Partial Differential Equations and Complex Variables
Homework 2

Due: Oct.17

18.4.5 For the problem

$$\alpha^2 u_{xx} = u_t \quad (-\infty < x < \infty, 0 < t < \infty)$$
$$u(x,0) = f(x) \quad (-\infty < x < \infty)$$

Use the Laplace transform instead, and obtain the ODE

$$\hat{u}_{xx} - \frac{s}{\alpha^2}\hat{u} = -\frac{1}{\alpha^2}f(x)$$

sol. We take LT with respect to t. Recall the derivation of differential property of Laplace transform

$$\int_0^\infty u_t(x,t)e^{-st}dt = u(x,t)e^{-st}\Big|_0^\infty - \int_0^\infty u(x,t)de^{-st} = -f(x) + s\int_0^\infty u(x,t)e^{-st}dt = \hat{u}s - f(x).$$

So the equation becomes

$$\alpha^2\hat{u}_{xx} = \hat{u}s - f(x)$$

and the result holds. ■

18.4.14. Consider the problem

$$\begin{cases} \alpha^2 u_{xx} = u_t, & (0 < x < \infty, \quad 0 < t < \infty) \\ u(x,0) = 0, & (0 < x < \infty) \\ u_x(0,t) = -Q, & (0 < t < \infty) \\ u \to 0 \quad \text{as} \quad x \to \infty \end{cases}$$

(a) Using the Laplace transform, derive the result

$$u(x,t) = \frac{\alpha Q}{\sqrt{\pi}}\int_0^t \frac{e^{-\frac{x^2}{4\alpha^2\tau}}}{\sqrt{\tau}}d\tau \qquad (\star)$$

and use $(\star)$ to solve $u(0,t)$, and sketch its graph.
(b)Show that

$$u(x,t) = Q\left(2\alpha\sqrt{\frac{t}{\pi}}e^{-\frac{x^2}{4\alpha^2 t}} - xerfc(\frac{x}{2\alpha\sqrt{t}})\right)$$

sol. (a) Take Laplace transform with respect to t. Let $U(x,s) = \mathscr{L}\{u(x,t)\}$, we have (you should show the process):

$$\begin{cases} \alpha^2 U_{xx} = sU - U(x,0) \\ U(x,0) = 0, & (0 < x < \infty) \\ U_x(0,s) = -\frac{Q}{s} \\ U \to 0 \quad \text{as} \quad x \to \infty \end{cases}$$

Since $U_{xx} - \frac{s}{\alpha^2}U = 0$, we have $U(x,s) = A(s)e^{\frac{\sqrt{s}}{\alpha}x} + B(s)e^{-\frac{\sqrt{s}}{\alpha}x}$. Apply B.C., $U(\infty,0) = 0$ leads to that $A(s) = 0$, so $U = B(s)e^{-\frac{\sqrt{s}}{\alpha}x}$. Also, since

$$U_x(s,t) = B(s)e^{-\frac{\sqrt{s}}{\alpha}x}\cdot\left(-\frac{\sqrt{s}}{\alpha}\right),$$

$U_x(0,s) = -\frac{Q}{s}$ implies that $B(s) = \frac{\alpha Q}{s^{\frac{3}{2}}}$. Hence

$$U(x,s) = \frac{\alpha Q}{s^{\frac{3}{2}}}e^{-\frac{\sqrt{s}}{\alpha}x}$$

Recall that (ref. Appendix C.)

$$\mathscr{L}\{1\} = \frac{1}{s} \qquad \text{(Prop. 1)}$$

$$\mathscr{L}\{\frac{e^{-\frac{-x^2}{4\alpha^2 t}}}{\sqrt{\pi t}}\} = \frac{e^{-\frac{\sqrt{(s)}}{\alpha}x}}{\sqrt{s}} \qquad \text{(Prop. 2)}$$

and the convolution property

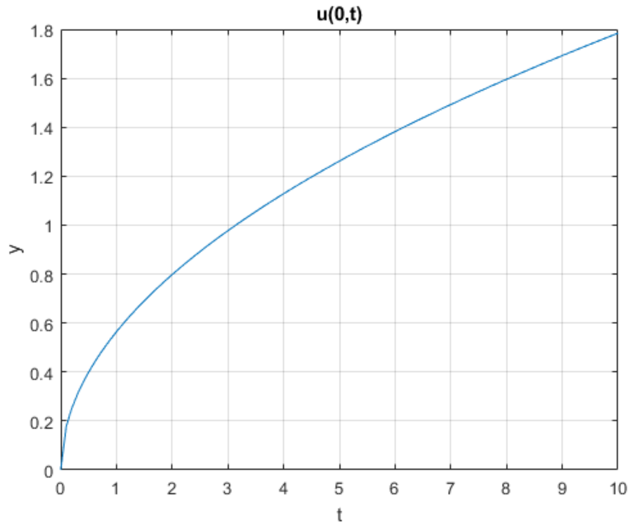
$$\mathscr{L}\{f(t) * g(t)\} = F(s)G(s). \qquad \text{(Prop. 3)}$$

Then,

$$\begin{aligned}\mathscr{L}^{-1}\{U(x,s)\} &= \mathscr{L}^{-1}\left\{\frac{\alpha Q}{s^{\frac{3}{2}}}e^{-\frac{\sqrt{s}}{\alpha}x}\right\} \\ &= \mathscr{L}^{-1}\left\{\alpha Q \cdot \frac{1}{s} \cdot \frac{e^{-\frac{\sqrt{s}}{\alpha}x}}{\sqrt{s}}\right\} \quad \text{(by Prop. 1 and 2.)} \\ &= \alpha Q \cdot \left(1 * \frac{e^{-\frac{-x^2}{4\alpha^2 t}}}{\sqrt{\pi t}}\right) \quad \text{(by Prop. 3)} \\ &= \frac{\alpha Q}{\sqrt{\pi}}\int_0^t \frac{e^{-\frac{-x^2}{4\alpha^2\tau}}}{\sqrt{\tau}}d\tau.\end{aligned} \tag{1}$$

Moreover,

$$u(0,t) = \frac{\alpha Q}{\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{\tau}}d\tau = 2\alpha Q\sqrt{\frac{t}{\pi}}$$



(b) Change of variables by letting $v = \frac{x}{2\alpha\sqrt{\tau}}$, then $dv = \frac{-x}{4\alpha\tau} \cdot \frac{1}{\sqrt{\tau}}d\tau$. Hence

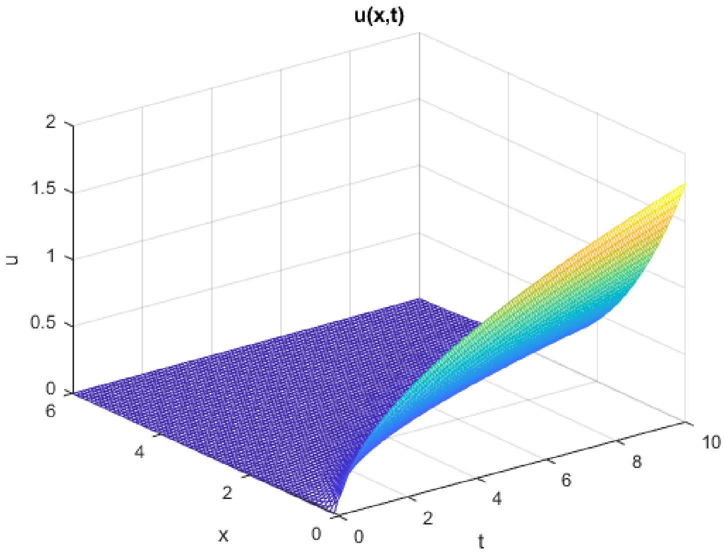
$$\begin{aligned}\frac{\alpha Q}{\sqrt{\pi}}\int_0^t \frac{e^{-\frac{-x^2}{4\alpha^2\tau}}}{\sqrt{\tau}}d\tau &= \frac{\alpha Q}{\sqrt{\pi}}\int_{\infty}^{\frac{x}{2\alpha t}} -e^{-v^2}\frac{4\alpha\tau}{x}dv \\ &= \frac{Qx}{\sqrt{\pi}}\int_{\frac{x}{2\alpha t}}^{\infty} \frac{e^{-v^2}}{v^2}dv\end{aligned} \tag{2}$$

Treat the integral separately (and ignore the constant of integration).

$$\begin{aligned}\int \frac{e^{-v^2}}{v^2}dv &= \int e^{-v^2}d(-\frac{1}{v}) \\ &= -\frac{e^{-v^2}}{v} + \int \frac{1}{v}e^{-v^2}(-2v)dv \\ &= -\frac{e^{-v^2}}{v} - 2\int e^{-v^2}dv.\end{aligned} \tag{3}$$

So

$$\begin{aligned}\frac{Qx}{\sqrt{\pi}}\int_{\frac{x}{2\alpha t}}^{\infty} \frac{e^{-v^2}}{v^2}dv &= \frac{Qx}{\sqrt{\pi}}\left(\frac{e^{-\frac{-x^2}{4\alpha^2 t}}}{x/2\alpha\sqrt{t}} - 2\int_{\frac{x}{2\alpha\sqrt{t}}}^{\infty} e^{-v^2}dv\right) \\ &= 2\alpha Q\sqrt{\frac{t}{\pi}}e^{-\frac{x^2}{4\alpha^2 t}} - Qxerfc(\frac{x}{2\alpha\sqrt{t}})\end{aligned} \tag{4}$$



■

Check for orthogonality The professor had claimed that the set

$$\{\phi_i(x) = 2\kappa_i \cos(\kappa_i x) + \sin(\kappa_i x)\}$$

forms an orthogonal basis, where κ_i is the i^{th} smallest root of the equation $\tan\kappa = -2\kappa$. Prove that

$$\int_0^1 \phi_i(x)\phi_j(x) = 0$$

if $i \neq j$. What is the value of the integral if $i = j$?

sol.

$$\begin{aligned}
&\int_0^1 \phi_i(x)\phi_j(x) \\
&= \int_0^1 [2\kappa_i \cos(\kappa_i x) + \sin(\kappa_i x)][2\kappa_j \cos(\kappa_j x) + \sin(\kappa_j x)]dx \\
&= \int_0^1 [4\kappa_i\kappa_j \cos(\kappa_i x)\cos(\kappa_j x) + \sin(\kappa_i x)\sin(\kappa_j x) + 2\kappa_i \cos(\kappa_i x)\sin(\kappa_j x) + 2\kappa_j \cos(\kappa_j x)\sin(\kappa_i x)]dx \\
&= \int_0^1 \cos(\kappa_i x)\cos(\kappa_j x)[4\kappa_i\kappa_j + \tan(\kappa_i x)\tan(\kappa_j x) + 2\kappa_i \tan(\kappa_j x) + 2\kappa_j \tan(\kappa_i x)]dx \\
&= \int_0^1 \cos(\kappa_i x)\cos(\kappa_j x)[4\kappa_i\kappa_j + (-2\kappa_i)(-2\kappa_j) + 2\kappa_i(-2)\kappa_j + 2\kappa_j(-2)\kappa_i]dx \\
&= \int_0^1 \cos(\kappa_i x)\cos(\kappa_j x)[8\kappa_i\kappa_j - \kappa_i\kappa_j]dx \\
&= 0
\end{aligned} \tag{5}$$