Partial Differential Equations and Complex Variables
Homework 1

Due: Oct.1

18.2.4. In deriving the diffusion equation, we assumed that the cross-sectional shape of the rod does not vary with x. Reconsider our derivation for the basic case where there is no Newton cooling (i.e., the lateral surface is insulated, $h = 0$) and no translation of the rod ($\nu = 0$), but allow for the cross-sectional area A to vary with x. Show that the revised diffusion equation is

$$\frac{\alpha^2}{A(x)}[A(x)u_x]_x = u_t$$

pf.

Let $u = u(x,t)$ be the temperature at certain point x at moment t, and $h(x,t)$ be the heat flow. We know that

$$\Delta h = h(x+\Delta x, t) - h(x,t) = mC(\frac{\partial u}{\partial t}),$$

provided

$$h(x,t) = kA(x)(-\frac{\partial u}{\partial x}).$$

Hence it leads to

$$k(A(x_0)\Big(-\frac{\partial u}{\partial x})\Big|_{x_0} - A(x_0+\Delta x)(-\frac{\partial u}{\partial x})\Big|_{x_0+\Delta x}\Big) = mC(\frac{\partial u}{\partial t})$$

Writing

$$m = \rho\int_{x_0}^{x_0+\Delta x} A(x)dx$$

and define $\alpha^2 = \frac{k}{\rho C}$ we derived

$$\alpha^2(A(x_0)(\frac{\partial u}{\partial x})\Big|_{x_0+\Delta x} - A(x_0)(\frac{\partial u}{\partial x})\Big|_{x_0}) = \frac{\partial u}{\partial x}\int_{x_0}^{x_0+\Delta x} A(x)dx \tag{1}$$

Divide (1) by Δx and taking limits on the both sides of the equality, we have

$$\frac{\alpha^2}{A(x)}[A(x)u_x]_x = u_t$$

■

18.3.9. The temperature distribution $u(x,t)$ in a 2-m long brass rod is govern by the problem

$$\begin{aligned} &\alpha^2 u_{xx} = u_t \quad (0 < x < 2, \quad 0 < t < \infty) \\ &u(0,t) = u(2,t) = 0, \quad (t > 0) \\ &u(x,0) = \begin{cases} 50x, & (0 < x < 1) \\ 100 - 50x, & (1 < x < 2) \end{cases} \end{aligned} \tag{2}$$

where $\alpha^2 = 2.9 \times 10^{-5}$ m^2/sec.

(a) Determine the solution for $u(x,t)$

(b) Compute the temperature at the midpoint of the rod at the end of 1 hour.

(c) Compute the time it will take for the temperature at the point to diminish to $5°C$

(d) Compute the time it will take for the temperature at the point to diminish to $1°C$

sol.

(a) Applying separation of variables (You should show the details),

$$u(x,t) = u_s(x) + \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{2} e^{-(\frac{n\pi\alpha}{2})^2 t}$$

with formula (You should show your calculation)

$$C_n = \int_0^2 f(x)\sin\frac{n\pi x}{2}dx = ... = \frac{400}{n^2\pi^2}\sin\frac{n\pi}{2}$$

So

$$u(x,t) = \sum_{n=1}^{\infty} \frac{400}{n^2\pi^2} \sin\frac{n\pi}{2} \sin\frac{n\pi x}{2} e^{-(\frac{n\pi\alpha}{2})^2 t}$$

(b) Notice that the power of n in the exponential is 2, the first term of the series is deterministic. We may compute 2 terms in (b).

$$\begin{aligned} u(1,3600) &= \sum_{n=1}^{\infty} \frac{400}{n^2\pi^2} \sin\frac{n\pi}{2} \sin\frac{n\pi x}{2} e^{-(\frac{n\pi}{2})^2(2.9\times10^{-5})t} \\ &\approx \frac{400}{\pi^2} \times (0.7729 + 0.01904 + ...) \\ &\approx 31.77^\circ\text{C} \end{aligned} \tag{3}$$

(c) For convenience, we may just compute the first term.

$$\begin{aligned} u(1,t) &= \sum_{n=1}^{\infty} \frac{400}{n^2\pi^2} \sin\frac{n\pi}{2} \sin\frac{n\pi x}{2} e^{-(\frac{n\pi}{2})^2(2.9\times10^{-5})t} \\ &\approx 127.3239 \times e^{-7.155\times10^{-5}t} \\ &= 5 \end{aligned} \tag{4}$$

So

$$t \approx -0.1398 \times 10^5 \times \ln\frac{5}{127.3239} \approx 4.5 \times 10^4(\text{s}) \approx 12.5(\text{hours})$$

(d) Use the result of (b). Since $u(1,t) = 1 \approx 127.3239 \times e^{-7.155\times10^{-5}t}$, we have $t \approx 6.7 \times 10^4$s≈ 18.6h.

18.3.13. Consider a cylindrical compressed-gas container of length L, divided in half by a baffle. To the left of the baffle is a gas of species A, and to the right of it is a different gas of species B. Suppose they are at the same pressure, so that when the baffle is removed at the time $t = 0$ the two gases proceed to mix by diffusion alone. Considering species A, say, its concentration $c_A(x,t)$ is governed by the problem

$$\text{PDE.} \quad D\frac{\partial^2 c_A}{\partial x^2} = \frac{\partial c_A}{\partial t}, \quad (0 < x < L, 0 < t < \infty) \tag{13.1}$$

$$\text{B.C.} \quad \frac{\partial c_A}{\partial x}(0,t) = \frac{\partial c_A}{\partial x}(L,t) = 0, \quad (0 < t < \infty) \tag{13.2}$$

$$\text{I.C.} \quad c_A(x,0) = \begin{cases} c_0, & 0 < x < L/2 \\ 0, & L/2 < x < L \end{cases} \tag{13.3}$$

where D is the diffusion coefficient and D and c_0 are constants
(a) Solve for $c_A(x,t)$. From $c_A(x,t)$ determine the steady-state solution

$$c_{As}(x) = \lim_{t\to\infty} c_A(x,t) \tag{13.4}$$

(b) Integrating equation (13.1) w.r.t. x, from 0 to L, show that

$$\int_0^L c_A(x,t)dx = \text{constant}. \tag{13.5}$$

(c) Solve for $c_{As}(x)$ directly, i.e., by solving

$$Dc''_{As}(x) = 0; \quad c'_{As}(0) = c'_A s(L) = 0$$

and using equation (13.5). Your result should be the same as in (a).

sol.

(a) Separate the variables, we'll have

$$c_A(x,t) = X(x)T(t) = A + Bx + (C\cos\kappa x + E\sin\kappa x)e^{-\kappa^2 Dt},$$

where the linear terms come from the eigenvalue $\kappa_0 = 0$. Applying B.C., it's easy to show that $B = E = 0$ and the eigenvalues $\kappa_n = \frac{n\pi}{L}$, $n \in \mathbb{N}$. Hence,

$$c_A(x,t) = A + \sum_{n=0}^{\infty} C_n \cos(\frac{n\pi x}{L})e^{-(\frac{n\pi}{L})^2 Dt}$$

Apply I.C., $\forall x \in [0,L]$,

$$c_A(x,0) = f(x) = A + \sum_{n=0}^{\infty} C_n \cos(\frac{n\pi x}{L})$$

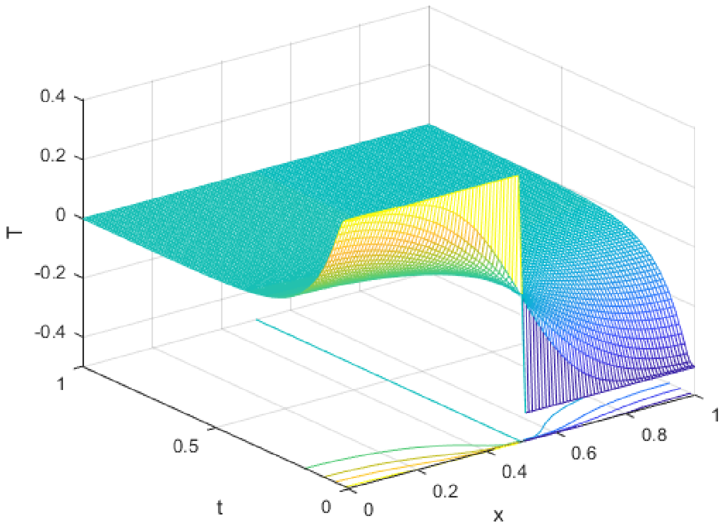
This is Fourier Series. Compute the coefficients.

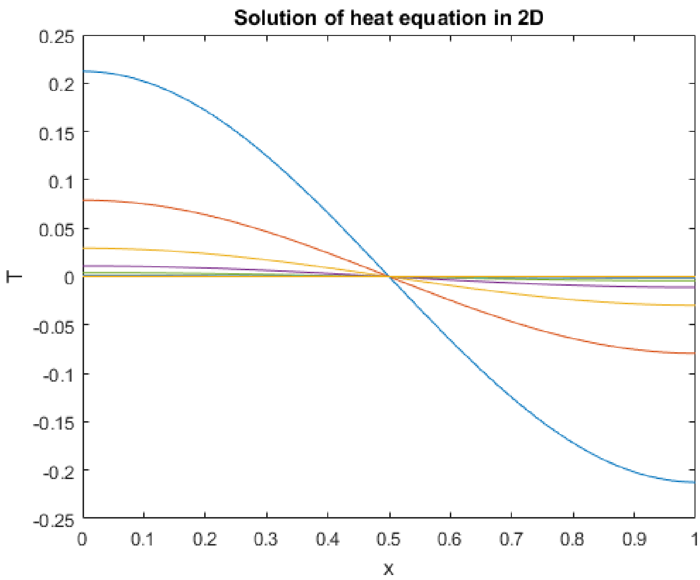
$$A = \int_0^L c_A(x,0)dx = \frac{c_0}{2}$$

$$C_n = \int_0^L c_A(x,0)\cos(\frac{n\pi x}{L})dx = \frac{2c_0}{n\pi}\sin\frac{n\pi}{2}$$

Combine the results, the solution is

$$c_A(x,t) = \frac{c_0}{2} + \sum_{n=0}^{\infty} \frac{2c_0}{n\pi}\sin\frac{n\pi}{2}\cos(\frac{n\pi x}{L})e^{-(\frac{n\pi}{L})^2 Dt}$$





(b) May assume $c_A(x,t)$ is continuous on $x \in [0,L]$ to guarantee the interchange of the derivative symbol and integral symbol is valid for x. Integrating the left hand side of the equality of (13.1)

$$\begin{aligned}\int_0^L D\frac{\partial^2 c_A}{\partial x^2}dx &= \int_0^L D\frac{\partial}{\partial x}(\frac{\partial c_A}{\partial x})dx \\ &= D\left(\frac{\partial c_A}{\partial x}\bigg|_{x=L} - \frac{\partial c_A}{\partial x}\bigg|_{x=0}\right) \\ &= D(0-0) \\ &= 0 \text{ (by B.C.)}\end{aligned} \tag{5}$$

On the other side,

$$\int_0^L D\frac{\partial c_A}{\partial t}dx = D\frac{\partial}{\partial t}\int_0^L c_A dx = 0 \text{ (by(2).)} \tag{6}$$

Since $\frac{\partial}{\partial t}\int_0^L c_A dx = 0$, $\int_0^L c_A dx$ remains a constant.

(c) plug $c_A(x)$ back into heat equation, we'll get $D(c_{As})_{xx} = 0$. Only if c_{As} is a linear function would it vanish after differentiating twice. Apply B.C. on it, $\frac{\partial c_{As}}{\partial x} = a = 0$, hence $c_{As} = b$. Then plug c_{As} into (13.5)

$$\int_0^L c_A(x,t)dx = bL$$

also, integrate I.C. as (13.5),

$$\int_0^L c_A(x,0)dx = \int_{\frac{L}{2}}^L c_0 dx = \frac{c_0 L}{2}$$

It's guaranteed that the values of two integrals are same by (b). i.e. $bL = \frac{c_0 L}{2}$ hence $b = c_{As}(x,t) = \frac{c_0}{2}$ ■

18.3.15. Solve the problem

$$\alpha^2 u_{xx} = u_t - F, (0 < x < L, 0 < t < \infty), \text{F is constant}$$
$$u(0,t) = 0, u(L,t) = 50, (0 < t < \infty)$$
$$u(x,0) = f(x), (0 < x < L)$$

by letting

$$u(x,t) = u_s(x) + X(x)T(t),$$

where u_s is the steady state solution.

sol.

Let $u(x,t) = u_s(x) + v(x,t)$, where $v(x,t) = X(x)T(t)$. Since $u(x,t)$ and $u_s(x)$ satisfies the PDE, By principle of superposition (see equation (23) in section 18.3), $v(x,t)$ also satisfies the given PDE but subjects to different B.C. and I.C. We shall handle $u_s(x)$ first. A steady state solution should satisfy the PDE and the given boundary conditions (not initial condition since u_s is independent from t). Hence it suffices to solve the two-point boundary value problem

$$\alpha^2 u_s''(x) = -F, u_s(0) = 0, u_s(L) = 50$$

Therefore u_s is a quadratic polynomial

$$u_s(x) = \frac{100\alpha^2 - FL^2}{2\alpha^2 L}x + \frac{F}{2\alpha^2}x^2$$

On top of that, since $u(x,t)$ and $u_s(x)$ share the same boundary conditions, $v(x,t) = u(x,t) - u_s(x,t)$ should have B.C. and I.C. as follow:

$$\begin{cases} v(0,t) = 0 - 0 = 0 \\ v(L,t) = 50 - 50 = 0 \\ v(x,0) = f(x) - u_s(x) \end{cases}$$

Thus, Apply formulae (22), (26)-(28) in section 18.3, we have

$$v(x,t) = \sum_{n=1}^{\infty} e^{-\alpha^2(\frac{n\pi}{L})^2 t}(D_n \sin\frac{n\pi x}{L})$$

where

$$D_n = \frac{2}{L}\int_0^L (f(x) - u_s(x))\sin\frac{n\pi x}{L}dx$$

Adding $u_s(x)$ and $v(x,t)$, we have derived the answer $u(x,t)$. ■

18.3.19. Solve $u(x,t)$ for the conducting problem

$$\alpha^2 u_{xx} = u_t, \quad (0 < x < L, 0 < t < \infty)$$
$$u_x(0,t) = -1, u_x(L,t) = 0, u(x,0) = 0$$

sol.

By some brilliant guess, we may write $u(x,t) = \frac{(x-L)^2}{2L} + v_1(x,t) + v_2(x,t)$, then plug back into the PDE.

$$\begin{cases} \alpha^2 u_{xx} = -\frac{\alpha^2}{L} + \alpha^2 v_{1xx}(x,t) + \alpha^2 v_{2xx}(x,t) \\ u_t = v_{1t}(x,t) + v_{2t}(x,t) \end{cases}$$

We may pick v_1 as the solution of the following problem:

$$\begin{cases} \alpha^2 v_{1xx} = v_{1t}, \quad (0 < x < L, 0 < t < \infty) \\ v_{1x}(0,t) = v_{1x}(L,t) = 0 \\ v_1(x,0) = -\frac{(x-L)^2}{2L} \end{cases}$$

and v_2 as the solution of the following problem:

$$\begin{cases} \alpha^2 v_{2xx} = v_{2t} - \frac{\alpha^2}{L}, & (0 < x < L, 0 < t < \infty) \\ v_{2x}(0,t) = v_{2x}(L,t) = v_{2x}(x,0) = 0 \end{cases}$$

So v_1 can be solved by separation of variable and eigenfunction expansion. We omit the process.

Observe that since its B.C. and I.C. vanishes, the homogeneous solution (in terms of Fourier series) equals to $v_{2h}(x,t) = 0$. However since v_2 non-homogeneous, we need to find a particular solution $v_{2p}(x,t)$. Assume $v_{2p}(x,t)$ is a function of x and t, or a single variable function of x or t. After try and error, we found that only if $v_{2p}(x,t)$ is a function of t, i.e.,

$$v_{2p}(t) = \frac{\alpha^2 t}{L}$$

can v_{2p} satisfy the PDE of v_2. Hence the total solution equals to

$$u(x,t) = \frac{(x-L)^2}{2L} + v_1(x,t) + v_{2h}(x,t) + v_{2p}(t) = \frac{(x-L)^2}{2L} + \frac{\alpha^2 t}{L} + v_1(x,t)$$

. ■

Remark. Only if the PDE is homogeneous and all B.C. equals to 0 can we use method of separation of variables to solve the problem. If either a condition is not satisfied, we have to introduce a **particular solution** $u_p(x,t)$ which satisfy the original PDE and B.C. Usually we guess that the particular solution only varies with x or t, i.e., $u_p(x,t) = u_p(x)$ or $u_p(t)$. If not, besides purely guessing, we have to consider some more advanced concepts that won't be introduced in this course. If we somehow figure out u_p, then write $v(x,t) = u(x,t) - u_p$, and $v(x,t)$ would satisfy a homogeneous PDE subjects to all B.C.= 0 and I.C.=initial I.C.$-u_p$, which allow us to use method of separation of variables.

Remark 2. We define a **steady state solution** $u_\infty(x) = \lim_{t\to\infty} u(x,t)$. After solving the PDE, the solution might form like this:

$$u(x,t) = u_p + Ax + B + \sum(\text{trigonometric terms})(\text{exponential terms})$$

where u_p represent the particular solution and the linear terms $Ax + B$ (may equals to 0) comes from eigenvalues $\kappa_n = 0$. If u_p is a function of x, then apparently $u_\infty(x) = u_p(x) + Ax + B$. If the PDE is homogeneous and all B.C. equals to 0, then $u_p = 0$ and thus $u_\infty(x) = Ax + B$.