Partial Differential Equations and Complex Variables
2020 Fall
Homework 3

19.3.6. Solve two-dimensional heat equation on the rectangular area $0 < x < a, \quad 0 < y < b$

$$\alpha^2(u_{xx} + u_{yy}) = u_t$$
$$u(0,y) = u(a,y) = u(x,0) = u(x,b) = 0,$$
$$u(x,y,0) = f(x,y).$$

(a) Derive the solution

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} e^{-\kappa_{mn}^2 \alpha^2 t},$$

where

$$\kappa_{mn} = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right),$$

and (c) Evaluate the A_{mn}'s for the case $f(x,y) = 100$ ((b)小題不用做)

sol. (a) Since both the equation and B.C.'s are homogeneous. Let $u(x,y,t) = X(x)Y(y)T(t)$, then

$$\frac{X''}{X} + \frac{Y''}{Y} = \frac{T'}{\alpha^2 T} = -\kappa^2.$$

$$\Rightarrow \begin{cases} T' + \alpha^2\kappa^2 T = 0 \\ \frac{X''}{X} = -\kappa^2 - \frac{Y''}{Y} = -\beta^2 \end{cases}$$

$$\Rightarrow \begin{cases} T' + \alpha^2\kappa^2 T = 0 \\ X'' + \beta^2 X = 0 \\ Y'' + (\kappa^2 - \beta^2)Y = 0 \end{cases}$$

So, (show details) $u(x,t) = (A\cos\beta x + B\sin\beta x)(C\cos\sqrt{\kappa^2-\beta^2}y + D\sin\sqrt{\kappa^2-\beta^2}y)e^{-\kappa^2\alpha^2 t}$. We abandoned the linear term since it can't survive after homogeneous Dirichlet B.C. Apply B.C.'s on it,

$$\begin{aligned}
&u(0,y,t) = 0 = A(...)e^{-\kappa^2\alpha^2 t} \Rightarrow A = 0 \\
\Rightarrow &u(x,y,t) = \sin\beta x(C\cos\sqrt{\kappa^2-\beta^2}y + D\sin\sqrt{\kappa^2-\beta^2}y)e^{-\kappa^2\alpha^2 t} \\
&u(x,0,t) = 0 = sin\beta x \cdot C \cdot e^{-\kappa^2\alpha^2 t} \Rightarrow C = 0 \\
\Rightarrow &u(x,y,t) = D\sin\beta x\sin(\sqrt{\kappa^2-\beta^2}y)e^{-\kappa^2\alpha^2 t} \\
&u(a,y,t) = 0 = D\sin\beta a(...) \Rightarrow \beta a = m\pi, \quad m\in\mathbb{N} \\
&u(x,b,t) = D\sin\beta x\sin(b\sqrt{\kappa^2-\beta^2})(...) \Rightarrow b\sqrt{\kappa^2-\beta^2} = n\pi, \quad n\in\mathbb{N}. \\
\Rightarrow &\kappa^2 - \beta^2 = \left(\frac{n\pi}{b}\right)^2 \\
\Rightarrow &\kappa = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} := \kappa_{mn}
\end{aligned} \tag{1}$$

So we let

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} D_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{-\kappa_{mn}^2\alpha^2 t}.$$

Using equation (17) to (22) in textbook leads to that

$$D_{mn} = \frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy.$$

(b) We begin with verifying B.C.'s. Clearly, B.C.'s are all satisfied since $\sin 0 = 0$ and $\sin m\pi = 0$ if m is odd. Also, if $t = 0$, it turns out that

$$u(x,y,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} D_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

It's double series expansion of $f(x,y)$. For the PDE, since the behavior of double series is more complicated than a single one. Hence we may assume the behavior of our solution is well-behaved so that we can change the order of summation and partial differentiation arbitrarily.

$$
\begin{aligned}
u_{xx} &= \frac{\partial^2}{\partial x^2}\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} D_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{-\kappa_{mn}^2\alpha^2 t}\right)\\
&= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} D_{mn}\left(\frac{\partial^2}{\partial x^2}\sin\frac{m\pi x}{a}\right)\sin\frac{n\pi y}{b}e^{-\kappa_{mn}^2\alpha^2 t}\\
&= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} D_{mn}\left(-\frac{m^2\pi^2}{a^2}\sin\frac{m\pi x}{a}\right)\sin\frac{n\pi y}{b}e^{-\kappa_{mn}^2\alpha^2 t}\\
&= \left(-\frac{m^2\pi^2}{a^2}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} D_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{-\kappa_{mn}^2\alpha^2 t}
\end{aligned}
\tag{2}
$$

$$
\begin{aligned}
u_{yy} &= \frac{\partial^2}{\partial y^2}\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} D_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{-\kappa_{mn}^2\alpha^2 t}\right)\\
&= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} D_{mn}\sin\frac{m\pi x}{a}\left(\frac{\partial^2}{\partial x^2}\sin\frac{n\pi y}{b}\right)e^{-\kappa_{mn}^2\alpha^2 t}\\
&= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} D_{mn}\sin\frac{m\pi x}{a}\left(-\frac{n^2\pi^2}{b^2}\sin\frac{n\pi y}{b}\right)e^{-\kappa_{mn}^2\alpha^2 t}\\
&= \left(-\frac{n^2\pi^2}{b^2}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} D_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{-\kappa_{mn}^2\alpha^2 t}
\end{aligned}
\tag{3}
$$

$$
\begin{aligned}
u_t &= \frac{\partial}{\partial t}\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} D_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{-\kappa_{mn}^2\alpha^2 t}\right)\\
&= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} D_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\frac{\partial}{\partial t}\left(e^{-\kappa_{mn}^2\alpha^2 t}\right)\\
&= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} D_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\frac{\partial}{\partial t}\left(-\kappa_{mn}^2\alpha^2\cdot e^{-\kappa_{mn}^2\alpha^2 t}\right)\\
&= \kappa_{mn}^2\alpha^2\cdot\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} D_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{-\kappa_{mn}^2\alpha^2 t}
\end{aligned}
\tag{4}
$$

So,

$$
\begin{aligned}
\alpha^2(u_{xx}+u_{yy}) &= \alpha^2\left(-\frac{m^2\pi^2}{a^2}-\frac{n^2\pi^2}{b^2}\right)u\\
&= -\alpha^2\kappa_{mn}^2 u\\
&= u_t
\end{aligned}
\tag{5}
$$

(c) DIY.

$$
D_{mn} = \frac{1600}{mn\pi^2}, \quad \text{m,n are odd.}
$$

$$
u(x,y,t) = \frac{1600}{\pi^2}\sum_{m=1,3,\ldots}^{\infty}\ \sum_{n=1,3,\ldots}^{\infty}\frac{1}{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{-\kappa_{mn}^2\alpha^2 t}.
$$

■

20.3.5 Consider the domain to be the whole plane minus a circular hole of of radius a.

$$
\nabla^2 u = 0, \quad (a < r < \infty)
$$

$$
u(a,\theta) = f(\theta), \quad u \text{ bounded as } r\to\infty
$$

Solve for $u(r,\theta)$, leaving expansion coefficients in integral form. What is the valur of u at $r=\infty$?

sol. Laplace equation in polar coordinate:

$$
u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0
$$

Let $u(r,\theta) = R(r)\Theta(\theta)$ Applying separation of variables,

$$
\begin{aligned}
&R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0\\
\Rightarrow &\frac{r^2R''}{R} + \frac{rR'}{R} + \frac{\Theta''}{\Theta} = 0\\
\Rightarrow &\frac{r^2R''}{R} + \frac{rR'}{R} = -\frac{\Theta''}{\Theta} = \kappa^2
\end{aligned}
\tag{6}
$$

We obtain,

$$\begin{cases} r^2R'' + rR' - \kappa^2 R = 0 \\ \Theta'' + \kappa^2\Theta = 0 \end{cases}$$

The R equation is a Cauchy-Euler equation, and the Θ one is our familiarity.

$$R(r) = \begin{cases} A + B\ln r, & \kappa = 0 \\ Cr^k + Dr^{-k}, & \kappa \neq 0 \end{cases}$$

$$\Theta(\theta) = \begin{cases} E + F\theta, & \kappa = 0 \\ G\cos\kappa\theta + H\sin\kappa\theta, & \kappa \neq 0 \end{cases}$$

So we can write

$$u(r,\theta) = (A + B\ln r)(E + F\theta) + (Cr^k + Dr^{-k})(G\cos\kappa\theta + H\sin\kappa\theta)$$

When $r \to \infty$, to avoid diverging, $B = C = 0$. May assume $E = D = 1$, so

$$u(r,\theta) = A + F\theta + r^{-k}(G\cos\kappa\theta + H\sin\kappa\theta)$$

By periodicity of Θ, $F = 0$ and

$$\begin{cases} \cos\kappa(\theta + 2\pi) = \cos\kappa\theta \\ \sin\kappa(\theta + 2\pi) = \sin\kappa\theta \end{cases}$$

$$\Rightarrow \begin{cases} -2\sin\kappa(\theta + \pi)\sin\kappa\theta = 0 \\ 2\cos\kappa(\theta + \pi)\sin\kappa\theta = 0 \end{cases}$$

So $\sin\kappa\pi = 0$, $\kappa = 1, 2, 3....$ Hence

$$u(r,\theta) = E + \sum_{n=1}^{\infty} r^{-n}(G_n\cos\kappa\theta + H_n\sin\kappa\theta)$$

apply B.C.,

$$u(b,\theta) = E + \sum_{n=1}^{\infty} b^{-n}(G_n\cos\kappa\theta + H_n\sin\kappa\theta)$$

So,

$$\begin{aligned} E &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)d\theta \\ G_n &= \frac{b^n}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta d\theta \\ H_n &= \frac{b^n}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta d\theta \end{aligned} \tag{7}$$

and therefore $\lim_{r\to\infty} u(r,\theta) = E$ ■

Supplement problem 1. By the principles used in modeling the string it can be shown that small free vertical vibrations of a uniform elastic beam are modeled by the forth-order PDE.

$$\frac{\partial^2 u}{\partial t^2} = -c^2\frac{\partial^4 u}{\partial x^4}$$

where $c^2 = \frac{EI}{\rho A}$ (E = Young's modulus of elasticity, I = moment of inertia of the cross section with respect to rotation axis, A = cross-section area)

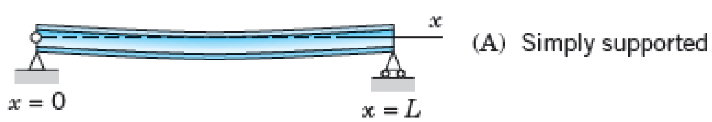
(a) Substituting $u = F(x)G(t)$ into the given equation, attain the results below

$$\frac{F^{(4)}}{F} = \frac{-1}{c^2}\frac{\ddot{G}}{G} = \beta^4 = const$$

$$F(x) = A\cos\beta x + B\sin\beta x + C\cosh\beta x + D\sinh\beta x$$

$$G(t) = a\cos c\beta^2 t + b\sin c\beta^2 t$$

(b) Find solutions $u_n = F_n(x)G_n(t)$ corresponding to zero initial velocity and satisfying the boundary conditions (see the photo below)



$$\begin{cases} u(0,t)=0, u(L,t)=0 \text{ (ends simply supported for all time t)} \\ u_{xx}(0,t)=0, u_{xx}(L,t)=0 \text{ (zero moments, hence zero curvature, at the ends)} \end{cases}$$

(c) Find the solution satisfying the initial condition

$$u(x,0)=f(x)=x(L-x)$$

sol. (a) After attain the results (Show your details) shown in the paragraph, we obtain

$$\begin{cases} F^{(4)}-\beta^4 F=0 \\ G''+c^2\beta^4 G=0 \end{cases}$$

By theory of ODE, we obtain

$$F(x)=A\cos(\beta x)+B\sin(\beta x)+C\cosh(\beta x)+D\sinh(\beta x),$$

and

$$G(t)=a\cos(c\beta^2 t)+b\sin(c\beta^2 t).$$

(b) Observe that β will be our eigenvalue. Moreover, we may abandon the terms u_0 correspond to $\beta=0$. (Why?) Then, write

$$u(x,t)=(A\cos(\beta x)+B\sin(\beta x)+C\cosh(\beta x)+D\sinh(\beta x))(a\cos(c\beta^2 t)+b\sin(c\beta^2 t)),$$

and

$$u_{xx}(x,t)=\beta^2(-A\cos(\beta x)-B\sin(\beta x)+C\cosh(\beta x)+D\sinh(\beta x))(a\cos(c\beta^2 t)+b\sin(c\beta^2 t))$$

Since $u(0,t)=0$ and $u_x x(0,t)=0$,

$$\begin{cases} (A+C)(a\cos(c\beta^2 t)+b\sin(c\beta^2 t))=0 \\ \beta^2(-A+C)(a\cos(c\beta^2 t)+b\sin(c\beta^2 t))=0 \end{cases}$$

$$\Rightarrow \begin{cases} A+C=0 \\ -A+C=0 \end{cases}$$

We have $A=C=0$. On the other hand, since $u(L,t)=0$ and $u_x x(L,t)=0$,

$$\begin{cases} (B\sin\beta L+D\sinh\beta L)(a\cos(c\beta^2 t)+b\sin(c\beta^2 t))=0 \\ \beta^2(B\sin\beta L-D\sinh\beta L)(a\cos(c\beta^2 t)+b\sin(c\beta^2 t))=0 \end{cases}$$

$$\Rightarrow \begin{cases} B\sin\beta L+D\sinh\beta L=0 \\ B\sin\beta L-D\sinh\beta L=0 \end{cases}$$

$$\Rightarrow \begin{cases} B\sin\beta L=0 \\ D\sinh\beta L=0 \end{cases}$$

$$\Rightarrow \begin{cases} \beta=n\pi/L \\ D=0 \end{cases}$$

Merge B into a and b, we have

$$u_n(x,t)=\sin\frac{n\pi x}{L}\left(a_n\cos\frac{n^2\pi^2 ct}{L}+b_n\sin\frac{n^2\pi^2 ct}{L}\right).$$

(c) First we assume

$$u(x,t)=\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\left(a_n\cos\frac{n^2\pi^2 ct}{L}+b_n\sin\frac{n^2\pi^2 ct}{L}\right),$$

We have two initial conditions, $u(x,0)=x(L-x)$ and $u_t(x,0)=0$ (since the initial velocity is zero). First, apply $u(x,0)=x(L-x)$, we have

$$u(x,0)=x(L-x)=\sum_{n=1}^{\infty}a_n\sin\frac{n\pi x}{L}$$

so

$$a_n=\int_0^L x(L-x)\sin\frac{n\pi x}{L}dx=\ldots=\begin{cases} \frac{4L^3}{\pi^3 n^3}, \text{ n odd} \\ 0, \text{ n even} \end{cases}$$

Apply another I.C.,

$$u_t(x,0) = 0 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L},$$

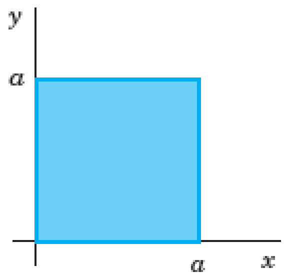
so obviously $b_n = 0$.

Therefore, the final answer becomes

$$u(x,t) = \sum_{n=1,3,5,\ldots} \frac{4L^3}{\pi^3 n^3} \sin \frac{n\pi x}{L} \cos \frac{n^2\pi^2 ct}{L}.$$

$\blacksquare$

Supplement problem 2. The faces of the thin square plate, shown below, with side a = 24 are perfectly insulated. The upper side is kept at 25° and the others are kept at 0°. Find the steady-state temperature $u(x,y)$ in the plate.



sol. It turns out that this is a common Dirichlet problem with nothing special.

$$\nabla^2 u = 0$$

$$\begin{cases} u(0,y) = 0, \\ u(y,0) = 0, \\ u(x,0) = 0, \\ u(x,24) = 0. \end{cases}$$

The procedures are same with example 1 in section 20.2 in the textbook (p.1059) (Again, you still need to show your steps or would receive no points).

$$u(x,y) = \frac{100}{\pi} \sum_{n=1,3,5,\ldots} \frac{1}{n \sinh n\pi} \sin \frac{n\pi x}{24} \sinh \frac{n\pi y}{24}$$

$\blacksquare$