Partial Differential Equations and Complex Variables
Homework 6

21.5.11 Given $f(z)$, determine $f'(z)$, where it exists, and state where f is analytic and where it is not.

(b) $\frac{x+iy}{x^2+y^2}$

(c) $|z|\sin z$

(f) $x + i\sin y$

sol.

(b) Let $u = x/(x^2+y^2)$ and $v = y/(x^2+y^2)$. Ut's obvious that f is not defined at $z = 0$. Since

$$u_x = \frac{y^2 - x^2}{(x^2+y^2)^2} \quad \text{for} z \neq 0$$

and

$$v_y = \frac{x^2 - y^2}{(x^2+y^2)^2} \quad \text{for} z \neq 0$$

and that both u_x and v_y do not exist at $z = 0$, the Cauchy-Riemann condition $u_x = v_y$ is satisfied only along the lines $y = \pm x$ except at the origin.

Moreover, Since

$$u_y = \frac{-2xy}{(x^2+y^2)^2} \quad \text{for} z \neq 0$$

and

$$v_y = \frac{-2xy}{(x^2+y^2)^2} \quad \text{for} z \neq 0,$$

so $u_y = -v_x$ only along the lines $x = 0$ and $y = 0$ but not at the origin.

Hence, f is differentiable and analytic nowhere on $\mathbb{C}$.

(c) Write

$$\begin{aligned} f(x) &= |z|\sin z \\ &= \sqrt{x^2+y^2}(\sin x \cos iy + \sin iy \cos x) \\ &= \sqrt{x^2+y^2}\sin x \cosh y + i\sqrt{x^2+y^2}\sinh y \cos x \end{aligned} \tag{1}$$

So

$$u = \sqrt{x^2+y^2}\sin x \cosh y$$
$$v = \sqrt{x^2+y^2}\sinh y \cos x$$

and

$$\begin{cases} u_x = (\frac{x}{\sqrt{x^2+y^2}}\sin x + \sqrt{x^2+y^2}\cos x)\cosh y \\ u_y = (\frac{y}{\sqrt{x^2+y^2}}\cosh y + \sqrt{x^2+y^2}\sinh y)\sin x \\ v_x = (\frac{x}{\sqrt{x^2+y^2}}\cos x - \sqrt{x^2+y^2}\sin x)\sinh y \\ v_y = (\frac{y}{\sqrt{x^2+y^2}}\sinh x - \sqrt{x^2+y^2}\cosh y)\cos x \end{cases}$$

for all $x, y \neq 0$. So the Cauchy-Riemann condition does not hold anywhere. Hence, f is differentiable and analytic nowhere on $\mathbb{C}$.

(f) Let $u = x$ and $v = \sin y$, so $u_x = v_y$ gives $1 = \cos y$. However, $u_y = -v_x$ holds for all real x, y. So f is differentiable all along the lines $y = n\pi/2$, where $n = \pm1, \pm3, \pm5, ...$ on $\mathbb{C}$.

■

21.5.15 Determine whether or not the given function u is harmonic and, if so, in what region. If it is, find the most general conjugate function v and corresponding analytic function $f(z)$. Express f in terms of z.

(a) $e^x \cos y$

(c) $x^3 - 3xy^2$

(f) r

sol.

(a) Since $\nabla^2(e^x\cos y) = e^x\cos y - e^x\cos y = 0$, u is harmonic. By Theorem 21.5.1 and Cauchy-Riemann equation, $u_x = e^x\cos y = v_y$, integrate the both side of the equation,

$$v = \int e^x \cos y dy = e^x \sin y + A(x).$$

Differentiate it w.r.t. x, we have $A'(x) = 0$ and thus $A(x) = C$, where C is arbitrary constant. So

$$\begin{aligned} f(z) &= u + iv \\ &= e^x \cos y + ie^x \sin y + C \\ &= e^z + C \end{aligned} \tag{2}$$

(c) Similar to (a), it's easy to derive that $f(z) = z^3 + C$.

(f) Recall the Laplace equation in the polar coordinate system:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

So r is not harmonic and hence not analytic, by the inverse of Theorem 21.5.2.

The problem here is that, since the graph of $f(z) = r$ is a cone whose vertex is located at the origin, it's not differential there. So Cauchy-Riemann equation fell to hold at 0 and thus Theorem 21.5.1 does not apply. ■

21.5.16 (Orthogonality of u = constant and v = constant curves)

(a) Prove that if $f(z) = u + iv$ is analytic in a region D, then the two families of level curves u = constant and v = constant are mutually orthogonal at all points in D at which $f'(z) \neq 0$

(b) Illustrate the idea contained in part (a) by sketching the u and v level curves for the case $f(z) = z = x + iy$.

(c) Repeat part (b) for the case $f(z) = z^2 = (x^2 - y^2) + i2xy$

(d) Repeat part (b) for the case

$$f(z) = \frac{1}{z} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

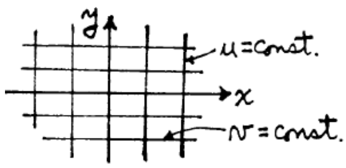
sol.

(a) Since f is analytic on D, u and v is differentiable on D. Let Γ_1 and Γ_2 be the curves that orthogonal to the curves u = constant and v = constant.

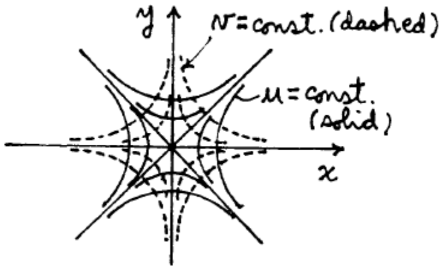
By concept of gradient, the normal vector of Γ_1 is given by $\hat{n_1} = \nabla u = u_x\hat{i} + u_y\hat{j}$, while the normal vector of Γ_2 is given by $\hat{n_2} = \nabla u = v_x\hat{i} + v_y\hat{j}$, where $u_x^2 + u_y^2 \neq 0$, and $v_x^2 + v_y^2 \neq 0$.

Then, since f is analytic, Cauchy-Riemann is satisfied. Thus we have $\hat{n_1} \cdot \hat{n_2} = u_x v_x + u_y v_y = v_x v_y - v_x v_y = 0$.

(b)



(c) If $f(z) = z = x + iy$, then $u = x^2 - y^2$, $v = 2xy$. So the u = constant and v = constant curves are hyperbolas:



(d) Since

$$u = \frac{x}{x^2+y^2}$$

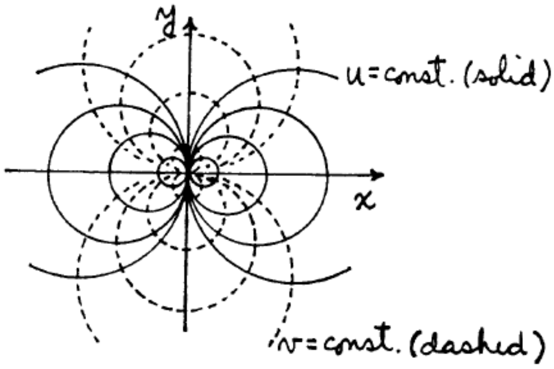
and

$$v = \frac{y}{x^2+y^2},$$

u = constant and v = constant curves are

$$\left(x - \frac{1}{2u}\right)^2 + y^2 = \left(\frac{1}{2u}\right)^2$$

$$x^2 + \left(y - \frac{1}{2v}\right)^2 = \left(\frac{1}{2v}\right)^2.$$



■

23.3.1 According to Example 2.

$$\oint_C \frac{dz}{z^2} = 0,$$

where C is a counterclockwise circle of radius R. centered at the origin. Yet $f(z) = 1 - z^2$ is not analytic within C; it is singular at $z = 0$. Explain why this result does not violate Cauchy's theorem.

sol. Cauchy theorem does not say that if $f(z)$ is not analytic inside C then $\oint_C f(z)dz \neq 0$. That is, the theorem does not contain a converse. ■

23.3.7 Evaluate $\int_C \bar{z}dz$, where C is

(a) a straight line from $z = 0$ to $z = 1 + i$

(b) the parabola $y = x^2$ from $z = 0$ to $z = 1 + i$

(c) C is the rectilinear path from $z = 0$ to $z = 1$ to $z = 1 + i$

(d) Are the answers the same? Is there any violation of Theorem 23.3.2? Explain.

sol. This time Path Independence Theorem does not help.

(a)

$$\int_{C_1} \bar{z}dz = \int_{C_1} (x - iy)(dx + idy) = \int_0^1 ((1 - i)(1 + i)xdx = 1$$

(b)

$$\int_{C_2} \bar{z}dz = \int_{C_2} (x - iy)(dx + idy) = \int_0^1 (x - ix^2)(1 + 2xi)dx = 1 + \frac{i}{3}$$

(c)

$$\int_{C_3} \bar{z}dz = \int_{C_2} (x - iy)(dx + idy) = \int_0^1 xdx + \int_0^1 (1 - iy)idy = 1 + i$$

(d) No, since $\bar{z}$ is not analytic. ■

23.3.9 Evaluate the following integrals. where in each case C is the circle $|z| = 3$, counterclockwise.

(a)

$$\oint_C \frac{dz}{z(z-1)}$$

(d)

$$\oint_C \frac{zdz}{z^2 - 3z + 2}$$

sol.

(a)

$$\begin{aligned}\oint_C \frac{dz}{z(z-1)} &= \oint_C \frac{dz}{z} + \oint_C \frac{dz}{z-1} \quad \text{(partial fraction decomposition)} \\ &= -2\pi i + 2\pi i \quad \text{(by formula (16) in section 23.3.)} \\ &= 0\end{aligned} \tag{3}$$

(d)

$$
\begin{aligned}
\oint\limits_C \frac{dzz}{z^2-3z+2} &= 2\oint\limits_C \frac{dz}{z-2} - \oint\limits_C \frac{dz}{z-1} \quad \text{(partial fraction decomposition)} \\
&= 2(2\pi i) - 2\pi i \quad \text{(by formula (16) in section 23.3.)} \\
&= 2\pi i
\end{aligned} \tag{4}
$$

■