γ that connect z_0 and z_1, then by fundamental theorem of complex integral calculus (Theorem 23.4.1),

$$\int_\gamma f'(\omega)d\omega = f(z_1) - f(z_0).$$

so $f(z_1) = f(z_0)$, f is a constant.

Now using the assertion of (a). Letting $\rho \to \infty$, we find $f'(z_0) = 0$ on $\mathbf{C}$. So by the claim, f is a constant.

(c) On imagine axis, $\sin z = \sin iy = i \sinh y$ is unbounded.

(d) Suppose $P(z)$ is nonzero everywhere, $f(z) = 1/P(z)$ is analytic everywhere. By Liouville's theorem, it must be a constant, which is contradict to the form of $P(z)$ (unless $n = 0$). ■

23.5.4 (Dirichlet problems) As mentioned in the text, just as the Cauchy integral formula

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z}d\zeta$$

express an analytic function $f(z) = u + iv$ in terms of its boundary values, we would expect there to exist a similar integral formula expressing a harmonic function $u(x, y)$ in a formula for two important cases: the case where the domain is a circular disk, and the case where the domain is the upper half plane

(a) (**Poisson integral formula for the circular disk**) Let C be the counterclockwise circle $|\zeta| = R$. If we seek the desired expression for u by equating real parts of the left-and right-hand sides of Cauchy integral formula,we find that the right-hand side involves both u and v, whereas the additional unknown v is not welcome.

The reason that v enters is that $1/(\zeta - z)$ is not purely real. With $\zeta = Re^{i\phi}$, show that we can re-express Cauchy integral formula as

$$f(z) = \frac{1}{2\pi i}\oint_C \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - R^2/\bar{z}}\right)f(\zeta)d\zeta = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\zeta}{\zeta - z} + \frac{\bar{z}}{\bar{\zeta} - \bar{z}}\right)f(\zeta)d\phi,$$

where the bracketed quantity is real. In particular, show that

$$\frac{\zeta}{\zeta - z} + \frac{\bar{z}}{\bar{\zeta} - \bar{z}} = \frac{R^2 - r^2}{|\zeta - z|^2},$$

and hence that

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)u(R,\phi)}{R^2-2Rr\cos(\phi-\theta)+r^2}d\phi$$

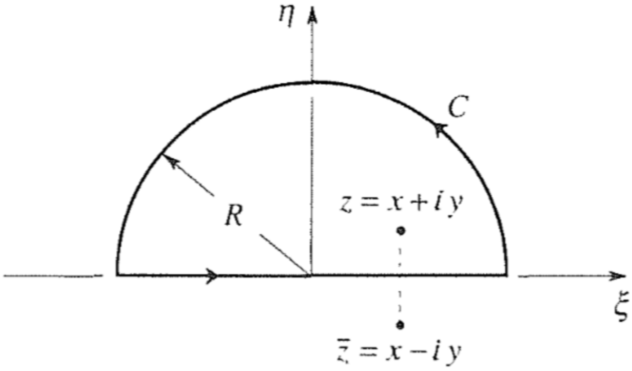
where $z = re^{i\theta}$ and $\zeta = Re^{i\theta}$.

This result is also derived by separation of variables in section 20.3.

(b) (**Poisson integral formula for the upper half plane**) This time let C be the contour shown here. Show that Cauchy integral formula can be re-expressed as

$$f(z) = \frac{1}{2\pi i}\oint_C \left(\frac{1}{\xi - z} - \frac{1}{\xi - \bar{z}}\right) f(\xi)d\xi$$

for all $R > |z|$.



Suppose that, as our boundary condition at infinity, $f(z) \to 0$ as $z \to \infty$. Letting $R \to \infty$ in the above equation, show that the semicircle integral tends to zero, leaving us with

$$f(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \left(\frac{1}{\xi - z} - \frac{1}{\xi - \bar{z}}\right) f(\xi)d\xi.$$

Finally, equating real parts in this equation, show that

$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{u(\xi,0)}{(\xi-x)^2+y^2}d\xi$$

is the solution to the Dirichlet problem for the upper half plane, with the boundary condition $u(x,y) \to 0$ as $r = \sqrt{x^2+y^2} \to \infty$.

sol.

(a) Since $f(\zeta)$ is analytic inside and on C, and the only pole of $\frac{1}{\zeta - R^2/\bar{z}}$ is $\zeta - R^2/\bar{z}$, which is outside of the circle, the integral

$$\frac{1}{2\pi i}\oint_C \frac{1}{\zeta - r^2/\bar{z}} f(\zeta)d\zeta = 0$$

by Cauchy-Goursat theorem. We make two assertion first.

(1) Since $\zeta = Re^{i\theta}$, we have $d(Re^{i\phi} = iRe^{i\phi}d\phi = i\zeta d\phi$

(2) Since we will take ζ in (1) into the bracket of the integral, we evaluate the following previously.

$$\frac{1}{\zeta - R^2/\bar{z}}\cdot\zeta = \frac{\bar{z}\zeta}{\zeta\bar{z} - R^2} = \frac{\bar{z}}{\bar{z} - \frac{R^2}{Re^{i\phi}}} = \frac{\bar{z}}{\bar{z} - Re^{-i\phi}} = \frac{\bar{z}}{\bar{z} - \bar{\zeta}}$$

So

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint_C \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - R^2/\bar{z}}\right) f(\zeta)d\zeta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{\zeta}{\zeta - z} - \frac{\zeta}{\zeta - R^2/\bar{z}}\right) f(\zeta)d\phi \quad \text{by (1)} \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\zeta}{\zeta - z} - \frac{\bar{z}}{\bar{\zeta} - \bar{z}} f(\zeta)d\phi \quad \text{by (2)} \end{aligned} \tag{1}$$

Evaluate the integrand,

$$\frac{\zeta}{\zeta - z} - \frac{\bar{z}}{\bar{\zeta} - \bar{z}} = \frac{\zeta\bar{\zeta} - \zeta\bar{z} + \bar{z}\zeta - z\bar{z}}{(\zeta - z)(\bar{\zeta} - \bar{z})} = \frac{R^2 - r^2}{\zeta\bar{\zeta} - \zeta\bar{z} - \bar{z}\zeta + z\bar{z}} = \frac{R^2 - r^2}{R^2 - 2Rr\cos(\phi - \theta) + r^2}$$

So

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - r^2)f(z)}{R^2 - 2Rr\cos(\phi - \theta) + r^2} d\phi$$

Taking the real part of the integral and $f(z)$ in polar form,

$$u(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - r^2)u(r,\theta)}{R^2 - 2Rr\cos(\phi - \theta) + r^2} d\phi$$

(b) This time $\bar{z}$ is located at the lower half plane. Thus, the integral

$$\oint_C \frac{f(\xi)}{\xi - \bar{z}} d\xi = 0$$

by Cauchy-Goursat theorem. Next, we have to give a bound to the integral. Let $C' = \{z = x + iy : |z| = R, x \geqslant 0\}$,

$$\begin{aligned} \left| \int_{C'} \frac{f(\xi)}{\xi - z} d\xi \right| &\leqslant \max_{z \in C'} \left| \frac{f(\xi)}{\xi - z} \right| \cdot \pi R \quad \text{(by ML bound)} \\ &\leqslant \frac{M}{R - r} \cdot \pi R \\ &= (\frac{\pi}{1 - r/R}) M \to 0 \text{ as } R \to \infty \end{aligned} \tag{2}$$

Similarly,

$$\left| \int_{C'} \frac{f(\xi)}{\xi - z} d\xi \right| \leqslant \frac{M\pi R}{R - r} \to 0 \text{ as } R \to \infty$$

So the part that contribute to the Cauchy integral formula we derived is the ξ axis, if we take $R \to \infty$. Also, in this way, we can regard the interior of the region surrounded by C as the whole complex plane. We have:

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left(\frac{1}{\xi - z} - \frac{1}{\xi - \bar{z}} \right) f(\xi) d\xi. \\ &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{z - \bar{z}}{(\xi - x - iy)(\xi - x + iy)} f(\xi) d\xi. \\ &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{2iy}{(\xi - x)^2 + y^2} [u(\xi,\eta) + iv(\xi,\eta)] d\xi \\ &= \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{u(\xi,0) + iv(\xi,0)}{(\xi - x)^2 + y^2} d\xi \quad \text{(on real axis).} \end{aligned} \tag{3}$$

Note that x and y are fixed numbers here. Taking the real parts

$$u(x,y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{u(\xi,0)}{(\xi - x)^2 + y^2} d\xi$$

■

24.3.9 The generating function for the Bessel function $J_n(x)$ is

$$\exp\left[\frac{x}{2}(z + \frac{1}{z})\right]$$

in as much as

$$e^{\frac{x}{2}\left(z + \frac{1}{z}\right)} = \sum_{n=-\infty}^{\infty} J_n(x) z^n. \tag{9.1}$$

(Here, x is not the real part of z, it is an independent real variable.

(a) Considering the analytic nature of the generating function in the left-hand side. show that (9.1) is valid in $z < |z| < \infty$.

(b) Use (3) in section 24.3, with C taken to be the unit circle, to derive the integral representation of $J_n(x)$,

$$J_n(x) = \frac{1}{\pi} \int_0^{\pi} \cos(n\theta - x\sin\theta) d\theta.$$

sol. (a) Let's introduce some concepts first

Product of infinite series. Given $\sum a_n$ and $\sum b_n$, we put

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} \quad (n = 0, 1, 2, ...)$$

and call $\sum c_n$ the product of the two given series.

Mertens Theorem. If $\sum_{n=0}^{\infty} a_n$ converges to A absolutely and $\sum_{n=0}^{\infty}$ converges to B, then $\sum_{n=0}^{\infty} c_n$ converges to AB. Let's put our faith in that the result holds for complex series "naturally".

$$\begin{aligned} e^{\frac{1}{2}z(x-\frac{1}{x})} &= e^{\frac{zx}{2}} e^{-\frac{z}{2x}} \\ &= \sum_{m=0}^{\infty} \frac{(\frac{x}{2})^m}{m!} z^m \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{x}{2})^k}{k!} z^{-k} \\ &= \sum_{n=-\infty}^{\infty} c_n z_n \end{aligned} \tag{4}$$

where

$$\begin{aligned} c_n &= \sum_{p=0}^{n} a_p b_{n-p} \\ &= \sum_{p=0}^{n} \frac{(\frac{x}{2})^n}{p!} \cdot \frac{(-1)^{n-p}}{(n-p)!} \end{aligned} \tag{5}$$

By some brilliant change of indexes (which I still can't figure it out), one can rewrite (6) as

$$\begin{aligned} c_n &= \sum_{m,k \geqslant 0} \frac{(\frac{x}{2})^{m+k}}{m!} \cdot \frac{(-1)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k)!k!} \left(\frac{x}{2}\right)^{2k+n} \\ &= J_n(x) \end{aligned} \tag{6}$$

(b) Apply (3) in section 24.3 to equation (9.1),

$$\begin{aligned} J_n(x) &= \frac{1}{2\pi i} \oint_C \frac{e^{\frac{x}{2}(\zeta - \frac{1}{\zeta})}}{(\zeta - 0)^{n+1}} d\zeta \quad \text{let } \zeta = e^{i\theta} \text{ on } \mathbb{C} \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{e^{\frac{x}{2}(e^{i\theta} - e^{-i\theta})}}{(e^{i\theta})^{n+1}} ie^{i\theta} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} e^{ix\sin\theta} e^{-in\theta} d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} [\cos(x\sin\theta - n\theta) + i\sin(x\sin\theta - n\theta)] d\theta \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(x\sin\theta - n\theta) d\theta \end{aligned} \tag{7}$$

■