CHAPTER 21

Section 21.2

2. $|z_1z_2| = |(x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)| = \sqrt{(x_1x_2-y_1y_2)^2 + (x_1y_2+x_2y_1)^2}$
$= \sqrt{x_1^2x_2^2 + y_1^2y_2^2 + x_1^2y_2^2 + x_2^2y_1^2}$

$|z_1||z_2| = \sqrt{x_1^2+y_1^2}\,\sqrt{x_2^2+y_2^2} = \sqrt{x_1^2x_2^2 + x_1^2y_2^2 + x_2^2y_1^2 + y_1^2y_2^2} = |z_1z_2|$ ✓

4.(a) Quadratic formula gives $z = (2 \pm \sqrt{4-8})/2 = 1 \pm i$. Consider $z = 1+i$, say. Then

$z^2 = (1+i)(1+i) = 0 + 2i$ per (4)

$-2z = (-2+0i)(1+i) = -2-2i$ per (4)

$z^2 - 2z = (z^2) + (-2z) = (0+2i) + (-2-2i) = -2$ per (3)

$z^2 - 2z + 2 = (z^2-2z) + (2+0i)$ per (6)

$= (-2+0i) + (2+0i) = 0$ per (3). ✓

5.(b) Using induction, we first observe that the equality holds for $n=1$. Next, suppose it holds for $n=k$. Then

$|z^{k+1}| = |z^k z| = |z^k||z|$ per (9)

$= |z|^k|z|$ per assumption

$= |z|^{k+1}$, which completes the proof by induction.

(c) $|z_1z_2z_3| = |(z_1z_2)(z_3)| = |z_1z_2||z_3|$ per (9)

$= |z_1||z_2||z_3|$ per (9) again.

6.(e) Using induction, first observe that the equality holds for $n=1$. Next, suppose it holds for $n=k$. Then

$\overline{z^{k+1}} = \overline{z^k z} = \overline{z^k}\,\overline{z}$ per (14b)

$= \bar{z}^k\,\bar{z}$ per assumption

$= \bar{z}^{k+1}$, which completes the proof.

8. $z_1z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) = 0$ gives $\left.\begin{array}{l} x_1x_2 - y_1y_2 = 0 \\ x_1y_2 + x_2y_1 = 0. \end{array}\right\} \star$

Regarding $\star$ as a linear system on x_1, y_1, if x_1, y_1 are not both 0 then we must have the determinant $= x_2^2 + y_2^2 = 0$. Thus, if $z_1 \neq 0$ then we must have $z_2 = 0$. Similarly, if $z_2 \neq 0$ then we need $z_1 = 0$. Thus, z_1 and z_2 cannot both be nonzero.

9. (a) $(2-i)^3 = (4-4i-1)(2-i) = (3-4i)(2-i) = 6 - 11i - 4 = 2 - 11i$

(e) $\left(\frac{1+i}{2-i}\right)^3 = \left(\frac{1+i}{2-i}\right)^2\left(\frac{1+i}{2-i}\right) = \frac{2i}{3-4i}\cdot\frac{1+i}{2-i} = \frac{-2+2i}{+2-11i}\;\frac{+2+11i}{+2+11i} = \frac{-26-18i}{125} = -\frac{26}{125} - \frac{18}{125}i$

(g) $\text{Im}\,(1+i)^3 = \text{Im}\;2i(1+i) = \text{Im}\,(-2+2i) = 2$

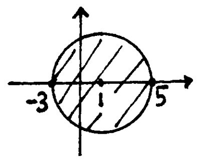
(h) $\left(\text{Re}\,\frac{1}{1+i}\right)^3 = \left(\text{Re}\,\frac{1}{1+i}\,\frac{1-i}{1-i}\right)^3 = \left(\text{Re}\,\frac{1-i}{2}\right)^3 = \left(\frac{1}{2}\right)^3 = 1/8$

10. (a) $\left|\frac{1-i}{1+i}\right| = \left|\frac{1-i}{1+i}\,\frac{1-i}{1-i}\right| = \left|\frac{-2i}{2}\right| = 1$

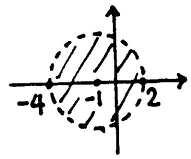
11.(a) $|z_1+z_2| = |(2+3i)+(4-i)| = |6+2i| = \sqrt{36+4} = \sqrt{40} = 6.324$
$|z_1|+|z_2| = |2+3i|+|4-i| = \sqrt{13}+\sqrt{17} = 7.729$ is ≥ 6.324. ✓

Section 21.3

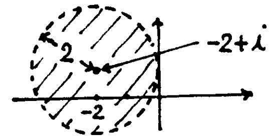
1. (a)



(b)

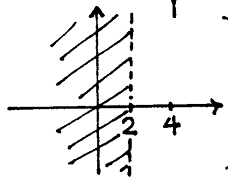


(c) $|z-(-2+i)|<2$



(d) The circle of radius 2, centered at $-2+i$.

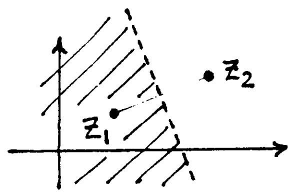
(e)



The open half plane $\operatorname{Re} z<2$, i.e., $x<2$.

(f) $(x+1)^2+y^2 \leq x^2+y^2$ gives $2x+1 \leq 0$, hence, the half-plane $x \leq -1/2$. Could also have seen this by noting that $|z+1|=|z|$ is the locus of points equidistant from $z=-1$ and $z=0$, i.e., the line $x=-1/2$. Then $|z+1|\leq|z|$ is the half-plane $x \leq -1/2$.

(g)



(h) $\operatorname{Re}(z-i) = \operatorname{Re}(x+iy-i) = x$, so it is the half-plane $x>3$.

(i) $\sqrt{(x+1)^2+y^2} = \sqrt{x^2+y^2}+1$

$(x+1)^2+y^2 = x^2+y^2+2\sqrt{x^2+y^2}+1$

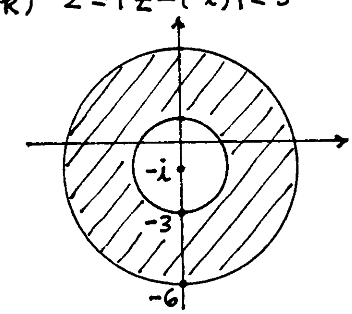
$\cancel{x^2}+2x+\cancel{1}+\cancel{y^2} = \cancel{x^2}+\cancel{y^2}+2\sqrt{x^2+y^2}+\cancel{1}$

$x = \sqrt{x^2+y^2} \quad (*)$

Squaring gives $y=0$. However, we see from $(*)$ that we need $x \geq 0$. Thus, the set is comprised of the nonnegative x-axis.

(j) The half-plane $x<2$.

(k) $2 \leq |z-(-i)| \leq 5$



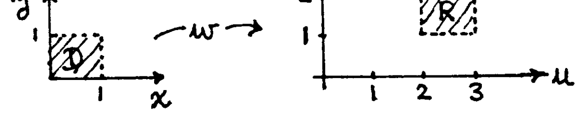
(l) $\operatorname{Im}(z-i)>1$

$\operatorname{Im}(x+i(y-1))>1$

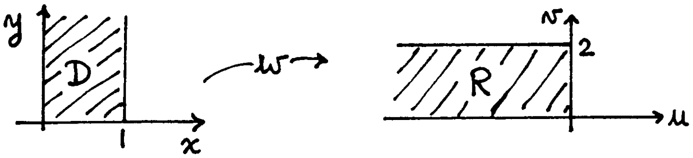
$y-1>1$, $y>2$.

The half-plane $y>2$.

2. (a) $w = z+2+i$



(b) $w = 2iz = 2i(x+iy) = \underbrace{-2y}_{u}+i\underbrace{2x}_{v}$. $\quad 0<x<1 \Rightarrow 0<v<2$
$0<y<\infty \Rightarrow -\infty<u<0$, so D and R are as shown:

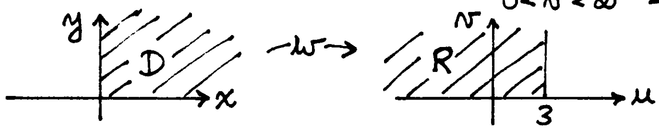


(open regions)

(c) $w = iz+3 = i(x+iy)+3 = \underbrace{3-y}_{u} + i\underbrace{x}_{v}$. $\quad 0<x<\infty,\ 0<y<\infty$

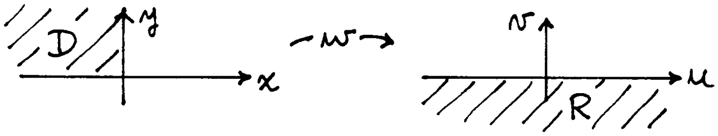
$\Downarrow \qquad \Downarrow$

$0<v<\infty \qquad -\infty<u<3$



(open regions)

(d) $w = z^2 = \underbrace{(x^2-y^2)}_{u} + i\underbrace{2xy}_{v}$. $\quad -\infty<x<0,\ 0<y<\infty \Rightarrow -\infty<u<\infty,\ -\infty<v<0$



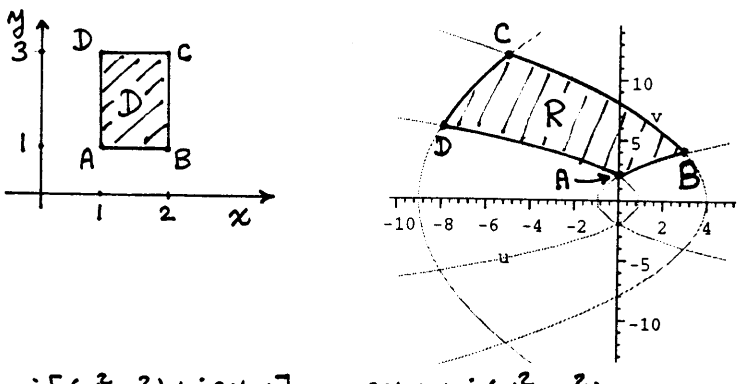
(e) $w = z^2 = \underbrace{(x^2-y^2)}_{u} + i\underbrace{2xy}_{v}$. Image of $x=1$: $\left.\begin{array}{l} u=1-y^2 \\ v=2y \end{array}\right\}\ u = 1-\frac{v^2}{4}$

Image of $x=2$: $\left.\begin{array}{l} u=4-y^2 \\ v=4y \end{array}\right\}\ u = 4-\frac{v^2}{16}$

Image of $y=1$: $\left.\begin{array}{l} u=x^2-1 \\ v=2x \end{array}\right\}\ u = \frac{v^2}{4}-1$

Image of $y=3$: $\left.\begin{array}{l} u=x^2-9 \\ v=6x \end{array}\right\}\ u = \frac{v^2}{36}-9$

maple:
```
> with(plots):
> implicitplot({u=1-v^2/4,u=4-v^2/16,u=
  -1+v^2/4,u=-9+v^2/36},u=-10..5,v=-14..14);
```

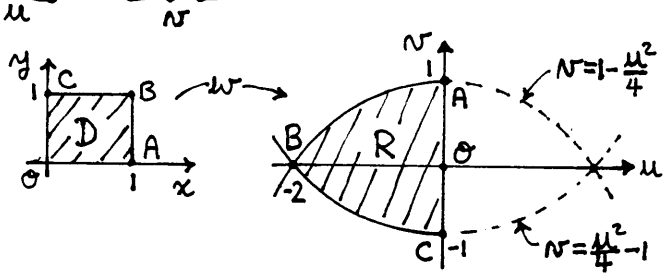


(f) $w = iz^2 = i[(x^2-y^2)+i2xy] = \underbrace{-2xy}_{u} + i\underbrace{(x^2-y^2)}_{v}$

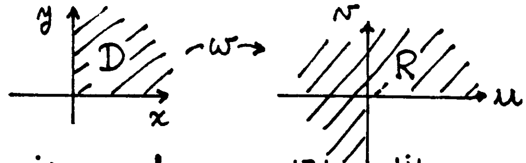
$x=0$: $u=0,\ v=-y^2$

$x=1$: $\left.\begin{array}{l} u=-2y \\ v=1-y^2 \end{array}\right\} \to v = 1-\frac{u^2}{4}$

$\left.\begin{array}{l} y=0:\ u=0,\ v=x^2 \\ y=1:\ u=-2x,\ v=x^2-1 \end{array}\right\} \to v = \frac{u^2}{4}-1$



(g) $w = z^3 = (x+iy)^3 = \underbrace{(x^3-3xy^2)}_{u} + i(\underbrace{3x^2y-y^3}_{v})$



3. $|e^z| \neq e^{|z|}$. For example, if $z=i$ then $|e^z| = |e^i| = 1$ whereas $e^{|z|} = e^{|i|} = e$. It follows from this single counterexample that, in general, $|w(z)| \neq w(|z|)$.

4. $\overline{e^z} = e^x(\cos y - i\sin y)$ from (8)
$e^{\bar{z}} = e^{x-iy} = e^x(\cos y - i\sin y) = \overline{e^z}$. However, in general $\overline{w(z)} \neq w(\bar{z})$. For example, if $w(z) = i$ then $\overline{w(z)} = -i$ but $w(\bar{z}) = i$.

5. (20a)-(20d) are important. Their derivation is simple, following immediately from the definitions of $\cos z, \sin z, \cosh z, \sinh z$. For ex.,
$$\cos iz = (e^{i(iz)} + e^{-i(iz)})/2 = (e^{-z}+e^{z})/2 = \cosh z.$$

6.(a)
$$e^{z_1}e^{z_2} = e^{x_1}(\cos y_1 + i\sin y_1)\,e^{x_2}(\cos y_2 + i\sin y_2)$$
$$= e^{x_1+x_2}(\cos y_1\cos y_2 - \sin y_1 \sin y_2 + i(\sin y_1 \cos y_2 + \cos y_1 \sin y_2))$$
$$= e^{x_1+x_2}[\cos(y_1+y_2) + i\sin(y_1+y_2)]$$
$$= e^{(x_1+x_2)+i(y_1+y_2)} = e^{z_1+z_2}.$$

(b) Let us use induction and the result established in part (a). Surely the proposition holds for $n=1$. Assume it holds for $n=k$. Then
$(e^z)^{k+1} = (e^z)^k e^z = e^{kz}e^z$ by assumption
$= e^{kz+z}$ by (a)
$= e^{(k+1)z}$, so we have proof by induction.

7.(b)
$$\cos z_1 \cos z_2 - \sin z_1 \sin z_2 = \frac{e^{iz_1}+e^{-iz_1}}{2}\,\frac{e^{iz_2}+e^{-iz_2}}{2} - \frac{e^{iz_1}-e^{-iz_1}}{2i}\,\frac{e^{iz_2}-e^{-iz_2}}{2i}$$
$$= \tfrac{1}{4}\big(e^{i(z_1+z_2)} + \cancel{e^{i(z_1-z_2)}} + \cancel{e^{-i(z_1-z_2)}} + e^{-i(z_1+z_2)}$$
$$+\, e^{i(z_1+z_2)} - \cancel{e^{i(z_1-z_2)}} - \cancel{e^{-i(z_1-z_2)}} + e^{-i(z_1+z_2)}\big)$$
$$= \tfrac{1}{4}(2e^{i(z_1+z_2)} + 2e^{-i(z_1+z_2)})$$

(e)
$$\cos x \cosh y - i\sin x \sinh y = \frac{e^{ix}+e^{-ix}}{2}\,\frac{e^{y}+e^{-y}}{2} - i\,\frac{e^{ix}-e^{-ix}}{2i}\,\frac{e^{y}-e^{-y}}{2}$$
$$= \tfrac{1}{4}(\cancel{e^{y+ix}} + e^{-y+ix} + e^{y-ix} + \cancel{e^{-y-ix}} - \cancel{e^{y+ix}} + e^{-y+ix} + e^{y-ix} - \cancel{e^{-y-ix}})$$
$$= \tfrac{1}{4}(2e^{-y+ix} + 2e^{y-ix}) = \tfrac{1}{2}(e^{i(x+iy)} + e^{-i(x+iy)}) = \cos(x+iy).$$

8.(b)
$$\cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2 = \frac{e^{z_1}+e^{-z_1}}{2}\,\frac{e^{z_2}+e^{-z_2}}{2} + \frac{e^{z_1}-e^{-z_1}}{2}\,\frac{e^{z_2}-e^{-z_2}}{2}$$
$$= \tfrac{1}{4}(e^{z_1+z_2} + \cancel{e^{z_1-z_2}} + \cancel{e^{-z_1+z_2}} + e^{-(z_1+z_2)} + e^{z_1+z_2} - \cancel{e^{z_1-z_2}} - \cancel{e^{-z_1+z_2}} + e^{-(z_1+z_2)})$$
$$= \tfrac{1}{4}(2e^{(z_1+z_2)} + 2e^{-(z_1+z_2)}) = \cosh(z_1+z_2).$$

(e) $\cosh x\cos y + i\sinh x\sin y = \frac{e^x+e^{-x}}{2}\,\frac{e^{iy}+e^{-iy}}{2} + i\,\frac{e^x-e^{-x}}{2}\,\frac{e^{iy}-e^{-iy}}{2i}$

$= \frac{1}{4}\left(e^{x+iy}+\cancel{e^{x-iy}}+\cancel{e^{-x+iy}}+e^{-(x+iy)}+e^{x+iy}-\cancel{e^{x-iy}}-\cancel{e^{-x+iy}}+e^{-(x+iy)}\right)$

$= \frac{1}{4}\left(2e^{(x+iy)}+2e^{-(x+iy)}\right) = \cos(x+iy)$

9. (a) $e^{2+\pi i} = e^2(\cos\pi + i\sin\pi) = -e^2$ (b) $e^{1-i} = e[\cos 1 - i\sin 1]$

(c) $e^{-\pi i/4} = \cos\pi/4 - i\sin\frac{\pi}{4} = \frac{1}{\sqrt2} - i\frac{1}{\sqrt2}$ (d) $\sin(3+\pi i) = \sin 3\cos\pi i + \sin\pi i\cos 3$
$= \sin 3\cosh\pi + i\sinh\pi\cos 3$

(e) $\cos(-2+3\pi i) = \cos 2\cos 3\pi i + \sin 2\sin 3\pi i = \cos 2\cosh 3\pi + i\sin 2\sinh 3\pi$

(f) $\sec(1+i) = 1/\cos(1+i) = 1/(\cos 1\cos i - \sin 1\sin i) = 1/(\cos 1\cosh 1 - i\sin 1\sinh 1)$

$= \frac{1}{\cos 1\cosh 1 - i\sin 1\sinh 1}\,\frac{\cos 1\cosh 1 + i\sin 1\sinh 1}{\cos 1\cosh 1 + i\sin 1\sinh 1}$

$= \frac{\cos 1\cosh 1}{(\cos 1\cosh 1)^2 + (\sin 1\sinh 1)^2} + i\,\frac{\sin 1\sinh 1}{(\cos 1\cosh 1)^2+(\sin 1\sinh 1)^2}$ [See (g), below.]

(g) $\csc(1-i) = 1/\sin(1-i) = \frac{2i}{e^{i(1-i)}-e^{-i(1-i)}} = \frac{2i}{e^{1+i}-e^{-1-i}}$ (Now multiply top and bottom by complex conj. of denominator)

$= \frac{2i}{e^{1+i}-e^{-1-i}}\,\frac{e^{1-i}-e^{-1+i}}{e^{1-i}-e^{-1+i}} = \frac{2i[e(\cos 1 - i\sin 1) - e^{-1}(\cos 1 + i\sin 1)]}{e^2 - e^{2i} - e^{-2i} + e^{-2}}$

$= \frac{2i[(e\cos 1 - e^{-1}\cos 1) - i(e\sin 1 + e^{-1}\sin 1)]}{(e^2+e^{-2}) - (e^{2i}+e^{-2i})} = \frac{i\cos 1\,(e-e^{-1}) + \sin 1\,(e+e^{-1})}{\cosh 2 - \cos 2}$

$= \frac{2\sin 1\cosh 1}{\cosh 2 - \cos 2} + i\,\frac{2\cos 1\sinh 1}{\cosh 2 - \cos 2}$ NOTE: Our procedure has been different in (f) and (g); we could have used the method of (g) in (f) or visa versa. Note also that the Maple symbol for i is I. The command

```
evalf(csc(1-I));
```

gives .6215180172 + .3039310016I

(h) $\tan\left(-\frac{3\pi i}{4}\right) = \frac{\sin(-3\pi i/4)}{\cos(-3\pi i/4)} = -i\,\frac{\sinh 3\pi/4}{\cosh 3\pi/4} = -i\tanh\frac{3\pi}{4}$

(i) $\cot\left(\frac{\pi i}{4}\right) = \frac{\cos(\pi i/4)}{\sin(\pi i/4)} = \frac{\cosh\pi/4}{i\sinh\pi/4} = -i\coth\frac{\pi}{4}$

(j) $\sinh(3+\pi i) = \sinh 3\cosh\pi i + \sinh\pi i\cosh 3$ (Here I've used Exercise 8(c), but we could simply use the definition of sinh.)
$= \sinh 3\cos\pi + i\,\underbrace{\sin\pi}_{0}\cosh 3$
$= -\sinh 3$

(k) $\cosh(1-\pi i) = \cosh 1\cosh(-\pi i) + \sinh(1)\sinh(-\pi i)$ (by Exercise 8(b))
$= \cosh 1\cos\pi + (\sinh 1)(-i)\sin\pi = -\cosh 1.$

(l) $\tanh(2+4\pi i) = \frac{\sinh(2+4\pi i)}{\cosh(2+4\pi i)} = \frac{\sinh 2\cosh 4\pi i + \sinh 4\pi i\cosh 2}{\cosh 2\cosh 4\pi i + \sinh 2\sinh 4\pi i}$

$= \frac{\sinh 2\cos 4\pi + i\sin 4\pi\cosh 2}{\cosh 2\cos 4\pi + i\sinh 2\sin 4\pi} = \frac{\sinh 2}{\cosh 2} = \tanh 2$

10. The step $|\cos z + i\sin z| = \sqrt{\cos^2 z + \sin^2 z}$ is incorrect. It holds if $\cos z$ and $\sin z$ are both real, but they are not.

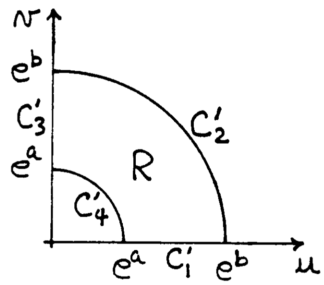
11. (a) $e^z = e^x(\cos y + i\sin y) = 1 = 1 + 0i \Rightarrow e^x\cos y = 1,$ ①
$e^x \sin y = 0.$ ②

Now, $e^x \neq 0$ for all x so ② $\Rightarrow \sin y = 0$ so $y = 0, \pm\pi, \pm 2\pi, \dots$. For $y = 0, \pm 2\pi, \pm 4\pi, \dots$ ① becomes $e^x = 1$ so $x = 0$; for $y = \pm\pi, \pm 3\pi, \dots$ ① becomes $-e^x = 1$ which has no real roots for x. Thus, $e^z = 1$ has only the roots $z = 0 + 2n\pi i$ where $n = 0, \pm 1, \pm 2, \dots$.

(b) $e^{z_1} = e^{z_2} \rightarrow e^{z_1 - z_2} = 1$, and (a) $\rightarrow z_1 - z_2 = 2n\pi i$ or $z_1 = z_2 + 2n\pi i$.

12. $w = e^z = \underbrace{e^x \cos y}_{u} + i\,\underbrace{e^x \sin y}_{v}$

C_1: $u = e^x,\ v = 0,\ a < x < b \rightarrow e^a < u < e^b$
C_2: $u = e^b \cos y,\ v = e^b \sin y,\ 0 < y < \pi/2$
or, $u^2 + v^2 = (e^b)^2$
C_3: $u = 0,\ v = e^x,\ a < x < b \rightarrow e^a < v < e^b$
C_4: $u = e^a \cos y,\ v = e^a \sin y,\ 0 < y < \pi/2$
or, $u^2 + v^2 = (e^a)^2$.
$\Rightarrow$

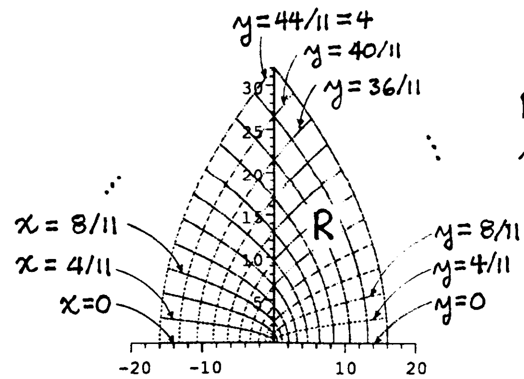


13. $\sin z = \sin(x + iy) = \sin x \cos iy + \sin iy \cos x = \underbrace{\sin x \cosh y}_{u} + i\,\underbrace{\sinh y \cos x}_{v}$

$y = 0, -\pi/2 < x < \pi/2 \rightarrow u = \sin x, v = 0$ gives the segment $-1 < u < 1$ of the u axis
$x = \pi/2, 0 < y < \infty \rightarrow u = \cosh y, v = 0$ gives the segment $1 < u < \infty$ of the u axis
$x = -\pi/2, 0 < y < \infty \rightarrow u = -\cosh y, v = 0$ gives the segment $-\infty < u < -1$ of the u axis
Further, $z = i \rightarrow w = i\sinh 1$, so R is evidently the upper half plane, not the lower half plane.

14. (a) $(4 + 4i)^2 = 16 + 32i - 16 = 32i$

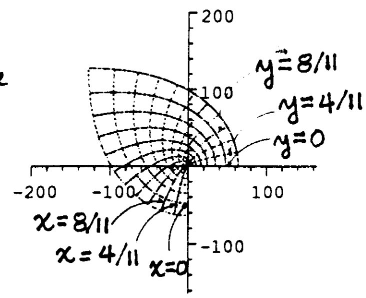
```
> with(plots):
> conformal(z^2,z=0..4+4*I,w=-20..20+32*I,grid=[12,12],numxy=[40,40]
  );
```



R is the image of the square $0 < x < 4, 0 < y < 4$.

15.(a)
```
> conformal(z^3,z=0..4+4*I,w=-200-164*I..164+200*I,grid=[12,12],numxy=[40,40]);
```

NOTE: The image of the region $0<x<\infty$, $0<y<\infty$ would be the 1st, 2nd, and 3rd quadrants.



16.(a) $I = \operatorname{Im}\int_0^\infty e^{-x}e^{i\omega x}\,dx = \operatorname{Im}\int_0^\infty e^{-(1-i\omega)x}\,dx$

$= \operatorname{Im} \left.\dfrac{e^{-(1-i\omega)x}}{-1+i\omega}\right|_0^\infty = \operatorname{Im}\left(0-\dfrac{1}{-1+i\omega}\right) = \operatorname{Im}\left(\dfrac{1}{1-i\omega}\,\dfrac{1+i\omega}{1+i\omega}\right)$

$= \operatorname{Im}\dfrac{1+i\omega}{1+\omega^2} = \omega/(1+\omega^2)$. NOTE: In more detail, $e^{-(1-i\omega)x}=0$ at $x=\infty$ because $\lim_{x\to\infty}|e^{-(1-i\omega)x}| = \lim_{x\to\infty}|e^{-x}e^{i\omega x}| = \lim_{x\to\infty} e^{-x}|e^{i\omega x}|$

$= \lim_{x\to\infty} e^{-x} = 0$. Now, if $|e^{-(1-i\omega)x}|\to 0$ as $x\to\infty$, then $e^{-(1-i\omega)x}\to 0$ as $x\to\infty$.

(b) $I = \int_0^\infty e^{-st}\cos\omega t\,dt$. Be careful; if s is complex (with nonzero imaginary part) then $e^{-st}\cos\omega t = \operatorname{Re}(e^{-st}e^{i\omega t}) = \operatorname{Re}(e^{-(s-i\omega)t})$ is <u>not true</u>.* To use the method let us assume that s is <u>real</u>, with $s>0$. Then

$I = \operatorname{Re}\int_0^\infty e^{-(s-i\omega)t}\,dt = \operatorname{Re}\left.\dfrac{e^{-(s-i\omega)t}}{-(s-i\omega)}\right|_0^\infty = \operatorname{Re}\left(0-\dfrac{1}{-s+i\omega}\right) = \operatorname{Re}\left(\dfrac{1}{s-i\omega}\,\dfrac{s+i\omega}{s+i\omega}\right)$

$= \operatorname{Re}\left(\dfrac{s+i\omega}{s^2+\omega^2}\right) = s/(s^2+\omega^2)$.

17. (a) The problem is that $\cos mx\cos nx = (\operatorname{Re} e^{imx})(\operatorname{Re} e^{inx}) \neq \operatorname{Re}(e^{i(m+n)x})$. Let us show that:

$$\operatorname{Re} e^{i(m+n)x} = \cos(m+n)x,$$

but
$$\cos mx\cos nx = \tfrac{1}{2}(e^{imx}+e^{-imx})\tfrac{1}{2}(e^{inx}+e^{-inx})$$
$$= \tfrac{1}{4}(e^{i(m+n)x}+e^{-i(m+n)x}+e^{i(m-n)x}+e^{-i(m-n)x})$$
$$= \tfrac{1}{2}\cos(m+n)x + \tfrac{1}{2}\cos(m-n)x \neq \cos(m+n)x.$$

18. (b) $v'+2v = 10e^{i3t}$. Seek $v_p = Ae^{i3t}$. $(3i+2)Ae^{i3t} = 10e^{i3t}$ so $A = 10/(2+3i)$.

$x_p(t) = \operatorname{Im} v_p(t) = \operatorname{Im}\dfrac{10e^{i3t}}{2+3i}\,\dfrac{2-3i}{2-3i} = \dfrac{10}{13}\operatorname{Im}[(\cos 3t+i\sin 3t)(2-3i)]$

$= \dfrac{10}{13}(2\sin 3t - 3\cos 3t)$

---

* Nevertheless, it is fortuitous that the result $I = s/(s^2+\omega^2)$ is correct even if s is complex (provided that $\operatorname{Re} s>0$).

(c) $x_p(t) = \frac{10}{17}(3\cos 5t + 5\sin 3t)$

(d) $v'' + v' = 100e^{i5t}$, $v_p = Ae^{i5t}$, $(-25+5i)A\cancel{e^{i5t}} = 100\cancel{e^{i5t}}$ so $A = 100/(-25+5i)$.
$x_p(t) = \operatorname{Im}\left(\frac{20}{-5+i}\,\frac{-5-i}{-5-i}\,e^{i5t}\right) = -\frac{20}{26}\operatorname{Im}[(5+i)(\cos 5t + i\sin 5t)]$
$= -\frac{10}{13}(\cos 5t + 5\sin 5t)$.

(e) $v'''' + 2v' + v = 10e^{it}$, $v_p = Ae^{it}$, $(1+2i+1)A\cancel{e^{it}} = 10\cancel{e^{it}}$ so $A = 10/(2+2i)$.
$x_p(t) = \operatorname{Im}\left(\frac{10}{2+2i}e^{it}\right) = 5\operatorname{Im}\left[\frac{(\cos t + i\sin t)}{1+i}\,\frac{1-i}{1-i}\right] = \frac{5}{2}(\sin t - \cos t)$.

(f) $v'''' - v' + 5v = 20e^{i2t}$, $v_p = Ae^{i2t}$, $(16-2i+5)A\cancel{e^{i2t}} = 20\cancel{e^{i2t}}$, $A = 20/(21-2i)$
$x_p(t) = \operatorname{Re}\left(\frac{20}{21-2i}\,\frac{21+2i}{21+2i}(\cos 2t + i\sin 2t)\right) = \frac{20}{445}(21\cos 2t - 2\sin 2t)$

(g) $v'''' - 2v' - 3v = 60e^{i3t}$, $v_p = Ae^{i3t}$, $(81-6i-3)A\cancel{e^{i3t}} = 60\cancel{e^{i3t}}$, $A = 60/(78-6i)$
$x_p(t) = \operatorname{Im}\frac{30}{39-3i}e^{i3t} = \operatorname{Im}\left(\frac{10}{13-i}\,\frac{13+i}{13+i}(\cos 3t + i\sin 3t)\right)$
$= \frac{10}{170}\operatorname{Im}[(13+i)(\cos 3t + i\sin 3t)] = \frac{1}{17}(13\sin 3t + \cos 3t)$.

Section 21.4

1. (a) $z = -3i$, $r = 3$, $\theta = -\pi/2$ rad $= -90°$
(b) $z = 8i$, $r = 8$, $\theta = \pi/2$ rad $= 90°$
(c) $z = -6$, $r = 6$, $\theta = \pi$ rad $= 180°$ (Recall that $-\pi < \theta \le \pi$ for principal arg.)
(d) $z = 1+5i$, $r = \sqrt{26}$, $\theta = \tan^{-1}5 = 1.373$ rad $= 78.69°$
(e) $z = -4-3i$, $r = 5$, $\theta = \tan^{-1}\frac{3}{4} = -2.498$ rad $= -143.13°$
(f) $z = 2-12i$, $r = \sqrt{148} = 2\sqrt{37}$, $\theta = \tan^{-1}\left(-\frac{12}{2}\right) = -1.406$ rad $= -80.538°$
(g) $z = -1+i$, $r = \sqrt{2}$, $\theta = 3\pi/4$ rad $= 135°$
(h) $z = -1-i$, $r = \sqrt{2}$, $\theta = -3\pi/4$ rad $= -135°$
(i) $z = 0.2+i$, $r = \sqrt{1.04}$, $\theta = \tan^{-1}\left(\frac{1}{.2}\right) = 1.373$ rad $= 87.433°$

2. $\operatorname{Re}(re^{i\theta}) = \operatorname{Re}(r\cos\theta + ir\sin\theta) = r\cos\theta$
$\operatorname{Im}(\;''\;) = \operatorname{Im}(\;\;''\;\;) = r\sin\theta$

3. Product: $z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)} = r_1r_2e^{i(\theta_{1_0}+2m\pi+\theta_{2_0}+2n\pi)}$
$= r_1r_2e^{i(\theta_{1_0}+\theta_{2_0})}\underbrace{e^{i2(m+n)\pi}}_{=1 \text{ for all integers } m \text{ and } n} = r_1r_2e^{i(\theta_{1_0}+\theta_{2_0})}$

Similarly for z_1/z_2.

4. (a) $(-1+i)^{10} = (2^{1/2} e^{(3\pi/4)i})^{10} = 2^5 e^{15\pi i/2} = 2^5 e^{6\pi i} e^{3\pi i/2} = \underbrace{32 e^{3\pi i/2}}_{\text{polar}} = \underbrace{-32i}_{\text{cartesian}}$

Of course the polar form $32e^{15\pi i/2}$ was OK too but usually we prefer the arg to be in $0 \le \theta < 2\pi$ or in $-\pi < \theta \le \pi$ (the latter for the principal arg)

$(-1+i)^{20} = 2^{10} e^{15\pi i} = \underbrace{1024 e^{\pi i}}_{\text{Polar}} = \underbrace{-1024}_{\text{Cartesian}}$

(b) $(1+i)^{10} = (2^{1/2} e^{\pi i/4})^{10} = 2^5 e^{5\pi i/2} = \underbrace{32 e^{\pi i/2}}_{\text{Polar}} = \underbrace{32i}_{\text{Cartesian}}$

$(1+i)^{20} = (2^{1/2} e^{\pi i/4})^{20} = 2^{10} e^{5\pi i} = \underbrace{1024 e^{\pi i}}_{\text{Polar}} = \underbrace{-1024}_{\text{Cartesian}}$

(c) $(1+2i)^{10} = (5^{1/2} e^{1.107i})^{10} = 5^5 e^{11.07i} = \underbrace{3125 e^{4.787i}}_{\text{Polar}} = \underbrace{237-3116i}_{\text{Cartesian}}$

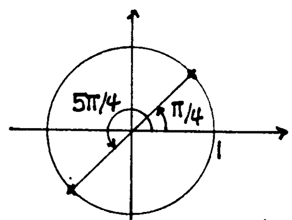
$(1+2i)^{20} = (5^{\frac{1}{2}} e^{1.107i})^{20}$ Before continuing, note that since we are multiplying the 1.107 by 20 we will not obtain very accurate results (i.e., say to 4 significant figures), so let us include more places to begin with: $\tan^{-1} 2 = 1.1071487$.

$= (5^{1/2} e^{1.1071487i})^{20} = 5^{10} e^{22.142974i} = 5^{10} e^{15.85979i} = 5^{10} e^{9.57660i}$

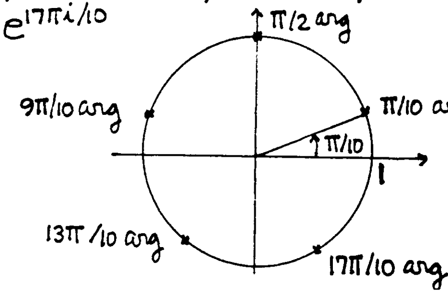
$= \underbrace{5^{10} e^{3.2934i}}_{\text{polar}} = \underbrace{5^{10}(-0.9885 - .1512i) = -9653320.3 - 1476562.5i}_{\text{Cartesian}}$

The maple command (1+2*I)^20 gives
$-9653287 - 1476984i$

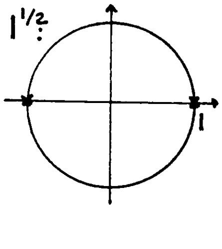
5. (a) $i^{1/2} = (1e^{\pi i/2})^{1/2}, (1e^{5\pi i/2})^{1/2}$
$= e^{\pi i/4}, e^{5\pi i/4}$



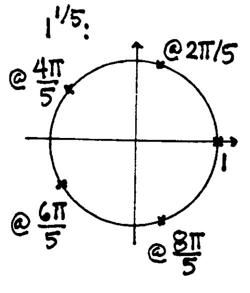
$i^{1/5} = (1e^{\pi i/2})^{1/5}, (1e^{5\pi i/2})^{1/5}, (1e^{9\pi i/2})^{1/5}, (1e^{13\pi i/2})^{1/5}, (1e^{17\pi i/2})^{1/5}$
$= e^{\pi i/10}, e^{\pi i/2}, e^{9\pi i/10}, e^{13\pi i/10}, e^{17\pi i/10}$



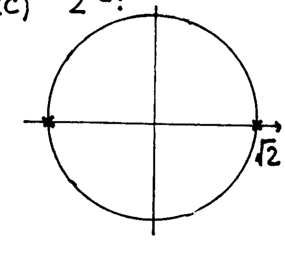
(b) $1^{1/2}$:



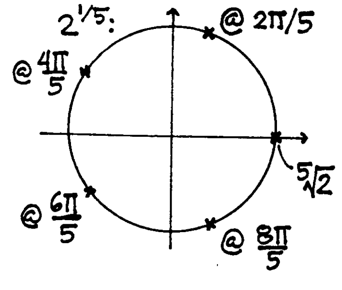
$1^{1/5}$:



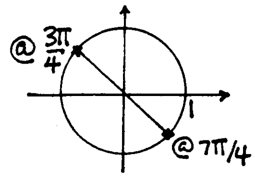
(c) $2^{1/2}$:



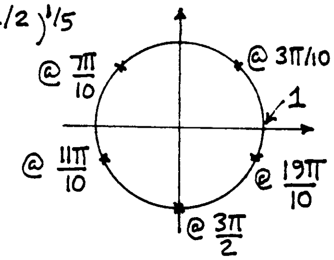
$2^{1/5}$:



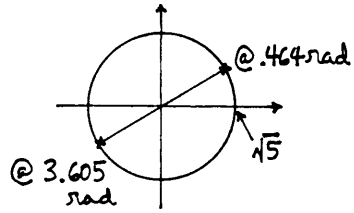
(d) $(-i)^{1/2} = (1e^{3\pi i/2})^{1/2}, (1e^{7\pi i/2})^{1/2}$
$= 1e^{3\pi i/4}, \ 1e^{7\pi i/4}$



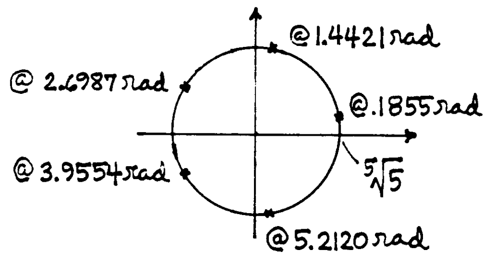
$(-i)^{1/5} = (1e^{3\pi i/2})^{1/5}, (1e^{7\pi i/2})^{1/5}, (1e^{11\pi i/2})^{1/5}, (1e^{15\pi i/2})^{1/5}, (1e^{19\pi i/2})^{1/5}$
$= 1e^{3\pi i/10}, 1e^{7\pi i/10}, 1e^{11\pi i/10}, 1e^{3\pi i/2}, 1e^{19\pi i/10}$



(g) $(3+4i)^{1/2} = (5e^{0.92730i})^{1/2}, (5e^{7.21048i})^{1/2}$
$= \sqrt{5}\, e^{.4637i}, \ \sqrt{5}\, e^{3.6052i}$



$(3+4i)^{1/5} = (5e^{i.92730})^{1/5}, (5e^{7.21048i})^{1/5}, (5e^{13.49367i})^{1/5}, (5e^{19.77686i})^{1/5}, (5e^{26.06004i})^{1/5}$
$= \sqrt[5]{5}\, e^{.1855i}, \ \sqrt[5]{5}\, e^{1.4421i}, \ \sqrt[5]{5}\, e^{2.6987i}, \ \sqrt[5]{5}\, e^{3.9554i}, \ \sqrt[5]{5}\, e^{5.2120i}$



6. (a) $\log(-2) = \log(2e^{(\pi+2n\pi)i}) = \ln 2 + (2n+1)\pi i \quad (n=0,\pm1,\pm2,\ldots)$
(b) $\log(1) = \log(1e^{2n\pi i}) = \ln 1 + 2n\pi i = 2n\pi i \quad (\quad " \quad)$
(c) $\log(i) = \log(1e^{(\frac{\pi}{2}+2n\pi)i}) = \ln 1 + (\frac{4n+1}{2})\pi i \quad (\quad " \quad)$

(d) $\log(-5i) = \log(5e^{(-\frac{\pi}{2}+2n\pi)i}) = \ln 5 + (\frac{4n-1}{2})\pi i \quad (\quad " \quad)$
(e) $\log(2-i) = \log(\sqrt{5}\, e^{(-.4636+2n\pi)i}) = \frac{1}{2}\ln 5 + (-.4636+2n\pi)i \quad (n=0,\pm1,\pm2,\ldots)$

7. $x_1 = x_2$ and $y_1 = y_2$ gives $\begin{cases} r_1\cos\theta_1 = r_2\cos\theta_2 \\ r_1\sin\theta_1 = r_2\sin\theta_2 . \end{cases}$
Squaring and adding gives $r_1^2 = r_2^2$ so $r_1 = r_2$. Then, $\cos\theta_1 = \cos\theta_2$ and $\sin\theta_1 = \sin\theta_2$ give $\theta_1 = \theta_2$ + arbitrary integer multiple of 2π.

8. (a) $(2i)^{2/3} \overset{\text{not an essential step}}{=} (-4)^{1/3} = (4e^{i(\pi+2k\pi)})^{1/3} = \sqrt[3]{4}\, e^{\pi i/3}, \ \sqrt[3]{4}\, e^{\pi i}, \ \sqrt[3]{4}\, e^{5\pi i/3}$
$(2i)^{3/2} = (-8i)^{1/2} = (8e^{i(\frac{3\pi}{2}+2k\pi)})^{1/2} = \sqrt{8}\, e^{3\pi i/4}, \ \sqrt{8}\, e^{7\pi i/4}$
$(2i)^{\pi} = e^{\pi \log 2i} = \exp\{\pi \log[2e^{(\frac{\pi}{2}+2k\pi)i}]\} = \exp\{\pi[\ln 2 + (\frac{\pi}{2}+2k\pi)i]\}$

$= e^{\pi \ln 2}\, e^{i(1+4k)\pi^2/2} \quad (k=0,\pm1,\pm2,\ldots)$

(b) $3^{2/3} = 9^{1/3} = (9e^{i2k\pi})^{1/3} = \sqrt[3]{9}, \ \sqrt[3]{9}\, e^{2\pi i/3}, \ \sqrt[3]{9}\, e^{4\pi i/3}$
$3^{3/2} = 27^{1/2} = (27e^{i2k\pi})^{1/2} = \sqrt{27}, -\sqrt{27}$
$3^{\pi} = e^{\pi \log 3} = e^{\pi(\ln 3 + 2k\pi i)} = e^{\pi \ln 3}\, e^{2k\pi^2 i}$

(e) $(1-i)^{2/3} = (-2i)^{1/3} = \left(2e^{(-\frac{\pi}{2}+2k\pi)i}\right)^{1/3} = \sqrt[3]{2}\,e^{-\pi i/6},\ \sqrt[3]{2}\,e^{\pi i/2},\ \sqrt[3]{2}\,e^{7\pi i/6}$

$(1-i)^{3/2} = (-2-2i)^{1/2} = \left(\sqrt{8}\,e^{(\frac{5\pi}{4}+2k\pi)i}\right)^{1/2} = \sqrt[4]{8}\,e^{5\pi i/8},\ \sqrt[4]{8}\,e^{13\pi i/8}$

$$(1-i)^{\pi} = e^{\pi \log(1-i)} = e^{\pi \log[\sqrt{2}\,e^{(-\frac{\pi}{4}+2k\pi)i}]} = e^{\pi \ln\sqrt{2} + \pi(8k\pi-1)i/4}$$
$$= e^{\pi\ln\sqrt{2}}\, e^{\pi(8k\pi-1)i/4} \quad (k=0,\pm1,\dots)$$

9. (a) $(2i)^i = e^{i\log 2i} = e^{i\log(2e^{(\frac{\pi}{2}+2k\pi)i})} = e^{i[\ln 2 + (4k+1)\pi i/2]}$
$$= e^{-(4k+1)\pi/2}[\cos(\ln 2) + i\sin(\ln 2)]$$

$$(2i)^{1-i} = e^{(1-i)\log(2i)} = e^{(1-i)\log[2e^{(\frac{\pi}{2}+2k\pi)i}]} = e^{(1-i)[\ln 2 + (\frac{\pi}{2}+2k\pi)i]}$$
$$= e^{\ln 2 + (\pi/2+2k\pi)}\, e^{i(\frac{\pi}{2}+2k\pi-\ln 2)}$$
$$= 2e^{(4k+1)\pi/2}\left[\cos(\tfrac{\pi}{2}+2k\pi-\ln 2) + i\sin(\tfrac{\pi}{2}+2k\pi-\ln 2)\right] \quad (k=0,\pm1,\dots)$$

(b) $3^i = e^{i\log 3} = e^{i\log[3e^{2k\pi i}]} = e^{i[\ln 3 + 2k\pi i]} = e^{-2k\pi}(\cos(\ln 3) + i\sin(\ln 3)),$ for $k=0,\pm1,\dots$

$3^{1-i} = 3e^{-i\log 3} = 3e^{-i[\ln 3 + 2k\pi i]} = 3e^{2k\pi}(\cos(\ln 3) + i\sin(\ln 3))$ for $k=0,\pm1,\dots$

(e) $(1-i)^i = e^{i\log(1-i)} = e^{i\log[\sqrt{2}\,e^{(-\frac{\pi}{4}+2k\pi)i}]} = e^{i[\ln\sqrt{2} + (-\frac{\pi}{4}+2k\pi)i]}$
$$= e^{(\frac{\pi}{4}-2k\pi)}[\cos(\ln\sqrt{2}) + i\sin(\ln\sqrt{2})]$$

$$(1-i)^{1-i} = e^{(1-i)\log(1-i)} = e^{(1-i)\log[\sqrt{2}\,e^{(-\frac{\pi}{4}+2k\pi)i}]} = e^{(1-i)[\ln\sqrt{2} + (-\frac{\pi}{4}+2k\pi)i]}$$
$$= e^{\ln\sqrt{2} - \frac{\pi}{4} + 2k\pi}\, e^{i[2k\pi - \frac{\pi}{4} - \ln\sqrt{2}]}$$
$$= \sqrt{2}\,e^{2k\pi-\pi/4}\left[\cos(2k\pi - \tfrac{\pi}{4} - \ln\sqrt{2}) + i\sin(2k\pi - \tfrac{\pi}{4} - \ln\sqrt{2})\right] \quad (k=0,\pm1,\dots)$$

10. With $c = 1+\sqrt{3}i$, $|c| = 2$ and $\operatorname{Arg} c = \pi/3$, (10.2) gives, for $z = 2-5i$,
$$c^z = e^{(2-5i)(\ln 2 + i\pi/3)} = e^{2\ln 2 + 5\pi/3}\, e^{i(2\pi/3 - 5\ln 2)}$$
$$= 4e^{5\pi/3}\left[\cos(\tfrac{2\pi}{3} - 5\ln 2) + i\sin(\tfrac{2\pi}{3} - 5\ln 2)\right]$$

11. (a) $\log(-3i) = \log(3e^{-\pi i/2}) = \ln 3 - \pi i/2$

$\sqrt{-3i} = (3e^{-\pi i/2})^{1/2} = \sqrt{3}\,e^{-\pi i/4} = \frac{\sqrt{3}}{2} - i\frac{\sqrt{3}}{2}$

(b) $\log 2 = \log(2e^{i0}) = \ln 2 + i0 = \ln 2$

$\sqrt{2} = (2e^{i0})^{1/2} = \sqrt{2}\,e^{i0} = \sqrt{2}$

(c) $\log(-4) = \log(4e^{i\theta})$. Is $\theta = +\pi$ or $-\pi$? If the point -4 is on top of the cut then $\theta = \pi$ and
$$\log(-4) = \log(4e^{i\pi}) = \ln 4 + i\pi,$$
and if -4 is on the bottom of the cut then $\theta = -\pi$ and
$$\log(-4) = \log(4e^{-i\pi}) = \ln 4 - i\pi$$

NOTE: We can't "figure out" whether -4 means the point -4 on top of the cut or the point -4 on the bottom of the cut; we need to specify whether it is on the top or bottom (and it does have to be one or the other!).

(d) $\log(2-i) = \log(\sqrt{5}\, e^{-.4636i}) = \ln\sqrt{5} - 0.4636i$

NOTE: What does Maple give for $\log(2-i)$?
The command log(2-I); merely gives the output "ln(2-I)",
the command eval(log(2-I)); does the same, but
the command evalf(log(2-I)); does give the principal value,
$.8047189562 - .4636476090I$.

$\sqrt{2-i} = (\sqrt{5}\, e^{-.4636i})^{1/2} = \sqrt[4]{5}\, e^{-.2318i} = \sqrt[4]{5}(\cos .2318 - i \sin .2318)$

Likewise, the Maple command evalf(sqrt(2-I)); gives this same value, namely, $1.455346690 - .3435607497I$.

(e) $\log(1+\sqrt{3}i) = .6931471807 + 1.047197551i$
$\sqrt{1+\sqrt{3}i} = 1.224744871 + .7071067813i$

(f) $\log(-1-i) = .3465735903 - 2.356194490i$
$\sqrt{-1-i} = .4550898606 - 1.098684113i$

(g) $\log(-5i) = 1.609437912 - 1.570796327i$
$\sqrt{-5i} = 1.581138830 - 1.581138830i$

(h) $\log(4-2i) = 1.497866137 - .4636476090i$
$\sqrt{4-2i} = 2.058171027 - .4858682718i$

12. (a) $\log(z_1 z_2) = \log(r_1 e^{i\theta_1} r_2 e^{i\theta_2}) = \log(r_1 r_2 e^{i(\theta_1+\theta_2)}) = \ln(r_1 r_2) + i(\theta_1+\theta_2)$
$= (\ln r_1 + i\theta_1) + (\ln r_2 + i\theta_2) = \log(r_1 e^{i\theta_1}) + \log(r_2 e^{i\theta_2}) = \log z_1 + \log z_2$ ✓

(b) Similar to (a).

(c) $\log z^c = \log e^{c \log z}$ (let $c = a+ib$, say)
$= \log e^{(a+ib)(\ln r + i\theta)} = \log(e^{a\ln r - b\theta} e^{i(b\ln r + a\theta)})$
$= (a\ln r - b\theta) + i(b\ln r + a\theta)$
$c\log z = (a+ib)(\ln r + i\theta) = (a\ln r - b\theta) + i(b\ln r + a\theta) = \log z^c$. ✓

13. (a) $z = \sin w = (e^{iw} - e^{-iw})/2i$ gives $(e^{iw})^2 - 2iz(e^{iw}) - 1 = 0$ so, by the quadratic formula, $e^{iw} = (2iz \pm \sqrt{-4z^2+4})/2 = iz + \sqrt{1-z^2}$, where there is no loss in dropping the $\pm$ since the $\sqrt{\ }$ always gives the $\pm$.
Then, log of both sides gives

$$iw = \log(iz + \sqrt{1-z^2})$$
$$w = -i\log(iz + \sqrt{1-z^2})$$
$$\sin^{-1} z = -i\log(iz + \sqrt{1-z^2}).$$

(b) $\sin^{-1}(\frac{1}{2}) = -i\log(\frac{1}{2}i \pm \frac{\sqrt{3}}{2})$

Using the upper (+) sign gives
$$\sin^{-1}(\tfrac{1}{2}) = -i\log(\tfrac{\sqrt{3}}{2} + \tfrac{i}{2}) = -i\log(1e^{i(\frac{\pi}{6}+2k\pi)}) = \tfrac{\pi}{6} + 2k\pi$$
and using the lower sign gives
$$\sin^{-1}(\tfrac{1}{2}) = -i\log(-\tfrac{\sqrt{3}}{2} + \tfrac{i}{2}) = -i\log(1e^{i(\frac{5\pi}{6}+2k\pi)}) = \tfrac{5\pi}{6} + 2k\pi,$$

for $k=0,\pm 1,\dots$.

(c) $\sin^{-1}2 = -i\log(2i \pm \sqrt{3}\,i) = -i\log[(2\pm\sqrt{3})i]$ ← both are positive and real

$$= -i\log[(2\pm\sqrt{3})e^{(\frac{\pi}{2}+2k\pi)i}] = -i\ln(2\pm\sqrt{3}) + (\tfrac{\pi}{2}+2k\pi)$$

$$= (\tfrac{\pi}{2}+2k\pi) - i\ln(2\pm\sqrt{3}) \quad (k=0,\pm1,\dots)$$

(d) $\sin^{-1}(2i) = -i\log[i(2i)\pm\sqrt{5}] = -i\log(-2\pm\sqrt{5})$.

The upper sign gives (since $-2+\sqrt{5}>0$)

$$\sin^{-1}(2i) = -i\log[(\sqrt{5}-2)e^{i(0+2k\pi)}]$$

$$= -i[\ln(\sqrt{5}-2)+2k\pi i] = 2k\pi - i\ln(\sqrt{5}-2)$$

and the lower sign gives (since $-2-\sqrt{5}<0$)

$$\sin^{-1}(2i) = -i\log[(\sqrt{5}+2)e^{i(\pi+2k\pi)}]$$

$$= -i[\ln(\sqrt{5}+2)+(2k+1)\pi i] = (2k+1)\pi - i\ln(\sqrt{5}+2)$$

14. (a) Let $w=\cos^{-1}z$. Then $z=\cos w = (e^{iw}+e^{-iw})/2$ so $(e^{iw})^2 - 2z(e^{iw})+1=0$ and the quadratic formula gives

$$e^{iw} = (2z\pm\sqrt{4z^2-4})/2 = z+\sqrt{z^2-1}$$

$$iw = \log(z+\sqrt{z^2-1})$$

$$w = -i\log(z+\sqrt{z^2-1}).$$

(b) Let $w=\tan^{-1}z$. Then $z=\tan w = \dfrac{\sin w}{\cos w} = \dfrac{1}{i}\dfrac{e^{iw}-e^{-iw}}{e^{iw}+e^{-iw}} = \dfrac{1}{i}\dfrac{e^{i2w}-1}{e^{i2w}+1}$

and, by algebra, $e^{i2w} = \dfrac{1+iz}{1-iz} = \dfrac{i-z}{i+z}$,

so $\quad i2w = \log\dfrac{i-z}{i+z}, \quad w=\tan^{-1}z = -\dfrac{i}{2}\log\dfrac{i-z}{i+z}$.

15. (a) Let $w=\sinh^{-1}z$. Then $z=\sinh w = (e^{w}-e^{-w})/2$ so $(e^{w})^2-2z(e^{w})-1=0$ and the quadratic formula gives

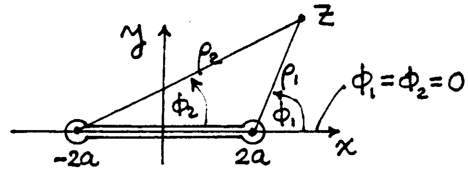
$$e^{w} = (2z\pm\sqrt{4z^2+4})/2 = z+\sqrt{z^2+1}$$

$$w = \sinh^{-1}z = \log(z+\sqrt{z^2+1})$$

(c) Let $w=\tanh^{-1}z$. Then $z=\tanh w = \dfrac{\sinh w}{\cosh w} = \dfrac{e^{w}-e^{-w}}{e^{w}+e^{-w}} = \dfrac{e^{2w}-1}{e^{2w}+1}$ and, by algebra,

$$e^{2w} = \frac{1+z}{1-z}, \quad 2w = \log\left(\frac{1+z}{1-z}\right)$$

$$w = \tanh^{-1}z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right).$$

16. 

On the top of the plate (i.e., on the top of the cut) $\rho_1 = 2a-x$, $\phi_1=\pi$
$\rho_2 = 2a+x$, $\phi_2 = 0$

so $$u(x,0+)-iv(x,0+) = \frac{iV_0x}{\sqrt{(z-2a)(z+2a)}} = \frac{iV_0x}{\sqrt{\rho_1e^{i\phi_1}\rho_2e^{i\phi_2}}} = \frac{iV_0x}{\sqrt{(4a^2-x^2)e^{i(\pi+0)}}}$$
$$= \frac{V_0x}{\sqrt{4a^2-x^2}}$$

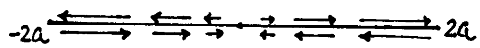
so $u(x,0+) = V_0x/\sqrt{4a^2-x^2}$ and $v(x,0+)=0$ (as it should!).

On the bottom of the plate $\rho_1 = 2a-x$, $\phi_1 = -\pi$
$\rho_2 = 2a+x$, $\phi_2 = 0$

so $$u(x,0-)-iv(x,0-) = \frac{iV_0x}{\sqrt{(z-2a)(z+2a)}} = \frac{iV_0x}{\sqrt{\rho_1e^{i\phi_1}\rho_2e^{i\phi_2}}} = \frac{iV_0x}{\sqrt{(4a^2-x^2)e^{i(-\pi+0)}}}$$
$$= -\frac{V_0x}{\sqrt{4a^2-x^2}}$$

so $u(x,0-) = -V_0x/\sqrt{4a^2-x^2}$ and $v(x,0-)=0$ (as it should).

Observe that $u=v=0$ both on top of the plate and on the bottom of the plate - at the origin. The x-velocity increases as we move away from the origin and $\to\infty$ as $x\to\pm 2a$, so we say that the flow is "singular" at $z=\pm 2a$. When the flow turns the 180° corners, at $z=\pm 2a$ it slows down as it approaches the origin which, again, is a stagnation point.



Section 21.5

1. Is there a $\delta(\epsilon)$ such that $|3iz-3i|<\epsilon$ for all $0<|z-1|<\delta$? Well, $|3iz-3i|<\epsilon$ gives $|z-1|<\epsilon/3$, so we can choose $\delta=\epsilon/3$ or smaller. Besides $\lim_{z\to 1} 3iz = 3i$ it is also true that $(3iz)|_{z=1} = 3i$. Thus, $w(z)=3iz$ is continuous at $z=1$.

2. (a) $|z^2|<\epsilon$ gives $|z|^2<\epsilon$, $|z|<\sqrt{\epsilon}$, so with $\delta(\epsilon)=\sqrt{\epsilon}$ (or smaller) it follows that $|z^2|<\epsilon$ for all $|z|<\sqrt{\epsilon}$. Further, z^2 is $=0$ at $z=0$ so $w(z)=z^2$ is continuous at $z=0$.

3. (a) $\lim_{z\to z_0} f(z)=A$ implies that for any $\epsilon>0$, no matter how small, there is a δ_1 such that $|f(z)-A|<\epsilon/2$ for all $0<|z-z_0|<\delta_1$. Similarly, $|g(z)-B|<\epsilon/2$ for all $0<|z-z_0|<\delta_2$. Thus, $|f(z)+g(z)-A-B| \le |f(z)-A|+|g(z)-B| < \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon$ for all $0<|z-z_0|<\min(\delta_1,\delta_2)$.

(b) $\lim_{z\to z_0} f(z)=A$ implies that for any $\epsilon'>0$ (no matter how small) there is a δ_1 such that $|f(z)-A|<\epsilon'$ for all $0<|z-z_0|<\delta_1$. Similarly, $|g(z)-B|<\epsilon'$ for all $0<|z-z_0|<\delta_2$. Thus,

$$\begin{aligned}|f(z)g(z)-AB| &= |(f(z)-A)(g(z)-B)+Ag(z)+Bf(z)-AB-AB| \\ &= |(f(z)-A)(g(z)-B)+B(f(z)-A)+A(g(z)-B)| \\ &\le |f(z)-A||g(z)-B|+|B||f(z)-A|+|A||g(z)-B| \\ &< \epsilon'^2+|B|\epsilon'+|A|\epsilon' \equiv \epsilon\end{aligned}$$

Now, for any value of ϵ (no matter how small) we can solve

$$\epsilon'^2+(|A|+|B|)\epsilon'=\epsilon$$

for ϵ', namely, $\epsilon'=\left[-(|A|+|B|)+\sqrt{(|A|+|B|)^2+4\epsilon}\,\right]/2>0$. Thus, for any given value of ϵ (no matter how small) there exists a $\delta=\min(\delta_1,\delta_2)$ such that $|f(z)g(z)-AB|<\epsilon$ for all z in $0<|z-z_0|<\delta$, so $\lim_{z\to z_0} f(z)g(z)=AB$.

4. No. As proof, a single counterexample will suffice. Here is one: $f(x)=1/(1+x^2)$ is continuous for all x (its graph is a bell-shaped curve), but $f(z)=1/(1+z^2)$ is not continuous everywhere, for it is discontinuous at $z=\pm i$, where it $\to\infty$.

5. (a) $\frac{d}{dz}z^3=\lim_{\Delta z\to0}\frac{(z+\Delta z)^3-z^3}{\Delta z}=\lim_{\Delta z\to0}\frac{\cancel{z^3}+(\Delta z)^3+3z^2\Delta z+3z(\Delta z)^2-\cancel{z^3}}{\Delta z}=3z^2$

(b) $\frac{d}{dz}\frac{1}{z}=\lim_{\Delta z\to0}\frac{\frac{1}{z+\Delta z}-\frac{1}{z}}{\Delta z}=\lim_{\Delta z\to0}\frac{z-(z+\Delta z)}{z(z+\Delta z)\Delta z}=-\lim_{\Delta z\to0}\frac{1}{z(z+\Delta z)}=-\frac{1}{z^2}$ (if $z\neq0$)

(c) $\frac{d}{dz}\frac{1}{z^2}=\lim_{\Delta z\to0}\frac{\frac{1}{(z+\Delta z)^2}-\frac{1}{z^2}}{\Delta z}=\lim_{\Delta z\to0}\frac{z^2-(z^2+2z\Delta z+(\Delta z)^2)}{z^2(z+\Delta z)^2\Delta z}=\lim_{\Delta z\to0}\frac{-2z-\Delta z}{z^2(z+\Delta z)^2}=-\frac{2}{z^3}$ (if $z\neq0$)

6. (b) $\lim_{\Delta z\to0}\frac{f(z+\Delta z)g(z+\Delta z)-f(z)g(z)}{\Delta z}=\lim_{\Delta z\to0}\frac{f(z+\Delta z)[g(z+\Delta z)-g(z)]+[f(z+\Delta z)-f(z)]g(z)}{\Delta z}$

$=\lim_{\Delta z\to0}f(z+\Delta z)\lim_{\Delta z\to0}\frac{g(z+\Delta z)-g(z)}{\Delta z}+g(z)\lim_{\Delta z\to0}\frac{f(z+\Delta z)-f(z)}{\Delta z}=f(z)g'(z)+f'(z)g(z).$

(c) $\lim_{\Delta z\to0}\frac{f(g(z+\Delta z))-f(g(z))}{\Delta z}=\lim_{\Delta z\to0}\frac{f(g+\Delta g)-f(g)}{\Delta g}\frac{\Delta g}{\Delta z}$ where $g(z+\Delta z)\equiv g+\Delta g$

$=\lim_{\Delta g\to0}\frac{f(g+\Delta g)-f(g)}{\Delta g}\lim_{\Delta z\to0}\frac{\Delta g}{\Delta z}=\underbrace{f'(g(z))}_{df/dg}\,g'(z)$

7. $\lim_{z\to z_0}f(z)=\lim_{z\to z_0}\left[\frac{f(z)-f(z_0)}{z-z_0}(z-z_0)+f(z_0)\right]=f'(z_0)(0)+f(z_0)=f(z_0)$

8. $\lim_{z\to z_0}\frac{f(z)}{g(z)}=\lim_{z\to z_0}\frac{[f(z)-f(z_0)]/(z-z_0)}{[g(z)-g(z_0)]/(z-z_0)}$ since $f(z_0)=g(z_0)=0$

$=\frac{\lim[f(z)-f(z_0)]/(z-z_0)}{\lim[g(z)-g(z_0)]/(z-z_0)}=\frac{f'(z)}{g'(z)}$

9. $u(x,y) = \begin{cases} (x^3-y^3)/(x^2+y^2), & z \neq 0 \\ 0 & , z=0 \end{cases}$ and $v(x,y) = \begin{cases} (x^3+y^3)/(x^2+y^2), & z \neq 0 \\ 0 & , z=0 \end{cases}$

$$\frac{\partial u}{\partial x}(0,0) = \lim_{\substack{x\to 0\\ y=0}} \frac{(x^3-y^3)/(x^2+y^2) - 0}{x} = 1, \quad \frac{\partial v}{\partial y}(0,0) = \lim_{\substack{y\to 0\\ x=0}} \frac{(x^3+y^3)/(x^2+y^2)-0}{y} = 1 \ \checkmark$$

and similarly for $\partial u/\partial y$ and $\partial v/\partial x$: we find that $\frac{\partial u}{\partial y}(0,0) = -1$ and $\frac{\partial v}{\partial x}(0,0) = 1$. ✓ (Note that to evaluate these derivatives we <u>must</u> use the difference quotient formula; for ex., we can't compute $\partial u/\partial x$ from $u(x,y) = (x^3-y^3)/(x^2+y^2)$ since the latter does not hold <u>at</u> $z=0$.) But, consider letting $\Delta z \to 0$ along any ray $y = \alpha x$. Then $\Delta z = x + i\alpha x = (1+i\alpha)x$, so

$$f'(z) = \lim_{x\to 0} \frac{\frac{(1-\alpha^3)x^3 + i(1+\alpha^3)x^3}{(1+\alpha^2)x^2} - 0}{(1+i\alpha)x} = \frac{(1-\alpha^3)+i(1+\alpha^3)}{(1+\alpha^2)(1+i\alpha)},$$ which is not

independent of α. For example, if $\alpha=0$ it gives $1+i$, but if $\alpha=1$ it gives $(1+i)/2$. Since the result is not unique for all possible paths of approach, f is not differentiable at $z=0$.

10. (a) $f(z) = \cos z = \cos(x+iy) = \cos x \cos iy - \sin x \sin iy = \cos x \cosh y - i \sin x \sinh y$

so $u(x,y) = \cos x \cosh y$, $v(x,y) = -\sin x \sinh y$.

$f'(z) = u_x + i v_x$, say

$\quad = -\sin x \cosh y - i \cos x \sinh y$.

We weren't asked to express the answer in terms of z, but we <u>can</u>:

$f'(z) = -(\sin x \cosh y + i \cos x \sinh y) = -\sin(x+iy) = -\sin z$.

(b) $f(z) = e^z = e^x(\cos y + i \sin y)$ so $u = e^x \cos y$, $v = e^x \sin y$

$f'(z) = u_x + i v_x = e^x \cos y + i e^x \sin y = e^x(\cos y + i \sin y) = e^z$

11. (a) $f(z) = (1-2z^3)^5$, $f'(z) = 5(-6z^2)(1-2z^3)^4 = -30z^2(1-2z^3)^4$ for all z; f is analytic for all z

(b) $f(z) = \frac{x+iy}{x^2+y^2} = \frac{z}{z\bar{z}} = \frac{1}{\bar{z}}$, but can't express it as a function of z itself, so let us use the $u+iv$ form,

$f(z) = \frac{x}{x^2+y^2} + i\frac{y}{x^2+y^2}$ so $u = x/(x^2+y^2)$ and $v = y/(x^2+y^2)$.

$f'(z) = u_x + i v_x$ [by (19)], but (19) holds only if f is indeed differentiable in the first place. Let's check that <u>first</u>:

$$u_x = \frac{1}{x^2+y^2} + \frac{x(-1)2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$ (for $z \neq 0$; at $z=0$, $f = (x+iy)/(x^2+y^2)$ does not even exist)

$$v_y = \frac{1}{x^2+y^2} + \frac{y(-1)2y}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$ (〃)

so the Cauchy-Riemann condition $u_x = v_y$ is satisfied only along the lines $y = \pm x$ (except at the origin, where f is not even defined). Next,

$u_y = \dfrac{x(-1)2y}{(x^2+y^2)^2}$ and $v_x = \dfrac{y(-1)2x}{(x^2+y^2)^2}$ so $u_y = -v_x$ only along the lines $x=0$ and $y=0$. The intersection of these lines with the lines $y = \pm x$ is only the origin. Thus, at best, f is differentiable only at the single point $z=0$ and therefore analytic nowhere. In fact, it is not differentiable at $z=0$ either since f is not even defined uniquely at $z=0$. Thus, f is differentiable nowhere, and analytic nowhere. NOTE: For generalization of this result see Exercise 14(c).

(c) $f(z) = |z| \sin z = (x^2+y^2)(\sin x \cos iy + \sin iy \cos x) = \underbrace{(x^2+y^2)\sin x \cosh y}_{u} + i\underbrace{(x^2+y^2)\sinh y \cos x}_{v}$

$u_x = [2x \sin x + (x^2+y^2)\cos x]\cosh y$

$v_y = [2y \sinh y + (x^2+y^2)\cosh y]\cos x$

so $u_x = v_y$ gives $x \sin x \cosh y = y \sinh y \cos x$ ①

$u_y = [2y \cosh y + (x^2+y^2)\sinh y]\sin x$

$v_x = [2x \cos x - (x^2+y^2)\sin x]\sinh y$

so $u_y = -v_x$ gives $y \cosh y \sin x = -x \cos x \sinh y$ ②

Solving ① and ② for y/x and equating those results gives

$$\frac{\sin x \cosh y}{\sinh y \cos x} = -\frac{\cos x \sinh y}{\cosh y \sin x}$$

or,
$$\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y = 0.$$

Since the latter is a sum of squares we need each of the two terms to be zero. Since $\cosh y \neq 0$ for all y, we need $\sin x = 0$ so $x = n\pi$. Then $\cos x \sinh y = 0$ becomes $(-1)^n \sinh y = 0$, so $y=0$. Indeed, $x = n\pi$ $(n = 0, \pm 1, ..)$ and $y=0$ does satisfy ① and ②, and u_x, u_y, v_x, v_y are all continuous at those points, so $f(z)$ is differentiable at those points only. Since it is not differentiable throughout any neighborhood of those points, f is not analytic at those points. Conclusion: f is

differentiable at $(n\pi, 0)$ for $n = 0, \pm 1, \pm 2, \ldots$,

analytic nowhere.

(d) Merely differentiating f gives $f'(z) = -\dfrac{2z+3i}{(z^2+3iz-2)^2}$, which is a unique finite number for all z's except where the denominator vanishes, namely, at $z = -i$ and $-2i$, at which points the numerator is nonzero. Thus, f is analytic for all z except at $z = -i, -2i$.

(e) $f' = \dfrac{(-1)3z^2}{(z^3+1)^2}$, except where $z^3+1=0$, namely, at $z = -1, \frac{1}{2}+\frac{\sqrt{3}}{2}i, \frac{1}{2}-\frac{\sqrt{3}}{2}i$.

(f) $u = x$ and $v = \sin y$, so $u_x = v_y$ gives $1 = \cos y$ and $u_y = -v_x$ gives $0=0$. Thus, $f(z)$ is differentiable all along the lines $y = \pm\pi/2, \pm 3\pi/2, \ldots$, and analytic nowhere.

13. (a) $f(z) = z^{100} = (x+iy)^{100}$ is too cumbersome to express in the form $u(x,y) + iv(x,y)$, so express $f(z) = r^{100} e^{i100\theta} = \underbrace{r^{100}\cos 100\theta}_{u(r,\theta)} + i\underbrace{r^{100}\sin 100\theta}_{v(r,\theta)}$

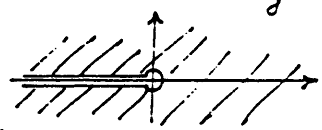
Then,

$u_r = 100 r^{99}\cos 100\theta$, $v_\theta = 100 r^{100}\cos 100\theta$

$u_\theta = -100 r^{100}\sin 100\theta$, $v_r = 100 r^{99}\sin 100\theta$.

$u, v, u_r, v_r, u_\theta, v_\theta$ are continuous everywhere, and the Cauchy-Riemann conditions (30) are satisfied everywhere, so z^{100} is analytic everywhere. (More simply, of course, $f' = 100 z^{99}$ exists, and is unique, everywhere, so f is analytic everywhere.)

(b) $f'(z) = \frac{1}{2} z^{-1/2} = 1/2\sqrt{z}$ where the $\sqrt{z}$ is defined by the branch cut in Fig. 6. Thus, f is analytic everywhere in the cut plane:



(c) As in (b), f is analytic everywhere in the cut plane.

14. (a) $f = u + iv$: $u_x = v_y$, $u_y = -v_x$

$\bar{f} = u - iv$: $u_x = -v_y$, $u_y = v_x$

Adding gives $2u_x = 0$ and $2u_y = 0$ so $u(x,y) = $ constant. Likewise, $v_x = v_y = 0$ gives $v = $ constant so, at most, f is a constant.

(b) If $f' = 0$ then (19) gives $u_x = v_x = u_y = v_y = 0$ so u and v are, at most, constants. Thus, $f(z)$ is at most a constant.

(c) Let $f = u + iv = f(z, \bar{z}) \equiv F(x,y)$

Note that $x = (z+\bar{z})/2$, $y = (z-\bar{z})/2i$.

Then
$$f_{\bar{z}} = F_x \frac{\partial x}{\partial \bar{z}} + F_y \frac{\partial y}{\partial \bar{z}} = (u_x + iv_x)\left(\frac{1}{2}\right) + (u_y + iv_y)\left(-\frac{1}{2i}\right)$$
$$= \frac{1}{2}\Big[\underbrace{(u_x - v_y)}_{0 \text{ by CR}} + i\underbrace{(v_x + u_y)}_{0 \text{ by CR}}\Big] = 0, \text{ so } f_{\bar{z}} = 0,\ f = f(z).$$

15. (a) $\nabla^2(e^x\cos y) = e^x\cos y - e^x\cos y = 0$ so u is harmonic. To find v,

$u_x = e^x\cos y = v_y$ gives $v = \int e^x\cos y\,\partial y = e^x\sin y + A(x)$

$u_y = -e^x\sin y = -v_x = -e^x\sin y - A'(x)$ gives $A'(x) = 0$, $A(x) = C$ so

$f(z) = u + iv = e^x\cos y + ie^x\sin y + C = e^z + \text{Constant}$.

(b) $\nabla^2(e^{2x}\sin 2y) = 4e^{2x}\sin 2y - 4e^{2x}\sin 2y = 0$ so u is harmonic. To find v,

$u_x = 2e^{2x}\sin 2y = v_y$ gives $v = \int 2e^{2x}\sin 2y\,\partial y = -e^{2x}\cos 2y + A(x)$,

$u_y = 2e^{2x}\cos 2y = -v_x = 2e^{2x}\cos 2y - A'(x)$ gives $A' = 0$, $A(x) = $ constant,

$f(z) = u + iv = e^{2x}\sin 2y - ie^{2x}\cos 2y + $ const.

$= -ie^{2x}(\cos 2y + i\sin 2y) + $ const. $= -ie^{2x}e^{i2y} + $ const. $= -e^{2z} + $ const

(c) $\nabla^2(x^3 - 3xy^2) = 6x - 6x = 0$ so u is harmonic. To find v,

$u_x = 3x^2 - 3y^2 = v_y$ gives $v = \int(3x^2 - 3y^2)\partial y = 3x^2y - y^3 + A(x)$,

$u_y = -6xy = -v_x = -6xy - A'(x)$ gives $A'(x) = 0$, $A(x) = $ const., so

$f(z) = u + iv = (x^3 - 3xy^2) + i(3x^2y - y^3) + \text{const.} = (x+iy)^3 + \text{const.} = z^3 + \text{const.}$

(d) $\nabla^2 u = \nabla^2(r^3 \sin 3\theta) = 6r\sin 3\theta + 3r \sin 3\theta - 9r\sin 3\theta = 0$ so, by (30),

$u_r = 3r^2 \sin 3\theta = \frac{1}{r} v_\theta$ gives $v = \int 3r^3 \sin 3\theta \, \partial\theta = -r^3 \cos 3\theta + A(r)$

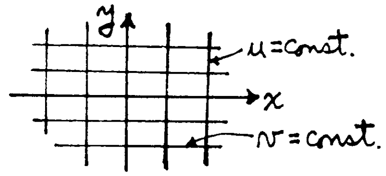
$-\frac{1}{r} u_\theta = -3r^2 \cos 3\theta = v_r = -3r^2 \cos 3\theta + A'(r)$ gives $A' = 0$, $A(r) = \text{const.}$,

$f(z) = u + iv = r^3 \sin 3\theta - ir^3 \cos 3\theta + \text{const} = -ir^3(\cos 3\theta + i \sin 3\theta) = -ir^3 e^{i3\theta}$
$= -iz^3$.

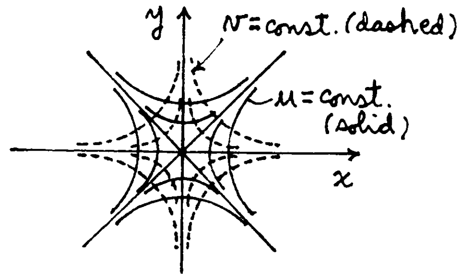
16.(a) Normals to the $u=$ const. curves are given by $\underset{\sim}{n} = \underset{\sim}{\nabla} u = u_x \hat{\imath} + u_y \hat{\jmath}$ (if $u_x^2 + u_y^2 \neq 0$)

" " " $v =$ " " " " " $\underset{\sim}{n} = \underset{\sim}{\nabla} v = v_x \hat{\imath} + v_y \hat{\jmath}$ (if $v_x^2 + v_y^2 \neq 0$)

(b) Simple: $u = x$, $v = y$, so:

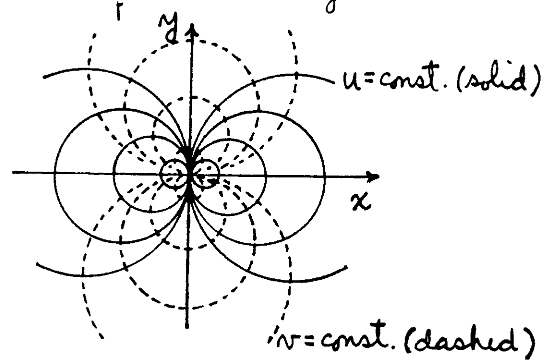


(c) $u = x^2 - y^2$, $v = 2xy$ so the $u=$ const and $v=$ const curves are hyperbolas:



Note that orthogonality does break down at $z=0$, where $f'(z) = 2z = 0$

(d) $u = x/(x^2+y^2)$ so $(x - \frac{1}{2u})^2 + y^2 = (\frac{1}{2u})^2$
$v = -y/(x^2+y^2)$ so $x^2 + (y - \frac{1}{2v})^2 = (\frac{1}{2v})^2$
so the $u=$ const and $v=$ const curves are families of circles through $z=0$:



17. $u_x = v_y \Rightarrow v(x,y) = \int_{y_0}^{y} u_x(x,y')\,\partial y' + A(x)$

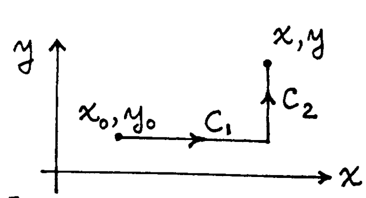
$u_y = -v_x = -\int_{y_0}^{y} u_{xx}(x,y')\,\partial y' - A'(x) = \int_{y_0}^{y} u_{y'y'}(x,y')\,\partial y' - A'(x)$ since $\nabla^2 u = 0$

or, $\cancel{u_y(x,y)} = \cancel{u_y(x,y)} - u_y(x,y_0) - A'(x)$, so $A(x) = -\int_{x_0}^{x} u_y(x',y_0)\,dx'$ and

$v(x,y) = \int_{y_0}^{y} u_x(x,y')\,\partial y' - \int_{x_0}^{x} u_y(x',y_0)\,dx'$

$= \int_{x_0}^{x} -u_y(x',y_0)\,dx' + \int_{y_0}^{y} u_x(x,y')\,\partial y'$

$= \int_{C_1 + C_2} \left[-\frac{\partial u}{\partial y}(x',y')\,dx' + \frac{\partial u}{\partial x}(x',y')\,dy' \right]$



Finally, the Cauchy-Riemann conditions imply that the vector field $-u_y\hat{\imath} + u_x\hat{\jmath}$ is irrotational so, by Theorem 16.10.1, the line integral is independent of path. That is, a unique value is obtained for any path C from x_0, y_0 to x, y, within D:

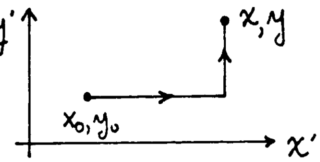
$$v(x,y) = \int_{x_0,y_0}^{x,y} \left[-\frac{\partial u}{\partial y}(x',y')\,dx' + \frac{\partial u}{\partial x}(x',y')\,dy' \right], \qquad (*)$$

which result is unique only up to an arbitrary additive constant since the initial point x_0, y_0 is arbitrary. That is, if we change x_0, y_0 in $(*)$ to x_1, y_1 then the difference between the two expressions for v is the line integral from x_0, y_0 to x_1, y_1, which is a constant.

To illustrate the use of $(*)$ let us use it to find v in Exercise 15(a).

$u(x,y) = e^x \cos y,$

$$v(x,y) = \int_{x_0,y_0}^{x,y} \left(e^{x'} \sin y'\,dx' + e^{x'} \cos y'\,dy' \right)$$



We can use the simple path shown at the right. Then

$$v(x,y) = \int_{x_0}^{x} e^{x'} \sin y_0\,dx' + 0 + 0 + \int_{y_0}^{y} e^{x} \cos y'\,dy'$$

$$= \cancel{e^x \sin y_0} - e^{x_0}\sin y_0 + e^x(\sin y - \cancel{\sin y_0}) = e^x \sin y - e^{x_0}\sin y_0$$

$$= e^x \sin y + \text{constant}.$$

18. $$df/dz = (u_x + iv_x)\frac{1}{1+iK} + (u_y + iv_y)\frac{K}{1+iK}$$

$$= (u_x + iv_x)\frac{1}{1+iK} + (-v_x + iu_x)\frac{K}{1+iK}$$

$$= \frac{u_x(1+iK) + iv_x(1+iK)}{1+iK} = u_x + iv_x$$

independent of K. However, we are still short of a proof of part (ii) of the Theorem 21.5.1 since we have not allowed for an arbitrary path of approach, only linear paths.

13. (a) $f(z)=z^{100}=(x+iy)^{100}$ is too cumbersome to express in the form $u(x,y)+iv(x,y)$, so express $f(z)=r^{100}e^{i100\theta}=\underbrace{r^{100}\cos 100\theta}_{u(r,\theta)}+i\underbrace{r^{100}\sin 100\theta}_{v(r,\theta)}$

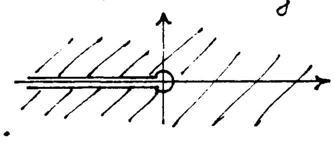
Then,

$u_r=100r^{99}\cos 100\theta$, $v_\theta=100r^{100}\cos 100\theta$

$u_\theta=-100r^{100}\sin 100\theta$, $v_r=100r^{99}\sin 100\theta$.

$u, v, u_r, v_r, u_\theta, v_\theta$ are continuous everywhere, and the Cauchy-Riemann conditions (30) are satisfied everywhere, so z^{100} is analytic everywhere. (More simply, of course, $f'=100z^{99}$ exists, and is unique, everywhere, so f is analytic everywhere.)

(b) $f'(z)=\frac{1}{2}z^{-1/2}=1/2\sqrt{z}$ where the $\sqrt{z}$ is defined by the branch cut in Fig. 6. Thus, f is analytic everywhere in the cut plane:



(c) As in (b), f is analytic everywhere in the cut plane.

14. (a) $f=u+iv$: $u_x=v_y,\ u_y=-v_x$

$\bar{f}=u-iv$: $\underline{u_x=-v_y,\ u_y=v_x}$

adding gives $2u_x=0$ and $2u_y=0$ so $u(x,y)=$ constant. Likewise, $v_x=v_y=0$ gives $v=$ constant so, at most, f is a constant.

(b) If $f'=0$ then (19) gives $u_x=v_x=u_y=v_y=0$ so u and v are, at most, constants. Thus, $f(z)$ is at most a constant.

(c)
$$\left.\begin{aligned} u_x&=u_z z_x+u_{\bar z}\bar z_x=u_z+u_{\bar z}\\ u_y&=u_z z_y+u_{\bar z}\bar z_y=iu_z-iu_{\bar z}\\ v_x&=v_z z_x+v_{\bar z}\bar z_x=v_z+v_{\bar z}\\ v_y&=v_z z_y+v_{\bar z}\bar z_y=iv_z-iv_{\bar z}\end{aligned}\right]$$
so the Cauchy-Riemann conditions give
$$u_z+u_{\bar z}=iv_z-iv_{\bar z} \quad ①$$
$$iu_z-iu_{\bar z}=-v_z-v_{\bar z} \quad ②$$

① + i times ② gives $2u_{\bar z}=-2iv_{\bar z}$ or, $u_{\bar z}+iv_{\bar z}=0$ or, $f_{\bar z}=0$.

15. (a) $\nabla^2(e^x\cos y)=e^x\cos y-e^x\cos y=0$ so u is harmonic. To find v,

$u_x=e^x\cos y=v_y$ gives $v=\int e^x\cos y\,\partial y=e^x\sin y+A(x)$

$u_y=-e^x\sin y=-v_x=-e^x\sin y-A'(x)$ gives $A'(x)=0$, $A(x)=C$ so

$f(z)=u+iv=e^x\cos y+ie^x\sin y+C=e^z+$ Constant.

(b) $\nabla^2(e^{2x}\sin 2y)=4e^{2x}\sin 2y-4e^{2x}\sin 2y=0$ so u is harmonic. To find v,

$u_x=2e^{2x}\sin 2y=v_y$ gives $v=\int 2e^{2x}\sin 2y\,\partial y=-e^{2x}\cos 2y+A(x)$,

$u_y=2e^{2x}\cos 2y=-v_x=2e^{2x}\cos 2y-A'(x)$ gives $A'=0$, $A(x)=$ constant,

$f(z)=u+iv=e^{2x}\sin 2y-ie^{2x}\cos 2y+$ const.

$=-ie^{2x}(\cos 2y+i\sin 2y)+$ const. $=-ie^{2x}e^{i2y}+$ const. $=-e^{2z}+$ const

(c) $\nabla^2(x^3-3xy^2)=6x-6x=0$ so u is harmonic. To find v,

$u_x=3x^2-3y^2=v_y$ gives $v=\int(3x^2-3y^2)\partial y=3x^2y-y^3+A(x)$,

$u_y=-6xy=-v_x=-6xy-A'(x)$ gives $A'(x)=0$, $A(x)=$ const., so