CHAPTER 18

Section 18.2

2. (b) $L[Au+Bv]-AL[u]-BL[v]$
$$= (Au+Bv)_t + \alpha(Au+Bv)(Au_x+Bv_x) + \beta(Au+Bv)_{xxx} - A(u_t+\alpha uu_x+\beta u_{xxx}) - B(v_t+\alpha vv_x+\beta v_{xxx})$$
$$= \alpha(A^2-A)uu_x + \alpha(B^2-B)vv_x + \alpha AB(uv)_x$$
is not identically zero for all constants A,B and functions u,v; e.g., if $A=0, B=2, u=8, v=x$, then it $= 0+\alpha(4-2)x = 2\alpha x \neq 0$. Thus, L is nonlinear. (We used A,B rather than α,β because of the α,β in the PDE.)

(c) linear

(d) $L[\alpha u+\beta v]-\alpha L[u]-\beta L[v]$
$$= (\cancel{\alpha u}+\cancel{\beta v})_{xx} + x(\cancel{\alpha u}+\cancel{\beta v})_{yy} - \alpha(\cancel{u_{xx}}+\cancel{xu_{yy}}) - \beta(\cancel{v_{xx}}+\cancel{xv_{yy}}) = 0;$$ linear

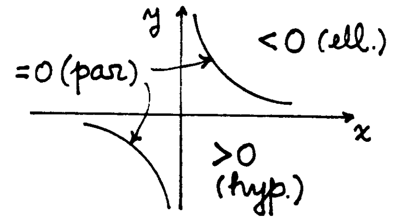
(e) nonlinear (due to the e^u term)

(f) linear (g) linear NOTE: $L = \partial^2/\partial x^2 + 5\partial^2/\partial x\partial y - x$ does not include the e^x.

(h) nonlinear (due to the uu_y term). Let's show it:
$L[\alpha u+\beta v]-\alpha L[u]-\beta L[v]$
$$= x(\cancel{\alpha u}+\cancel{\beta v})_x + (\alpha u+\beta v)(\alpha u+\beta v)_y - \alpha(\cancel{xu_x}+uu_y) - \beta(\cancel{xv_x}+vv_y)$$
$$= (\alpha^2-\alpha)uu_y + (\beta^2-\beta)vv_y + \alpha\beta(uv)_y$$
is not identically zero for all constants α,β and functions u,v; e.g. if $\alpha=0, \beta=3, u=\sin x, v=y^2$, it $= 0+6y^2\cdot 2y+0 = 12y^3 \neq 0$. Thus, L is nonlinear.

3. (a) $A=1, B=1/2, C=0$, so $B^2-AC = 1/4 > 0$, so hyperbolic (everywhere in the x,y plane)

(b) $A=x, B=-1/2, C=y$ so $B^2-AC = \frac{1}{4}-xy$ so elliptic in the two disjoint regions shown, hyperbolic in between, and parabolic on the hyperbolas $xy=1/4$.



(c) $A=C=0, B=1/2$, so $B^2-AC = 1/4 > 0$, so hyperbolic everywhere

(d) $A=x, B=0, C=-(\sin^2 y+1)$, so $B^2-AC = x(\sin^2 y+1)$, so elliptic in the left half-plane ($x<0$), hyperbolic in the right half-plane ($x>0$), and parabolic on the y axis ($x=0$).

(e) $A=1, B=1/2, C=1$, so $B^2-AC = -3/4 < 0$ so elliptic everywhere

(f) $A=1, B=0, C=\cos x$, so $B^2-AC = -\cos x$, so elliptic in the strips $\pi/2<x<3\pi/2$, $-3\pi/2<x<-\pi/2$, $5\pi/2<x<7\pi/2$, $-7\pi/2<x<-5\pi/2, \ldots$, hyperbolic in the strips $-\pi/2<x<\pi/2$, $-5\pi/2<x<-3\pi/2$, $3\pi/2<x<5\pi/2, \ldots$, and parabolic along the lines $x = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots$

(g) $A=1, B=0, C=0, B^2-AC=0$, parabolic everywhere

(h) $A=0, B=1/2, C=-1, B^2-AC=1/4>0$, hyperbolic everywhere

4. $$kAu_x\big|_{x+\Delta x} - kAu_x\big|_x = \frac{\partial}{\partial t}(A\Delta x\sigma c u),$$

$$k\frac{(Au_x)|_{x+\Delta x}-(Au_x)|_x}{\Delta x} = A\sigma c u_t, \quad \Delta x\to 0 \longrightarrow \frac{1}{A(x)}[A(x)u_x]_x = \sigma c u_t.$$

Section 18.3

2.(a) $\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = +\kappa^2$ gives $X''-\kappa^2X=0$, $X = \begin{cases} A\cosh\kappa x + B\sinh\kappa x, & \kappa\neq 0 \\ C+Dx, & \kappa=0 \end{cases}$

$T'-\kappa^2\alpha^2T=0$, $T = \begin{cases} E\exp(\kappa^2\alpha^2 t), & \kappa\neq 0 \\ F, & \kappa=0 \end{cases}$

$u = (C+Dx)F + (A\cosh\kappa x + B\sinh\kappa x)Ee^{\kappa^2\alpha^2 t} = C'+D'x+(A'\cosh\kappa x + B'\sinh\kappa x)e^{\kappa^2\alpha^2 t}$

$u(0,t) = u_1 = C' + A'\exp(\kappa^2\alpha^2 t) \Rightarrow C' = u_1, A'=0$

so $u(x,t) = u_1 + D'x + B'\sinh\kappa x\,\exp(\kappa^2\alpha^2 t)$

$u(L,t) = u_2 = u_1 + D'L + B'\sinh\kappa L\,\exp(\,\text{''}\,) \Rightarrow D' = (u_2-u_1)/L$, and $B'\sinh\kappa L = 0$.

Of the choices $B'=0$ and $\sinh\kappa L=0$ we choose the latter:

$\sinh\kappa L = \frac{1}{i}\sin i\kappa L = -i\sin i\kappa L = 0 \Rightarrow i\kappa L = n\pi \quad (n=1,2,\ldots)$

so $\kappa = -n\pi i/L$ or, equivalently, $\kappa = n\pi i/L$ since κ appears originally as κ^2, so we can never distinguish between $\pm$ values. Okay, $\kappa = n\pi i/L$ gives

$$u(x,t) = u_1 + (u_2-u_1)\frac{x}{L} + B'\sinh\left(i\frac{n\pi x}{L}\right)e^{-(n\pi\alpha/L)^2 t}$$
$$= u_1 + (u_2-u_1)\frac{x}{L} + iB'\sin\frac{n\pi x}{L}\,e^{-(n\pi\alpha/L)^2 t}$$

or, renaming iB' as G', say, and using superposition,

$$u(x,t) = u_1 + (u_2-u_1)\frac{x}{L} + \sum_1^\infty G_n'\sin\frac{n\pi x}{L}\exp[-(n\pi\alpha/L)^2 t],$$

which is the same as (22).

(b) Using $-\kappa^2$ in (6), as we did, the relevant St.-Liou. problem is

$$X''+\kappa^2X=0 \quad (0<x<L)$$
$$X(0)=0, \quad X(L)=0,$$

as noted in Example 3. Thus, $p(x)=1$, $q(x)=0$, $w(x)=1$, κ^2 is λ. Then $q(x)$ is ≤ 0 on $[0,L]$ and $[p(x)\phi_n(x)\phi_n'(x)]\big|_0^L = 0$ (because the ϕ_n's are 0 at 0 and L) is ≤ 0. Hence, by Theorem 17.7.2, $\lambda_n = \kappa_n^2 \geq 0$ so that κ_n^2 must be nonnegative and we see that our use of $-\kappa^2$ in (6) is justified.

3. Here are the only conditions under which the graph of $u(x,t)$, plotted versus x, does not change its shape (although its magnitude might vary with time):
   (i) If $f(x) = u_1 + (u_2 - u_1)x/L$ then $F(x) = 0$ in (28), so the solution simply remains a constant with time, namely, $u(x,t) = u_1 + (u_2 - u_1)x/L$.
   (ii) If $u_1 = u_2$ and $f(x)$ is of the form $C \sin n\pi x/L$ for some constant C and some integer n, then the solution is the single term
   $$u(x,t) = C \sin \frac{n\pi x}{L} \exp[-(n\pi\alpha/L)^2 t],$$
   which is of product form. Its shape is $C \sin \frac{n\pi x}{L}$, modulated in amplitude by the $\exp[-(n\pi\alpha/L)^2 t]$ factor.

4. (b) $u = XT$ gives $\frac{X'' + 2X'}{X} = \frac{T'}{T} = -k^2$, $X'' + 2X' + k^2 X = 0$, $T' + k^2 T = 0$.
Seeking $X = e^{\lambda x}$ gives $\lambda^2 + 2\lambda + k^2 = 0$, $\lambda = (-2 \pm \sqrt{4 - 4k^2})/2 = -1 \pm \sqrt{1-k^2}$
so we obtain distinct roots and hence the general solution — provided that $k \neq 1$;
if $k = 1$ then $\lambda = -1, -1$ and the solutions are e^{-x} and $\underline{x}e^{-x}$. Thus,
$$X(x) = \begin{cases} A e^{(-1+\sqrt{1-k^2})x} + B e^{(-1-\sqrt{1-k^2})x}, & k \neq 1 \\ (C + Dx)e^{-x}, & k = 1 \end{cases}$$
and
$$T(t) = \begin{cases} E e^{-k^2 t}, & k \neq 1 \\ F e^{-t}, & k = 1 \end{cases}$$
so we can form
$$u(x,t) = (C + Dx)e^{-x} F e^{-t} + e^{-x}(A e^{\sqrt{1-k^2}x} + B e^{-\sqrt{1-k^2}x}) E e^{-k^2 t}$$
$$= (C' + D'x) e^{-(x+t)} + e^{-x}\underbrace{(A' e^{\sqrt{1-k^2}x} + B' e^{-\sqrt{1-k^2}x})}_{*} e^{-k^2 t}$$
NOTE: Of course, we could use the form $A'' \cosh \sqrt{1-k^2}x + B'' \sinh \sqrt{1-k^2}x$ in place of $*$ if we wish. Further, note that the latter form is fine if k^2 turns out to be smaller than 1, but if it is greater than 1 then we are well-advised to re-express
$$A'' \cosh \sqrt{1-k^2}x + B'' \sinh \sqrt{1-k^2}x = A'' \cosh i\sqrt{k^2-1}x + B'' \sinh i\sqrt{k^2-1}x$$
$$= A'' \cos \sqrt{k^2-1}x + iB'' \sin \sqrt{k^2-1}x$$
$$= A'' \cos \sqrt{k^2-1}x + B''' \sin \sqrt{k^2-1}x$$
Anticipating an eventual Fourier (or, more generally, eigenfunction) expansion it is probably best to use the cosine and sine version rather than the cosh, sinh version.

(d) $u = XT$ gives $X''/X + 2X'T'/XT = T''/T$. Can't be separated due to the $X'T'/XT$ term.

NOTE: The following problem might be useful for lecture: $u_{xx} + 2u_x = u_{tt}$. Solution:

$u = XT$ gives $\frac{X'' + 2X'}{X} = \frac{T''}{T} = -k^2$, $X'' + 2X' + k^2 X = 0$, $T'' + k^2 T = 0$.
Proceeding as in (b), above, write
$$X(x) = \begin{cases} e^{-x}(A \cos \sqrt{k^2-1}x + B \sin \sqrt{k^2-1}x), & k \neq 1 \\ (C + Dx)e^{-x}, & k = 1 \end{cases}$$
Then, $T'' + k^2 T = 0$ gives $T(t) = E \cos kt + F \sin kt$. The latter is the general solution for all $k \neq 0$, since the sine term drops out if $k = 0$. The X solution dictated distinguishing the cases $k \neq 1$, $k = 1$ and the T solution dictates also distinguishing the case $k = 0$. Thus, we have

$$X(x)=\begin{cases} e^{-x}(A\cos\sqrt{k^2-1}\,x+B\sin\sqrt{k^2-1}\,x) \\ (C+Dx)e^{-x} \\ E+Fe^{-2x} \end{cases} \qquad T(t)=\begin{cases} G\cos kt+H\sin kt, & k\neq 0,1 \\ I\cos t+J\sin t, & k=1 \\ L+Mt, & k=0 \end{cases}$$

so

$$u(x,t)=(E+Fe^{-2x})(L+Mt)+(C+Dx)e^{-x}(I\cos t+J\sin t)$$
$$+e^{-x}(A\cos\sqrt{k^2-1}\,x+B\sin\sqrt{k^2-1}\,x)(G\cos kt+H\sin kt)$$

5. No, this is a serious error. We can superimpose various solutions of the same (linear) ODE, but here we would be superimposing solutions of different ODE's. Namely, $A\cos kx+B\sin kx$ is a solution of $X''+k^2X=0$ for $k\neq 0$, and $D+Ex$ is a solution of $X''=0$ (i.e., for $k=0$); these are different ODE's! If not convinced, put $(A\cos kx+B\sin kx+D+Ex)(Fe^{-k^2\alpha^2t}+G)$ into $\alpha^2u_{xx}=u_t$ and you will see that it does not work.

NOTE: This error is a common one, and is similar to the error in saying that if the eigenvalue problem $Ax=\lambda x$ has eigenpairs λ_1, e_1 and λ_2, e_2 then the solution of $Ax=\lambda x$ is $x=C_1e_1+C_2e_2$.

6. (b) $u(x,t)=A+Bx+(C\cos kx+D\sin kx)e^{-k^2\alpha^2t}$

$u(0,t)=10=A+Ce^{-k^2\alpha^2t} \rightarrow A=10, C=0$ so

$u(x,t)=10+Bx+D\sin kx\; e^{-k^2\alpha^2t}$

$u_x(2,t)=-5=B+kD\cos 2k\; e^{-k^2\alpha^2t} \rightarrow B=-5,\; 2k=n\pi/2\; (n=1,3,\dots)$ so

$u(x,t)=10-5x+\sum_{1,3,\dots}^{\infty} D_n\sin\frac{n\pi x}{4}\exp[-(n\pi\alpha/4)^2t]$ ①

$u(x,0)=f(x)=\cancel{10}=\cancel{10}-5x+\sum_{1,3,\dots}^{\infty} D_n\sin\frac{n\pi x}{4}$

or,

$5x=\sum_{1,3,\dots}^{\infty} D_n\sin\frac{n\pi x}{4}\quad (0<x<2)$ Here, $L=2$.

QRS: $D_n=\frac{2}{2}\int_0^2 5x\sin\frac{n\pi x}{4}\,dx=\frac{40}{n^2\pi^2}\left(2\sin\frac{n\pi}{2}-n\pi\cos\frac{n\pi}{2}\right)$ ②

Solution given by ① and ②. $u_s(x)=10-5x$.

(c) $u(x,t)=A+Bx+(C\cos kx+D\sin kx)e^{-k^2\alpha^2t}$

$u(0,t)=0=A+C\exp(-k^2\alpha^2t) \rightarrow A=C=0$ so

$u(x,t)=Bx+D\sin kx\,\exp(-k^2\alpha^2t)$

$u_x(2,t)=0=B+kD\cos 2k\,\exp(\;''\;) \rightarrow B=0,\; 2k=n\pi/2\; (n=1,3,\dots)$ so

$u(x,t)=\sum_{1,3,\dots}^{\infty} D_n\sin\frac{n\pi x}{4}\exp[-(n\pi\alpha/4)^2t]$ ①

$u(x,0)=f(x)=50\sin\frac{\pi x}{2}=\sum_{1,3,\dots}^{\infty} D_n\sin\frac{n\pi x}{4}\quad (0<x<2)\quad (L=2)$

QRS: $D_n=\frac{2}{2}\int_0^2 50\sin\frac{\pi x}{2}\sin\frac{n\pi x}{4}\,dx=-\frac{400}{\pi}\frac{\sin\frac{n\pi}{2}}{n^2-4}$ ②

Solution given by ① and ②. $u_s(x) = 0$.

(e) $u(x,t) = A + Bx + (C\cos\kappa x + D\sin\kappa x)e^{-\kappa^2\alpha^2 t}$

$u(0,t) = 25 = A + C\exp(-\kappa^2\alpha^2 t) \rightarrow A = 25,\ C = 0$ so

$u(x,t) = 25 + Bx + D\sin\kappa x \exp(-\kappa^2\alpha^2 t)$

$u_x(4,t) = 0 = B + \kappa D\cos 4\kappa \exp(\ \text{"}\ ) \rightarrow B = 0,\ 4\kappa = n\pi/2\ (n = 1,3,\dots)$ so

$u(x,t) = 25 + \sum_{1,3,\dots}^{\infty} D_n \sin\frac{n\pi x}{8} \exp[-(n\pi\alpha/8)^2 t]$ ①

$u(x,0) = 25 = 25 + \sum_{1,3,\dots}^{\infty} D_n \sin\frac{n\pi x}{8}$ or, $0 = \sum_{1,3,\dots}^{\infty} D_n \sin\frac{n\pi x}{8}$

QRS: D_n's $= 0$ by inspection! Thus,
$u(x,t) = 25$

(With hindsight, this was "obvious.") $u_s(x) = 25$.

(f) $u(x,t) = A + Bx + (C\cos\kappa x + D\sin\kappa x)e^{-\kappa^2\alpha^2 t}$

$u(0,t) = 25 = A + C\exp(-\kappa^2\alpha^2 t) \rightarrow A = 25,\ C = 0$ so, updating our solution,*

$u(x,t) = 25 + Bx + D\sin\kappa x \exp(-\kappa^2\alpha^2 t)$

$u_x(2,t) = 0 = B + \kappa D\cos 2\kappa \exp(\ \text{"}\ ) \rightarrow B = 0,\ 2\kappa = n\pi/2\ (n = 1,3,\dots)$ so

$u(x,t) = 25 + \sum_{1,3,\dots}^{\infty} D_n \sin\frac{n\pi x}{4} \exp[-(n\pi\alpha/4)^2 t]$ ①

$u(x,0) = f(x) = 25 + \sum_{1,3,\dots}^{\infty} D_n \sin\frac{n\pi x}{4}$ or, $f(x) - 25 = \sum_{1,3,\dots}^{\infty} D_n \sin\frac{n\pi x}{4}\quad (0 < x < 2)$

QRS: $D_n = \frac{2}{2}\int_0^2 [f(x) - 25]\sin\frac{n\pi x}{4}dx = \int_0^1 -25\sin\frac{n\pi x}{4}dx + \int_1^2 0\,dx = -\frac{100}{n\pi}\left(1 - \cos\frac{n\pi}{4}\right)$ ②

Solution given by ① and ②. $u_s(x) = 25$.

*** NOTE: As a procedural matter, we recommend "updating" the solution before moving on to the next boundary or initial condition. Also, it is helpful to write the arguments: for example, u(0,t) rather than just u, so we do not mistake a boundary conditon u(0,t) = etc. for the solution u(x,t) = etc.**

(h) $u(x,t) = A + Bx + (C\cos\kappa x + D\sin\kappa x)e^{-\kappa^2\alpha^2 t}$

$u_x(0,t) = 0 = B + \kappa D\exp(-\kappa^2\alpha^2 t) \rightarrow B = D = 0$ so

$u(x,t) = A + C\cos\kappa x \exp(-\kappa^2\alpha^2 t)$

$u_x(3\pi,t) = 0 = -\kappa C\sin 3\pi\kappa \exp(\ \text{"}\ ) \rightarrow 3\pi\kappa = n\pi\ (n = 1,2,\dots)$ so

$u(x,t) = A + \sum_1^{\infty} C_n \cos\frac{nx}{3} \exp[-(n\pi\alpha/3)^2 t]$ ①

$u(x,0) = f(x) = A + \sum_1^{\infty} C_n \cos\frac{nx}{3}\quad (0 < x < 3\pi)\ (L = 3\pi)$

HRC:

$A = \frac{1}{3\pi}\int_0^{3\pi} f\,dx = \frac{1}{3\pi}\int_{2\pi}^{3\pi} 60\,dx = 20,\ C_n = \frac{2}{3\pi}\int_0^{3\pi} f\cos\frac{nx}{3}dx = -\frac{120}{n\pi}\sin\frac{2n\pi}{3}$ ②

Solution given by ① and ②. $u_s(x) = 20$.

(i) $u(x,t) = A + Bx + (C\cos kx + D\sin kx)e^{-k^2\alpha^2 t}$
$u_x(0,t) = 5 = B + kD\exp(-k^2\alpha^2 t) \to B = 5, D = 0$ so
$u(x,t) = A + 5x + C\cos kx \exp(\text{''})$
$u_x(10,t) = 5 = 5 - kC\sin 10k \exp(\text{''}) \to 10k = n\pi \quad (n = 1,2,\dots)$ so
$u(x,t) = A + 5x + \sum_1^\infty C_n \cos\frac{n\pi x}{10}\exp[-(n\pi\alpha/10)^2 t]$

$u(x,0) = f(x) = 45 + \cancel{5x} = A + \cancel{5x} + \sum_1^\infty C_n\cos\frac{n\pi x}{10}$
$45 = A + \sum_1^\infty C_n \cos\frac{n\pi x}{10} \quad (0 < x < 10)$
HRC: By inspection (or by the integral formulas) $A = 45$, $C_n = 0$, so
$u(x,t) = 45 + 5x$

(j) $u(x,t) = A + Bx + (C\cos kx + D\sin kx)e^{-k^2\alpha^2 t}$
$u_x(0,t) = 3 = B + kD\exp(-k^2\alpha^2 t) \to B = 3, D = 0$ so
$u(x,t) = A + 3x + C\cos kx \exp(\text{''})$
$u_x(5,t) = \cancel{3} = \cancel{3} - kC\sin 5k \exp(\text{''}) \to 5k = n\pi \quad (n = 1,2,\dots)$ so
$u(x,t) = A + 3x + \sum_1^\infty C_n\cos\frac{n\pi x}{5}\exp[-(n\pi\alpha/5)^2 t]$ ①

$u(x,0) = 2x = A + 3x + \sum_1^\infty C_n \cos\frac{n\pi x}{5}$
or, $-x = A + \sum_1^\infty C_n\cos\frac{n\pi x}{5} \quad (0 < x < 5) \quad (L = 5)$
HRC: $A = \frac{1}{5}\int_0^5 (-x)\,dx = -5/2$, $C_n = \frac{2}{5}\int_0^5 (-5x)\cos\frac{n\pi x}{5}\,dx = \begin{cases} 0, & n = 2,4,\dots \\ \frac{100}{n^2\pi^2}, & n = 1,3,\dots \end{cases}$ ②
so ① and ② give $u(x,t) = -\frac{5}{2} + 3x + \frac{100}{\pi^2}\sum_{1,3,\dots}^\infty \frac{1}{n^2}\cos\frac{n\pi x}{5}\exp[-(n\pi\alpha/5)^2 t]$,
$u_s(x) = -\frac{5}{2} + 3x$.

(k) $u(x,t) = A + Bx + (C\cos kx + D\sin kx)e^{-k^2\alpha^2 t}$
$u(0,t) = 0 = A + C\exp(-k^2\alpha^2 t) \to A = C = 0$ so
$u(x,t) = Bx + D\sin kx \exp(\text{''})$
$u(5,t) = 0 = 5B + D\sin 5k \exp(\text{''}) \to B = 0, 5k = n\pi \quad (n = 1,2,\dots)$ so
$u(x,t) = \sum_1^\infty D_n \sin\frac{n\pi x}{5}\exp[-(n\pi\alpha/5)^2 t]$ ①

$u(x,0) = \sin\pi x - 37\sin\frac{\pi x}{5} + 6\sin\frac{9\pi x}{5} = \sum_1^\infty D_n \sin\frac{n\pi x}{5} \quad (0 < x < 5)$
HRS:
By inspection, $D_1 = -37$, $D_5 = 1$, $D_9 = 6$, all other D_n's $= 0$, so
$u(x,t) = -37\sin\frac{\pi x}{5}\exp[-(\pi\alpha/5)^2 t] + \sin\pi x\exp[-(\pi\alpha)^2 t]$
$+ 6\sin\frac{9\pi x}{5}\exp[-(9\pi\alpha/5)^2 t]$

and $u_s(x) = 0$.

(l) $u(x,t) = A + Bx + (C\cos kx + D\sin kx)e^{-k^2\alpha^2 t}$

$u(0,t) = 0 = A + C\exp(-k^2\alpha^2 t) \rightarrow A = C = 0$ so

$u(10,t) = 100 = 10B + D\sin 10k\exp(-k^2\alpha^2 t) \rightarrow B = 10,\ 10k = n\pi\ (n=1,2,\ldots)$ so

$u(x,t) = 10x + \sum_1^\infty D_n \sin\frac{n\pi x}{10}\exp[-(n\pi\alpha/10)^2 t]$ ①

$u(x,0) = 0 = 10x + \sum_1^\infty D_n\sin\frac{n\pi x}{10}$ or, $-10x = \sum_1^\infty D_n \sin\frac{n\pi x}{10}\quad (0<x<10)$

HRS:

$$D_n = \frac{2}{10}\int_0^{10}(-10x)\sin\frac{n\pi x}{10}\,dx = \frac{200}{n\pi}(-1)^n \quad ②$$

Solution given by (1) and (2). $u_s(x) = 10x$.

(m) $u(x,t) = A + Bx + (C\cos kx + D\sin kx)e^{-k^2\alpha^2 t}$

$u_x(0,t) = 2 = B + kD\exp(-k^2\alpha^2 t) \rightarrow B = 2, D = 0$ so

$u(x,t) = A + 2x + C\cos kx\exp(\text{''})$

$u(6,t) = 12 = A + 12 + C\cos 6k\exp(\text{''}) \rightarrow A = 0,\ 6k = n\pi/2\ (n = 1,3,\ldots)$ so

$u(x,t) = 2x + \sum_{1,3,\ldots}^\infty C_n\cos\frac{n\pi x}{12}\exp(\text{''})$ ①

$u(x,0) = 0 = 2x + \sum_{1,3,\ldots}^\infty C_n\cos\frac{n\pi x}{12}\quad (0<x<6)$

or,

$$-2x = \sum_{1,3,\ldots}^\infty C_n\cos\frac{n\pi x}{12}.\quad (0<x<6)\quad \#$$

NOTE: As usual, we move any known terms on the right-hand side of #, namely, the 2x term, to the left, and then seek to identify the series as HRC, HRS, QRC, or QRS. It helps to write, to the right of the equation, the interval on which the expansion is to hold (in this case $0 < x < 6$) since then we can see the $\cos(n\pi x/12)$ tearm as being of the form $\cos(n\pi x/2L)$. That fact, together with the absence of a constant term and the fact that the series is over n = 1,3,... tell us that the series is a QRC series.

QRC: $$C_n = \frac{2}{6}\int_0^6(-2x)\cos\frac{n\pi x}{12}\,dx = \frac{48}{n^2\pi^2}\left(2 - n\pi\sin\frac{n\pi}{2}\right)\quad ②$$

Solution given by ① and ②. $u_s(x) = 2x$.

(n) $u(x,t) = A + Bx + (C\cos kx + D\sin kx)e^{-k^2\alpha^2 t}$

$u_x(0,t) = 0 = B + kD\exp(-k^2\alpha^2 t) \rightarrow B = D = 0$ so

$u(x,t) = A + C\cos kx\exp(\text{''})$

$u(6,t) = 0 = A + C\cos 6k\exp(\text{''}) \rightarrow A = 0,\ 6k = n\pi/2\ (n=1,3,\ldots)$ so

$u(x,t) = \sum_{1,3,\ldots}^\infty C_n\cos\frac{n\pi x}{12}\exp(\text{''})$ ①

$u(x,0) = \sin x = \sum_{1,3,\ldots}^\infty C_n\cos\frac{n\pi x}{12}\quad (0<x<6)$

QRC: $C_n = \frac{2}{6}\int_0^6 \sin x \cos\frac{n\pi x}{12}\,dx = 4\,\frac{n\pi \sin 6 \sin\frac{n\pi}{2} - 12}{n^2\pi^2 - 144}$ ②

Solution given by ① and ②. $u_S(x) = 0$.

8. $K_n = \frac{2}{L}\int_0^L 40\sin\frac{\pi x}{L}\sin\frac{n\pi x}{L}\,dx = -80\,\frac{\sin n\pi}{n^2-1} = 0$ because $\sin n\pi = 0$. However, for $n=1$ it is 0/0 and hence indeterminate. L'Hôpital's rule gives $K_1 = -80 \lim_{n\to 1}\frac{\sin n\pi}{n^2-1} = -80\lim_{n\to 1}\frac{\pi\cos n\pi}{2n} = (-80)(-\frac{1}{2}) = 40$. Alternatively, we could work out K_1 separately: $K_1 = \frac{80}{L}\int_0^L \sin^2\frac{\pi x}{L}\,dx = \frac{80}{L}\frac{L}{2} = 40$.

9. (a) We obtain $u(x,t) = \sum_1^\infty D_n \sin\frac{n\pi x}{2}\, e^{-(n\pi\alpha/2)^2 t}$ ①

$u(x,0) = \sum_1^\infty D_n \sin\frac{n\pi x}{2} \quad (0<x<2)$

HRS:

$$D_n = \frac{2}{2}\int_0^2 u(x,0)\sin\frac{n\pi x}{2}\,dx = \int_0^1 50x\sin\frac{n\pi x}{2}\,dx + \int_1^2 (100-5x)\sin\frac{n\pi x}{2}\,dx$$
$$= -\frac{100}{n^2\pi^2}\left(n\pi\cos\frac{n\pi}{2} - 2\sin\frac{n\pi}{2}\right) - \frac{10}{n^2\pi^2}\left[18(-1)^n n\pi - 19n\pi\cos\frac{n\pi}{2} - 2\sin\frac{n\pi}{2}\right]$$
$$= -\frac{100}{n^2\pi^2}\left[1.8(-1)^n n\pi - 0.9n\pi\cos\frac{n\pi}{2} - 2.2\sin\frac{n\pi}{2}\right]$$

so $u(x,t) = -\frac{100}{\pi^2}\sum_1^\infty \frac{1}{n^2}\left[1.8(-1)^n n\pi - 0.9n\pi\cos\frac{n\pi}{2} - 2.2\sin\frac{n\pi}{2}\right]\sin\frac{n\pi x}{2}\, e^{-(n\pi\alpha/2)^2 t}$

(b) With $\alpha^2 = 2.9\times 10^{-5}$,

$u(1,3600) = -\frac{100}{\pi^2}\sum_1^\infty \frac{1}{n^2}\left[1.8(-1)^n n\pi - 0.9n\pi\cos\frac{n\pi}{2} - 2.2\sin\frac{n\pi}{2}\right]\sin\frac{n\pi}{2}\, e^{-(n\pi/2)^2(0.1044)}$

The Maple commands

```
S:= sum((1/i^2)*(1.8*(-1)^i*i*Pi-.9*i*Pi*cos(i*Pi/2)-2.2sin(i*Pi/2))
     *sin(i*Pi/2)*exp(-.1044*(i*Pi/2)^2), i=1..1):
u:= -100*S/Pi^2;
```

gives $u(1,3600) = 61.51290869$.

Changing $i=1..1$ to $i=1..3$ gives 59.87675495

" " " $i=1..5$ " 59.89647346

" " " $i=1..10$ " 59.89644798

and further increase of the upper limit of summation gives (to this many decimal places) no further change. (Of course, in most applications we don't need this many correct significant figures.)

(c) We wish to solve

$$u(1,t) = 5 = -\frac{100}{\pi^2}\sum_1^\infty \frac{1}{n^2}\left[1.8(-1)^n n\pi - \overset{*}{0.9n\pi\cos\frac{n\pi}{2}} - 2.2\sin\frac{n\pi}{2}\right]\overset{♀}{\sin\frac{n\pi}{2}} \times \exp\left[-(n\pi/2)^2(0.000029)t\right]$$

for t. Actually, the * term can be omitted since the ♀ term is nonzero only for n odd, and if n is odd then the $\cos n\pi/2$ is 0.

To solve, use the Maple commands

```
u: = -(100/Pi^2)*sum((1/i^2)*(1.8*(-1)^i*i*Pi-2.2*sin(i*Pi/2))
        *sin(i*Pi/2)* exp(-(i*Pi/2)^2*.000029*t, i=1..1):
fsolve(u=5,t);
```

and obtain $\quad t = 38675.42518$ seconds $\quad (\simeq 10.74$ hrs.)

To see how many significant figures can be believed, let us change $i=1..1$ to $i=1..3$ (which sums the first $\underline{3}$ terms — the 2nd term being 0 due to the $\sin n\pi/2$ factor). In that case we obtain $t=-2642$, which is obviously incorrect. To provide some help, include a search interval option in fsolve, such as fsolve(u=5,t, t=0..50000); and obtain

$$t = 38675.42517.$$

Evidently we already have 10 significant figure accuracy, the reason being that the $\exp[-(n\pi/2)^2(0.000029)t]$ factor causes faster and faster convergence as t increases.

(d) $t = 61167.86024$

10. (a) $\alpha^2 u_S'' = 0$, $u_S(x) = A + Bx$, $u_S(0) = u_1 = A$, $u_S'(L) = Q_2 = B$, so $u_S(x) = u_1 + Q_2 x$.

(b) $\alpha^2 u_S'' = 0$, $u_S(x) = A + Bx$, $u_S'(0) = Q_1 = B$, $u_S(L) = u_2 = A + BL$ gives $A = u_2 - Q_1 L$ and $B = Q_1$, so $u_S(x) = u_2 - Q_1 L + Q_1 x$

(c) $\alpha^2 u_S'' = 0$, $u_S(x) = A + Bx$, $u_S'(0) = Q_1 = B$, $u_S'(L) = Q_2 = B$, which give no solution if $Q_1 \neq Q_2$. Physically, Fourier's law of heat conduction tells us that

$$\text{Heat in at left end} = -u_x(0,t)kA = -Q_1 kA$$
$$\text{Heat out at right end} = u_x(L,t)kA = Q_2 kA$$

so $\quad$ net heat input $= (Q_2 - Q_1)kA$.

If the latter is nonzero then the temperature will increase indefinitely (if $Q_2 > Q_1$, and decrease indefinitely if $Q_2 < Q_1$) and a steady state will not exist! Only if $(Q_2 - Q_1)kA = 0$, i.e. if $Q_2 = Q_1$, will there exist a steady state. Let $Q_1 = Q_2 \equiv Q$. Then, from above, $B = Q$ and $u_S(x) = A + Qx$.

To determine A, integrate the PDE on x:

$$\alpha^2 \int_0^L u_{xx}\,dx = \int_0^L u_t\,dx$$
$$\alpha^2 u_x\big|_0^L = \frac{d}{dt}\int_0^L u(x,t)\,dx$$
$$\alpha^2 (Q - Q) = \quad ''$$
$$0 = \quad ''$$

so $\quad \int_0^L u(x,t)\,dx = \text{constant},$

which result gives us a connection between the steady state and the initial condition: $\quad \int_0^L u(x,\infty)\,dx = \int_0^L u(x,0)\,dx,$

$$\int_0^L (A + Qx)\,dx = \int_0^L f(x)\,dx,$$
$$AL + QL^2/2 = \int_0^L f(x)\,dx,$$

so $A = \frac{1}{L}\int_0^L f(x)\,dx - QL/2$ and
$$u_s(x) = \left(\frac{1}{L}\int_0^L f(x)\,dx - QL/2\right) + Qx.$$

(d) $u_s'' - \frac{H}{\alpha^2}u_s = 0$, $u_s(x) = A\sinh\frac{\sqrt{H}}{\alpha}x + B\cosh\frac{\sqrt{H}}{\alpha}x$

$$\left.\begin{array}{l} u_s(0) = u_1 = B \\ u_s(L) = u_2 = A\sinh\frac{\sqrt{H}}{\alpha}L + B\cosh\frac{\sqrt{H}}{\alpha}L \end{array}\right\} \begin{array}{l} B = u_1, \\ A = (u_2 - u_1\cosh\frac{\sqrt{H}}{\alpha}L)/\sinh\frac{\sqrt{H}}{\alpha}L \end{array}$$

so $u_s(x) = (u_2 - u_1\cosh\frac{\sqrt{H}}{\alpha}L)\dfrac{\sinh\sqrt{H}x/\alpha}{\sinh\sqrt{H}L/\alpha} + u_1\cosh\frac{\sqrt{H}}{\alpha}x$

(e) $u_s(x) = A\sinh\sqrt{H}x/\alpha + B\cosh\sqrt{H}x/\alpha$

$$\left.\begin{array}{l} u_s'(0) = Q_1 = \sqrt{H}A/\alpha \\ u_s(L) = u_2 = A\sinh\sqrt{H}L/\alpha + B\cosh\sqrt{H}L/\alpha \end{array}\right] \begin{array}{l} A = \alpha Q_1/\sqrt{H} \\ B = (u_2 - \frac{\alpha Q_1}{\sqrt{H}}\sinh\frac{\sqrt{H}L}{\alpha})/\cosh\frac{\sqrt{H}L}{\alpha} \end{array}$$

so $u_s(x) = \frac{\alpha Q_1}{\sqrt{H}}\sinh\frac{\sqrt{H}x}{\alpha} + (u_2 - \frac{\alpha Q_1}{\sqrt{H}}\sinh\frac{\sqrt{H}L}{\alpha})\dfrac{\cosh\sqrt{H}x/\alpha}{\cosh\sqrt{H}L/\alpha}$

(f) $u_s(x) = A\sinh\sqrt{H}x/\alpha + B\cosh\sqrt{H}x/\alpha$

$$\left.\begin{array}{l} u_s(0) = u_1 = B \\ u_s'(L) = Q_2 = \frac{\sqrt{H}}{\alpha}A\cosh\frac{\sqrt{H}L}{\alpha} + \frac{\sqrt{H}}{\alpha}B\sinh\frac{\sqrt{H}L}{2} \end{array}\right] \begin{array}{l} B = u_1 \\ A = (\frac{\alpha Q_2}{\sqrt{H}} - u_1\sinh\frac{\sqrt{H}L}{\alpha})/\cosh\frac{\sqrt{H}L}{\alpha} \end{array}$$

so $u_s(x) = (\frac{\alpha Q_2}{\sqrt{H}} - u_1\sinh\frac{\sqrt{H}L}{\alpha})\dfrac{\sinh\sqrt{H}x/\alpha}{\cosh\sqrt{H}L/\alpha} + u_1\cosh\frac{\sqrt{H}x}{\alpha}$

(g) $u_s(x) = A\sinh\sqrt{H}x/\alpha + B\cosh\sqrt{H}x/\alpha$

$$\left.\begin{array}{l} u_s'(0) = Q_1 = \sqrt{H}A/\alpha \\ u_s'(L) = Q_2 = \frac{\sqrt{H}}{\alpha}A\cosh\frac{\sqrt{H}L}{\alpha} + \frac{\sqrt{H}}{\alpha}B\sinh\frac{\sqrt{H}L}{\alpha} \end{array}\right] \begin{array}{l} A = \alpha Q_1/\sqrt{H} \\ B = (Q_2 - Q_1\cosh\frac{\sqrt{H}L}{\alpha})/\frac{\sqrt{H}}{\alpha}\sinh\frac{\sqrt{H}L}{\alpha} \end{array}$$

so $u_s(x) = \frac{\alpha Q_1}{\sqrt{H}}\sinh\sqrt{H}x/\alpha + (Q_2 - Q_1\cosh\frac{\sqrt{H}L}{\alpha})\frac{\alpha}{\sqrt{H}}\dfrac{\cosh\sqrt{H}x/\alpha}{\sinh\sqrt{H}L/\alpha}$

(h) $u_s'' - \frac{V}{\alpha^2}u_s' = 0$, $u_s(x) = A + Be^{Vx/\alpha^2}$

$$\left.\begin{array}{l} u_s(0) = u_1 = A + B \\ u_s(L) = u_2 = A + Be^{VL/\alpha^2} \end{array}\right] \begin{array}{l} B = (u_1 - u_2)/(1 - e^{VL/\alpha^2}) \\ A = u_1 - (u_1 - u_2)/(1 - e^{VL/\alpha^2}) \end{array}$$

so $u_s(x) = \dfrac{u_2 - u_1\exp(VL/\alpha^2)}{1 - \exp(VL/\alpha^2)} + \dfrac{u_1 - u_2}{1 - \exp(VL/\alpha^2)}\exp(Vx/\alpha^2)$

(i) $u_s(x) = A + Be^{Vx/\alpha^2}$,
$$\left.\begin{array}{l} u_s'(0) = Q_1 = BV/\alpha^2 \\ u_s(L) = u_2 = A + Be^{VL/\alpha^2} \end{array}\right] \begin{array}{l} B = \alpha^2 Q_1/V \\ A = u_2 - \frac{\alpha^2 Q_1}{V}e^{VL/\alpha^2} \end{array}$$

so $u_s(x) = u_2 - \frac{\alpha^2 Q_1}{V}e^{VL/\alpha^2} + \frac{\alpha^2 Q_1}{V}e^{Vx/\alpha^2} = u_2 - \frac{\alpha^2 Q_1}{V}(e^{VL/\alpha^2} - e^{Vx/\alpha^2})$

(j) $u_s(x) = A + Be^{Vx/\alpha^2}$,
$$\left.\begin{array}{l} u_s(0) + 5u_s'(0) = 3 = A + B + 5BV/\alpha^2 \\ u_s(L) = 10 = A + Be^{VL/\alpha^2} \end{array}\right] \begin{array}{l} B = 7/(e^{VL/\alpha^2} - 1 - 5V/\alpha^2) \\ A = 10 - 7e^{VL/\alpha^2}/(e^{VL/\alpha^2} - 1 - \frac{5V}{\alpha^2}) \end{array}$$

$$u_s(x) = \frac{3\exp(VL/\alpha^2) - 10 - 50V/\alpha^2 + 7\exp(Vx/\alpha^2)}{\exp(VL/\alpha^2) - 1 - 5V/\alpha^2}$$

11. If there is a steady state $u_S(x)$ then it satisfies $u_S''(x) = F(x)$. Integrating,

$$\alpha^2 \int_0^L u_S''(x)\,dx = \int_0^L F(x)\,dx,$$
$$\alpha^2 u_S'(x)\Big|_0^L = \quad ''$$
$$\alpha^2 (Q_2 - Q_1) = \int_0^L F(x)\,dx, \qquad ①$$

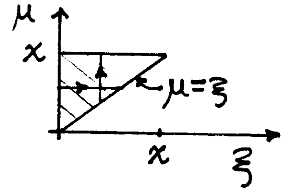
which relation must be satisfied by $Q_1, Q_2, F(x)$. In words, ① says that the net heat flux into the rod through its ends must equal the net absorption of heat by the distributed "sink" $F(x)$ if a steady state is to be maintained. Assuming that ① is satisfied let us solve for $u_S(x)$.

$$\alpha^2 u_S''(x) = F(x)$$
$$\alpha^2 u_S'(x) = \int_0^x F(\xi)\,d\xi + A \qquad ②$$

and setting $x=0$ in ② gives $A = \alpha^2 Q_1$. Integrating again,

$$u_S(x) = \frac{1}{\alpha^2}\int_0^x \int_0^\mu F(\xi)\,d\xi\,d\mu + Q_1 x + B \qquad ③$$

We can reduce the double integral in ③ to a single integral by reversing the order of integration:



$$\int_0^x \int_0^\mu F(\xi)\,d\xi\,d\mu = \int_0^x \int_\xi^x F(\xi)\,d\mu\,d\xi = \int_0^x (x-\xi)F(\xi)\,d\xi$$

so
$$u_S(x) = \frac{1}{\alpha^2}\int_0^x (x-\xi)F(\xi)\,d\xi + Q_1 x + B. \qquad ④$$

To evaluate B we establish a conservation principle relating $u_S(x)$ to the initial condition $u(x,0) = f(x)$, as we did in Exercise 10C. Integrating $\alpha^2 u_{xx} = u_t + F(x)$ on x from 0 to L and then using ① gives

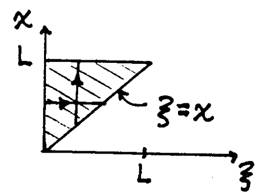
$$\alpha^2 \int_0^L u_{xx}\,dx = \int_0^L u_t\,dx + \int_0^L F(x)\,dx$$
$$\cancel{\alpha^2 (Q_2 - Q_1)} = \frac{d}{dt}\int_0^L u(x,t)\,dx + \cancel{\int_0^L F(x)\,dx}$$

so
$$\frac{d}{dt}\int_0^L u(x,t)\,dx = 0, \quad \text{or,} \quad \int_0^L u(x,t)\,dx = \text{constant}.$$

Hence,
$$\int_0^L u(x,\infty)\,dx = \int_0^L u(x,0)\,dx$$
$$\int_0^L u_S(x)\,dx = \int_0^L f(x)\,dx$$

or, from ④,
$$\frac{1}{\alpha^2}\int_0^L \int_0^x (x-\xi)F(\xi)\,d\xi\,dx + Q_1 \frac{L^2}{2} + BL = \int_0^L f(x)\,dx.$$

Solving for B and reducing the double integral to a single integral by reversing the order of integration,



$$B = \frac{1}{L}\int_0^L f(x)\,dx - \frac{Q_1 L}{2} - \frac{1}{\alpha^2 L}\int_0^L \int_\xi^L (x-\xi)F(\xi)\,dx\,d\xi$$
$$= \frac{1}{L}\int_0^L f(x) - \frac{Q_1}{2}L + \frac{1}{\alpha^2 L}\int_0^L \frac{(\xi-L)^2}{2}F(\xi)\,d\xi$$

so
$$u_S(s) = \frac{1}{\alpha^2}\int_0^x (x-\xi)F(\xi)\,d\xi + Q_1\left(x - \frac{L}{2}\right) + \frac{1}{L}\int_0^L f(x)\,dx + \frac{1}{2\alpha^2 L}\int_0^L (\xi-L)^2 F(\xi)\,d\xi.$$

12. (20) was $\frac{k}{c\sigma}u_{xx} - \frac{hs}{Ac\sigma}(u-u_\infty) - \frac{v}{c}u_x = u_t$. In steady state $u=u(x)$ so we have

$$u'' - \frac{hs}{Ak}(u-u_\infty) - \frac{v\sigma}{k}u' = 0.$$

With $v\sigma/k \equiv 2a$ and $h_f s/Ak \equiv b_f$, $h_a s/Ak \equiv b_a$, we have

$x<0$: $u'' - 2au' - b_f u = -b_f u_f$	$x>0$: $u'' - 2au' - b_a u = -b_a u_a$
$u(-\infty) = u_f$	$u(\infty) = u_a$

and at $x=0$ match u and u' from the two solutions.

Solving, $u(x) = e^{ax}\left(Be^{\sqrt{a^2+b_f}\,x} + \cancel{C}e^{-\sqrt{a^2+b_f}\,x}\right) + u_f$ in $x<0$

$u(x) = e^{ax}\left(\cancel{D}e^{\sqrt{a^2+b_a}\,x} + Ee^{-\sqrt{a^2+b_a}\,x}\right) + u_a$ in $x>0$

$u(x) \to u_f$ as $x \to -\infty \Rightarrow C=0$

$u(x) \to u_a$ as $x \to +\infty \Rightarrow D=0$.

Then, matching u and u' at $x=0$ gives

$$B + u_f = E + u_a,$$
$$(a+\sqrt{a^2+b_f})B = (a-\sqrt{a^2+b_a})E$$

Solving for E (we don't need B if we desire only the solution for $x>0$) gives

$$E = \frac{a+\sqrt{a^2+b_a}}{2a+\sqrt{a^2+b_a}-\sqrt{a^2+b_f}}(u_f - u_a)$$

so, over $0<x<\infty$,

$$u(x) = u_a + \frac{a+\sqrt{a^2+b_a}}{2a+\sqrt{a^2+b_a}-\sqrt{a^2+b_f}}(u_f-u_a)e^{(a-\sqrt{a^2+b_a})x}$$

$u(L) =$ etc.

13. (a) I will denote $c_A(x,t)$ as $c(x,t)$, and $c_{A_s}(x)$ as $c_s(x)$, for brevity.

$c(x,t) = A + Bx + (C\cos kx + D\sin kx)\,e^{-k^2Dt}$

$c_x(0,t) = 0 = B + kE\exp(-k^2Dt) \to B = E = 0$ so

$c(x,t) = A + C\cos kx \exp(\;"\;)$.

$c_x(L,t) = 0 = -kC\sin kL \exp(\;"\;) \to kL = n\pi \quad (n=1,2,\dots)$ so

$c(x,t) = A + \sum_1^\infty C_n \cos\frac{n\pi x}{L}\exp[-(n\pi/L)^2Dt]$ ①

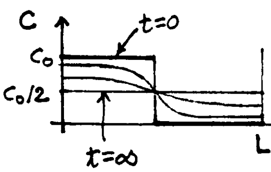
Then,

$c(x,0) = A + \sum_1^\infty C_n\cos\frac{n\pi x}{L} \quad (0<x<L)$

HRC: $A = \frac{1}{L}\int_0^L c(x,0)\,dx = c_0/2$ ②

$C_n = \frac{2}{L}\int_0^L c(x,0)\cos\frac{n\pi x}{L}dx = \frac{2}{L}\int_0^{L/2} c_0\cos\frac{n\pi x}{L}dx = \frac{2c_0}{n\pi}\sin\frac{n\pi}{2}$ ③

Solution is given by ①–③. Sketch:



NOTE that the slope u_x is 0 at $x=0$ and $x=L$, for all t.

(b) $D\int_0^L c_{xx}\,dx = \int_0^L c_t\,dx$,

$Dc_x(x,t)\Big|_0^L = \frac{d}{dt}\int_0^L c(x,t)\,dx$,

$0 = \frac{d}{dt}\int_0^L c(x,t)\,dx$,

so $\int_0^L c(x,t)\,dx =$ constant.

(c) $Dc_s''(x)=0$ gives $c_s(x)=C_1+C_2x$. $c_s'(0)=0$ and $c_s'(L)=0$ give $C_2=0$ so
$c_s(x)=C_1$.

Now use (13.5) between $t=0$ and $t=\infty$:

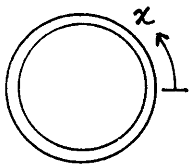
$$\int_0^L c(x,\infty)\,dx = \int_0^L c(x,0)\,dx$$

$$\int_0^L C_1\,dx = \int_0^{L/2} c_0\,dx + \int_{L/2}^L 0\,dx$$

gives $C_1L=\frac{c_0L}{2}$ so

$$c_s(x)=\frac{c_0}{2},$$

in agreement with the result obtained in (a).

14. 

(a) See Answers to Selected Exercises, in text.

(b) Integrating,

$$\alpha^2\int_0^L u_{xx}\,dx = \int_0^L u_t\,dx$$

$$\alpha^2 u_x\Big|_0^L = \frac{d}{dt}\int_0^L u\,dx$$

But $u_x(L,t)=u_x(0,t)$ so

$$0=\frac{d}{dt}\int_0^L u(x,t)\,dx, \text{ or, } \int_0^L u(x,t)\,dx = \text{constant}.$$

(c) $\alpha^2u_s''=0$ gives $u_s(x)=A+Bx$

$$u_s(0)-u_s(L)=0=A-(A+BL)=-BL \rightarrow B=0$$

$$u_s'(0)-u_s'(L)=0=B-B$$

so $u_s(x)=A$ (i.e., a constant). To evaluate A use (14.2):

$$\int_0^L u(x,\infty)\,dx = \int_0^L u(x,0)\,dx$$

$$\int_0^L u_s(x)\,dx = \int_0^L f(x)\,dx$$

$$AL = \quad '' \quad , \quad A=\frac{1}{L}\int_0^L f(x)\,dx$$

and $\quad c_s(x)=\frac{1}{L}\int_0^L f(x)\,dx$.

15. Seeking $u(x,t)=u_s(x)+X(x)T(t)$, we have, upon substitution,

$$\alpha^2(\cancel{u_s''})+\alpha^2X''T = XT'-\cancel{F}. \qquad ①$$

Ask u_s to be such that $\quad \alpha^2u_s''(x)=F$.

Then

$$u_s'(x)=\frac{F}{\alpha^2}x+A$$

$$u_s(x)=\frac{Fx^2}{2\alpha^2}+Ax+B$$

$$\left.\begin{aligned} u_s(0)&=0=B\\ u_s(L)&=50=\frac{FL^2}{2\alpha^2}+AL+B\end{aligned}\right\} \text{ give } B=0,\ A=(100\alpha^2-FL^2)/2\alpha^2L$$

so $u_s(x)=\dfrac{100\alpha^2-FL^2}{2\alpha^2L}x+\dfrac{F}{2\alpha^2}x^2 = \dfrac{50x}{L}+\dfrac{F}{2\alpha^2}x(x-L)$. ②

Next, ① gives X,T in the usual way, so

$$u(x,t)=u_s(x)+C+Dx+(P\cos kx+Q\sin kx)\,e^{-k^2\alpha^2t}$$

$$u(0,t)=0=0+C+P\exp(-k^2\alpha^2t) \rightarrow C=P=0 \text{ so}$$

$$u(x,t)=u_s(x)+Dx+Q\sin kx\,\exp(-k^2\alpha^2t)$$

$$u(L,t)=50=50+DL+Q\sin kL\,\exp('') \rightarrow D=0,\ kL=n\pi\ (n=1,2,\dots)$$

so $\quad u(x,t)=u_s(x)+\sum_1^\infty Q_n\sin\frac{n\pi x}{L}\exp[-(n\pi\alpha/L)^2t]$ ③

$$u(x,0) = f(x) = u_s(x) + \sum_1^\infty Q_n \sin\frac{n\pi x}{L}$$

or, $$f(x) - u_s(x) = \sum_1^\infty Q_n \sin\frac{n\pi x}{L} \quad (0<x<L)$$

HRS: $$Q_n = \frac{2}{L}\int_0^L [f(x) - u_s(x)] \sin\frac{n\pi x}{L}\,dx \qquad ④$$

Solution is given by ②-④.

16.(b) $\alpha^2 u_s'' = F$ gives $u_s(x) = \frac{F}{2\alpha^2}x^2 + Ax + B$

$$\left.\begin{array}{l} u_s'(0) = 0 = 0 + A \\ u_s(L) = 0 = \frac{FL^2}{2\alpha^2} + AL + B \end{array}\right\} \text{ give } A = 0,\ B = -FL^2/2\alpha^2$$

so

$$u_s(x) = \frac{F}{2\alpha^2}(x^2 - L^2) \qquad ①$$

$$u(x,t) = u_s(x) + C + Dx + (P\cos kx + Q\sin kx)e^{-k^2\alpha^2 t}$$

$$u_x(0,t) = 0 = 0 + D + kQ\exp(-k^2\alpha^2 t) \rightarrow D = Q = 0 \text{ so}$$

$$u(x,t) = u_s(x) + C + P\cos kx \exp(-k^2\alpha^2 t)$$

$$u(L,t) = 0 = 0 + C + P\cos kL \exp(\ '' \ ) \rightarrow C = 0,\ kL = n\pi/2\ (n = 1,3,\dots), \text{ so}$$

$$u(x,t) = u_s(x) + \sum_{1,3,\dots}^\infty P_n \cos\frac{n\pi x}{2L} \exp[-(n\pi\alpha/2L)^2 t] \qquad ②$$

$$u(x,0) = 0 = u_s(x) + \sum_{1,3,\dots}^\infty P_n \cos\frac{n\pi x}{2L}$$

so $$P_n = -\frac{2}{L}\int_0^L u_s(x)\cos\frac{n\pi x}{2L}\,dx. \qquad ③$$

Solution is given by ①-③.

(d) $\alpha^2 u_s'' = F$ gives $u_s(x) = \frac{F}{2\alpha^2}x^2 + Ax + B$

$$\left.\begin{array}{l} u_s(0) = 0 = B \\ u_s'(L) = -20 = \frac{F}{\alpha^2}L + A \end{array}\right\} \text{ so } B = 0,\ A = -20 - FL/\alpha^2, \text{ so}$$

$$u_s(x) = -20x + \frac{Fx}{2\alpha^2}(x - 2L) \qquad ①$$

$$u(x,t) = u_s(x) + C + Dx + (P\cos kx + Q\sin kx)e^{-k^2\alpha^2 t}$$

$$u(0,t) = 0 = 0 + C + P\exp(-k^2\alpha^2 t) \rightarrow C = 0, P = 0 \text{ so}$$

$$u(x,t) = u_s(x) + Dx + Q\sin kx \exp(-k^2\alpha^2 t)$$

$$u_x(L,t) = -20 = -20 + D + kQ\cos kL \exp(\text{"}) \rightarrow D = 0,\ kL = n\pi/2,$$

$$k = n\pi/2L \ (n \text{ odd}), \text{ so}$$

$$u(x,t) = u_s(x) + \sum_{1,3,\dots}^\infty Q_n \sin\frac{n\pi x}{2L} \exp[-(n\pi\alpha/2L)^2 t] \qquad ②$$

$$u(x,0) = 0 = u_s(x) + \sum_{1,3,\dots}^\infty Q_n \sin\frac{n\pi x}{2L},$$

$$Q_n = -\frac{2}{L}\int_0^L u_s(x)\sin\frac{n\pi x}{2L}\,dx \qquad ③$$

Solution given by ①-③.

17. (a) Putting (17.2) and (17.3) into the PDE gives
$$\alpha^2 \sum_1^\infty -\left(\frac{n\pi}{L}\right)^2 g_n \sin\frac{n\pi x}{L} = \sum_1^\infty g_n' \sin\frac{n\pi x}{L} - \sum_1^\infty F_n \sin\frac{n\pi x}{L}$$
so, equating coefficients of sines,
$$g_n'(t) + \left(\frac{n\pi\alpha}{L}\right)^2 g_n(t) = F_n(t). \qquad \text{(1a)}$$

Now, putting $t=0$ into (17.2) and using the initial condition $u(x,0)=0$ gives
$$0 = \sum g_n(0) \sin\frac{n\pi x}{L} \quad (0<x<L)$$
so that
$$g_n(0) = 0 \qquad \text{(1b)}$$
for each n. Solving (1a) subject to the initial condition (1b) gives [from (24) on page 24, with $b=0$]
$$g_n(t) = e^{-(n\pi\alpha/L)^2 t}\int_0^t e^{(n\pi\alpha/L)^2\tau} F_n(\tau)\,d\tau$$
$$= \int_0^t e^{(n\pi\alpha/L)^2(\tau-t)} F_n(\tau)\,d\tau$$
so
$$u(x,t) = \sum_1^\infty \left[\int_0^t F_n(\tau) e^{(n\pi\alpha/L)^2(\tau-t)} d\tau\right] \sin\frac{n\pi x}{L},$$

where the $F_n(\tau)$'s are given by (17.4).

(b) If $F(x,t) = e^{-t}$, then
$$F_n(\tau) = \frac{2}{L}\int_0^L e^{-t} \sin\frac{n\pi x}{L}\,dx = \frac{2e^{-t}}{L} \left.\frac{-\cos\frac{n\pi x}{L}}{n\pi/L}\right|_0^L = \begin{cases} 4e^{-t}/n\pi, & n \text{ odd} \\ 0, & n \text{ even}\end{cases}$$
and
$$u(x,t) = \sum_{1,3,\dots}^\infty \left[\int_0^t \frac{4e^{-\tau}}{n\pi} e^{(n\pi\alpha/L)^2(\tau-t)} d\tau\right] \sin\frac{n\pi x}{L}$$
$$= \frac{4}{\pi}\sum_{1,3,\dots}^\infty \frac{1}{n} \frac{e^{-t} - e^{-(n\pi\alpha/L)^2 t}}{(n\pi\alpha/L)^2 - 1} \sin\frac{n\pi x}{L}$$

(c) In this case the eigenfunctions from the relevant Sturm-Liouville problem
$$X'' + \kappa^2 X = 0; \quad X'(0)=0,\ X(L)=0$$
will be $\cos n\pi x/2L$ $(n=1,3,\dots)$, so this time seek
$$u(x,t) = \sum_{1,3,\dots}^\infty g_n(t) \cos\frac{n\pi x}{2L},$$
where
$$F(x,t) = \sum_{1,3,\dots}^\infty F_n(t) \cos\frac{n\pi x}{2L}.$$

Once again we obtain equations (1a) and (1b), as in (a), but with $L \to 2L$, so
$$u(x,t) = \sum_{1,3,\dots}^\infty \left[\int_0^t F_n(\tau) e^{(n\pi\alpha/2L)^2(\tau-t)} d\tau\right] \cos\frac{n\pi x}{2L}$$

18. $\alpha^2 u_{1xx} = u_{1t} + g(x,t)$

$\alpha^2 u_{2xx} = u_{2t}$

$\alpha^2 u_{3xx} = u_{3t}$

$\alpha^2 u_{4xx} = u_{4t}$

Addition gives $\alpha^2(u_{1xx} + \cdots + u_{4xx}) = (u_{1t} + \cdots + u_{4t}) + g(x,t)$, or,
$$\alpha^2(u_1 + \cdots + u_4)_{xx} = (u_1 + \cdots + u_4)_t + g(x,t) \qquad \text{(1)}$$

Likewise, add the boundary conditions and initial conditions:

$$\begin{array}{lll} u_1(0,t)=0 & u_1(L,t)=0 & u_1(x,0)=0 \\ u_2(0,t)=p(t) & u_2(L,t)=0 & u_2(x,0)=0 \\ u_3(0,t)=0 & u_3(L,t)=q(t) & u_3(x,0)=0 \\ \underline{u_4(0,t)=0} & \underline{u_4(L,t)=0} & \underline{u_4(x,0)=f(x)} \\ u_1(0,t)+\cdots+u_4(0,t)=p(t), & u_1(L,t)+\cdots+u_4(L,t)=q(t), & u_1(x,0)+\cdots+u_4(x,0)=f(x) \end{array}$$

so we see that $u(x,t)\equiv u_1(x,t)+\cdots+u_4(x,t)$ satisfies the PDE, boundary conditions, and initial condition in (18.1).

19. This problem represents the class of problems where $u_x(0,t)\neq u_x(L,t)$ so there is a net heat influx and a steady state does not exist. Let us begin with separation of variables, nonetheless, so we can see how it fails.

$$u(x,t) = A+Bx+(C\cos kx + D\sin kx)e^{-k^2\alpha^2 t}$$

$$\left.\begin{array}{l} u_x(0,t) = -1 = B + kD\exp(-k^2\alpha^2 t) \rightarrow B=-1,\ D=0 \\ u_x(L,t)=0 = B + kD\cos kL\exp('') \rightarrow B=0,\ kL=n\pi/2\ (n\text{ odd}) \end{array}\right\} \text{contradiction}$$

Following the hint, seek

$$u(x,t) = \frac{(x-L)^2}{2L} + v(x,t) \qquad (1)$$

That gives the following problem on v:

$$\alpha^2 v_{xx} = v_t - \frac{\alpha^2}{L}, \quad v_x(0,t)=0, \quad v_x(L,t)=0, \quad v(x,0) = -\frac{(x-L)^2}{2L}. \qquad (2)$$

The idea, then, is that the change of variables (1) led to <u>homogeneous</u> Neumann b.c.'s. It's true that we now have a nonzero source term and initial condition, but we can solve this problem by the method outlined in Exercise 15. Actually, it is nice to break (2) down first by superposition as

$$v(x,t) = v_1(x,t) + v_2(x,t)$$

where

$$\begin{array}{ll} \alpha^2 v_{1xx} = v_{1t} & \alpha^2 v_{2xx} = v_{2t} - \dfrac{\alpha^2}{L} \\ v_{1x}(0,t) = v_{1x}(L,t) = 0,\ v_1(x,t) = -\dfrac{(x-L)^2}{2L}, & v_{2x}(0,t) = v_{2x}(L,t) = v_2(x,0) = 0, \end{array}$$

because the v_2 problem is solved easily by inspection:

$$v_2(x,t) = \alpha^2 t/L.$$

20. $\alpha^2 u_{xx} = u_t;\quad u(0,t)=p(t),\ u(L,t)=q(t),\ u(x,0)=f(x)$

Setting $u(x,t) = v(x,t) + (1-\frac{x}{L})p(t) + \frac{x}{L}q(t)$,

$$u_{xx} = v_{xx}$$

$$u_t = v_t + (1-\tfrac{x}{L})p'(t) + \tfrac{x}{L}q'(t)$$

so the v problem is

$$\alpha^2 v_{xx} = v_t + \left[(1-\tfrac{x}{L})p'(t) + \tfrac{x}{L}q(t)\right] \leftarrow \text{call this } -F(x,t)$$

$v(0,t)=0$ because $u(0,t) = \cancel{p(t)} = v(0,t) + \cancel{p(t)}$

$v(L,t)=0$ because $u(L,t) = \cancel{q(t)} = v(L,t) + \cancel{q(t)}$

$v(x,t) = f(x) - (1-\frac{x}{L})p(0) - \frac{x}{L}q(0)$ because $u(x,0) = f(x) = v(x,0) + (1-\frac{x}{L})p(0) + \frac{x}{L}q(0)$.

21. (a) $v_s(x)$ satisfies $\alpha^2 v_s'' = h v_s$

$$v_s'' - \frac{h}{\alpha^2} v_s = 0; \quad v_s(0) = 50, \ v_s(L) = 50$$

Solving,

$$v_s(x) = A\cosh\frac{\sqrt{h}}{\alpha}x + B\sinh\frac{\sqrt{h}}{\alpha}x$$

$$v_s(0) = 50 = A$$

$$v_s(L) = 50 = A\cosh\frac{\sqrt{h}L}{\alpha} + B\sinh\frac{\sqrt{h}}{\alpha}L$$

so $A = 50$,

$$B = 50\left(1-\cosh\frac{\sqrt{h}L}{\alpha}\right)\Big/\sinh\frac{\sqrt{h}L}{\alpha},$$

$$v_s(x) = 50\cosh\frac{\sqrt{h}}{\alpha}x + 50\left(1-\cosh\frac{\sqrt{h}L}{\alpha}\right)\frac{\sinh\frac{\sqrt{h}}{\alpha}x}{\sinh\frac{\sqrt{h}}{\alpha}L}$$

Though not essential, we can simplify the latter a bit using the identity $\sinh(x+y) = \sinh x\cosh y + \sinh y\cosh x$:

$$v_s(x) = 50\,\frac{\sinh(\sqrt{h}(L-x)/\alpha) + \sinh(\sqrt{h}x/\alpha)}{\sinh(\sqrt{h}L/\alpha)}. \qquad ①$$

$$X'' + k^2X = 0 \longrightarrow X = \begin{cases} C\cos kx + D\sin kx, & k\neq 0 \quad ②a \\ E + Fx, & k = 0 \quad ②b \end{cases}$$

$$T' + (k^2\alpha^2 + h)T = 0 \rightarrow T = \begin{cases} Ge^{-(k^2\alpha^2+h)t}, & k\neq 0 \quad ③a \\ He^{-ht}, & k=0 \quad ③b \end{cases}$$

so $v(x,t) = v_s(x) + (E+Fx)He^{-ht} + (C\cos kx + D\sin kx)Ge^{-(k^2\alpha^2+h)t}$

$\quad = v_s(x) + (E'+F'x)e^{-ht} + (C'\cos kx + D'\sin kx)e^{-(k^2\alpha^2+h)t}$

$v(0,t) = 50 = 50 + E'e^{-ht} + C'\exp[-(k^2\alpha^2+h)t] \rightarrow E' = C' = 0$ so

$v(x,t) = v_s(x) + F'xe^{-ht} + D'\sin kx\,\exp[-(k^2\alpha^2+h)t]$

$v(L,t) = 50 = 50 + F'Le^{-ht} + D'\sin kL\,\exp[\quad " \quad] \rightarrow F' = 0,\ kL = n\pi\ (n=1,2,..)$

so $v(x,t) = v_s(x) + \sum_1^\infty D_n' \sin\frac{n\pi x}{L}\exp[-(n^2\pi^2\alpha^2/L^2 + h)t] \qquad ④$

Finally,

$$v(x,0) = f(x) = v_s(x) + \sum_1^\infty D_n'\sin\frac{n\pi x}{L},$$

$$f(x) - v_s(x) = \sum_1^\infty D_n'\sin\frac{n\pi x}{L} \qquad (0<x<L)$$

HRS:

$$D_n' = \frac{2}{L}\int_0^L [f(x) - v_s(x)]\sin\frac{n\pi x}{L}\,dx \qquad ⑤$$

and the solution is given by ④ and ⑤, where $v_s(x)$ is given by ①.

(b) Looking over the solution to part (a), it is tempting to believe that inclusion of the $v_s(x)$ term in the solution form $v(x,t) = v_s(x) + X(x)T(t)$ is essential. Actually, we can omit the $v_s(x)$ term — provided that we distinguish, in

② and ③, one more case, the case $k=(\sqrt{h}/\alpha)i$, because then $T'+(k^2\alpha^2+h)T=0$ reduces to $T'=0$, i.e., steady state. Thus, seeking $v(x,t)=X(x)T(t)$,

$$X=\begin{cases} A\cos kx+B\sin kx, & k\neq 0, \sqrt{h}i/\alpha \\ C+Dx, & k=0 \\ E\cosh\frac{\sqrt{h}}{\alpha}x+F\sinh\frac{\sqrt{h}}{\alpha}x, & k=\sqrt{h}i/\alpha \end{cases} \qquad T=\begin{cases} G\exp[-(k^2\alpha^2+h)t], & k\neq 0, \sqrt{h}i/\alpha \\ H\exp(-ht), & k=0 \\ I, & k=\sqrt{h}i/\alpha \end{cases}$$

so

$$v(x,t)=\underbrace{E\cosh\frac{\sqrt{h}}{\alpha}x+F\sinh\frac{\sqrt{h}}{\alpha}x}_{\text{this will give the "}v_s(x)\text{" part}}+(C+Dx)He^{-ht}+(A\cos kx+B\sin kx)Ge^{-(k^2\alpha^2+h)t}$$

22. $u(x,t)=u_\infty+e^{-ht}w(x,t)$

$u_{xx}=e^{-ht}w_{xx}$, $u_t=-he^{-ht}w+e^{-ht}w_t$

so (21.1) becomes

$$\alpha^2 e^{-ht}w_{xx}=-\cancel{he^{-ht}w}+e^{-ht}w_t+\cancel{he^{-ht}w}$$

or, $\alpha^2 w_{xx}=w_t$.

23. (a) The idea is that the initial condition for the $0<t<\infty$ problem is the steady-state solution for the $-\infty<t<0$ part, namely, the solution to

$$v_s''-rgv_s=0;\quad v_s(0)=12,\ v_s(L)=6 \qquad ①$$

Taking the solution of ① to be the initial condition $v(x,0)$ for the $0<t<\infty$ part, we have

$$v(x,0)=12\cosh\sqrt{rg}\,x+(6-12\cosh\sqrt{rg}\,L)\frac{\sinh\sqrt{rg}\,x}{\sinh\sqrt{rg}\,L}. \qquad ②$$

(b) Next, we will need the steady-state solution for the $0<t<\infty$ problem, namely, the solution of

$$v_s''-rgv_s=0;\quad v_s(0)=0,\ v_s(L)=6,$$

namely, $$v_s(x)=6\frac{\sinh\sqrt{rg}\,x}{\sinh\sqrt{rg}\,L}. \qquad ③$$

(c) $v(x,t)=v_s(x)+X(x)T(t)$ gives $\dfrac{X''}{X}=\dfrac{rCT'+rgT}{T}=-k^2$

$$X''+k^2X=0\to X=\begin{cases} A\cos kx+B\sin kx, & k\neq 0 \\ D+Ex, & k=0 \end{cases}$$

$$T'+\frac{k^2+rg}{rC}T=0\to T=\begin{cases} Fe^{-\beta t}, & k\neq 0 \\ Ge^{-gt/C}, & k=0 \end{cases} \quad \left(\beta=\frac{k^2+rg}{rC}\right)$$

so $v(x,t)=v_s(x)+(H+Ix)e^{-gt/C}+(J\cos kx+M\sin kx)e^{-\beta t}$

$v(0,t)=0=0+He^{-gt/C}+Je^{-\beta t}\to H=J=0$

$v(L,t)=6=6+ILe^{-gt/C}+M\sin kL\,e^{-\beta t}\to I=0,\ k=n\pi/L\ (n=1,2,\dots)$

so $v(x,t)=v_s(x)+\sum_1^\infty M_n\sin\frac{n\pi x}{L}e^{-\beta_n t}$. $\left(\beta_n=\frac{(n\pi/L)^2+rg}{rC}\right)$ ④

Finally,

$$v(x,0) = v_s(x) + \sum_1^\infty M_n \sin\frac{n\pi x}{L}$$

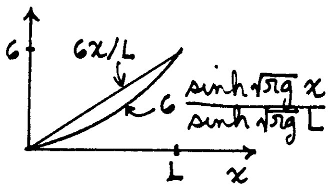
$$v(x,0) - v_s(x) = \sum_1^\infty M_n \sin\frac{n\pi x}{L} \qquad (0<x<L)$$

HRS:

$$M_n = \frac{2}{L}\int_0^L [v(x,0) - v_s(x)]\sin\frac{n\pi x}{L}\,dx \qquad ⑤$$

Solution given by ④ and ⑤, where $v(x,0)$ is given by ② and $v_s(x)$ by ③.
NOTE: It is interesting to examine the effects of the leakage g. The leakage

(i) reduces the steady state from $6x/L$ (for $g=0$) to $6\sinh\sqrt{rg}\,x/\sinh\sqrt{rg}\,L$, as sketched at the right. This makes sense, physically.



(ii) increase the β_n's and therefore speeds the decay of the transients. This makes sense too.

24. (a) $u = v/\rho$, $u_\rho = v_\rho/\rho - v/\rho^2$, $u_{\rho\rho} = v_{\rho\rho}/\rho - v_\rho/\rho^2 - v_\rho/\rho^2 + 2v/\rho^3$
so (24.1) becomes $\alpha^2\left(\frac{v_{\rho\rho}}{\rho} - \cancel{\frac{v_\rho}{\rho^2}} - \cancel{\frac{v_\rho}{\rho^2}} + \cancel{\frac{2v}{\rho^3}} + \frac{2}{\rho}\left(\cancel{\frac{v_\rho}{\rho}} - \cancel{\frac{v}{\rho^2}}\right)\right) = \frac{v_t}{\rho}$

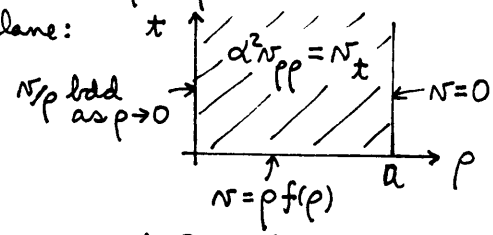
or, $\alpha^2 v_{\rho\rho} = v_t$.

(b) With $u(\rho,t) = v(\rho,t)/\rho$, the problem on v is

$$\alpha^2 v_{\rho\rho} = v_t, \quad (0<\rho<a,\ 0<t<\infty)$$
$$v(a,t) = 0, \quad (0<t<\infty)$$
$$v(\rho,0) = \rho f(\rho), \quad (0<\rho<a)$$

where $v(\rho,t)/\rho$ is bounded as $\rho\to 0$.

ρ, t plane:



$$v(\rho,t) = A + B\rho + (C\cos k\rho + D\sin k\rho)e^{-k^2\alpha^2 t}$$
$$v/\rho = \frac{A}{\rho} + B + \left(C\frac{\cos k\rho}{\rho} + D\frac{\sin k\rho}{\rho}\right)e^{-k^2\alpha^2 t}$$

bounded as $\rho\to 0 \Rightarrow A = 0$ and $C = 0$, so

$$v(\rho,t) = B\rho + D\sin k\rho\,\exp(-k^2\alpha^2 t)$$
$$v(a,t) = 0 = Ba + D\sin ka\,\exp(-k^2\alpha^2 t) \Rightarrow B = 0,\ ka = n\pi\ (n=1,2,\dots)$$
$$v(\rho,t) = \sum_1^\infty D_n \sin\frac{n\pi\rho}{a}\exp[-(n\pi\alpha/a)^2 t] \qquad ①$$

Finally,

$$v(\rho,0) = \rho f(\rho) = \sum_1^\infty D_n \sin\frac{n\pi\rho}{a} \qquad (0<\rho<a)$$

HRS: $D_n = \frac{2}{a}\int_0^a \rho f(\rho)\sin\frac{n\pi\rho}{a}\,d\rho \qquad ②$

Solution given by ① and ②, where $u(\rho,t) = v(\rho,t)/\rho$.

25. (a) $\frac{d}{dt}\int_0^L w^2(x,t)\,dx = \int_0^L 2ww_t\,dx$ by the Leibniz rule
$= 2\alpha^2\int_0^L ww_{xx}dx$ since $\alpha^2 w_{xx} = w_t$
$= 2\alpha^2\left(ww_x\big|_0^L - \int_0^L w_x^2\,dx\right)$
$= -2\alpha^2\int_0^L w_x^2\,dx$ since $w(0,t)=w(L,t)=0$

Now integrate on t from 0 to t:

$$d\int_0^L w^2(x,t)dx = -2\alpha^2\left(\int_0^L w_x^2\,dx\right)dt$$

$$\int_0^L w^2(x,t)\,dx - \int_0^L \underbrace{w^2(x,0)}_{0}\,dx = -2\alpha^2\int_0^t\int_0^L w_x^2(x,\tau)\,dx\,d\tau$$

so $\int_0^L w^2(x,t)dx = -2\alpha^2\int_0^t\int_0^L w_x^2(x,\tau)\,dx\,d\tau$.

The left-hand side is ≥ 0 and the right-hand side is ≤ 0, so they must both $=0$. Finally, if $w(x,t)$ is a continuous function of x, for each t, and $\int_0^L w^2(x,t)\,dx = 0$, then $w(x,t)=0$ over $0\leq x\leq L$ for each $t\geq 0$.

(b) Then $\alpha^2 w_{xx} = w_t$

$w(0,t)=0,\ w_x(L,t)=0,\ w(x,0)=0.$

$\frac{d}{dt}\int_0^L w^2(x,t)\,dx = \int_0^L 2ww_t\,dx$ (Leibniz)
$= 2\alpha^2\int_0^L ww_{xx}\,dx$ since $\alpha^2 w_{xx} = w_t$
$= 2\alpha^2\left(ww_x\big|_0^L - \int_0^L w_x^2\,dx\right)$
$= -2\alpha^2\int_0^L w_x^2\,dx$ since $w(0,t)=0$ and $w_x(L,t)=0$

Then proceed as in (a).

(c) This time consider a mixed boundary condition (i.e., of Robin type) at $x=L$.
$w = u_1 - u_2$ gives $\alpha^2 w_{xx} = w_t$; $w(0,t)=0$,
$w(L,t)+\beta w_x(L,t)=0$,
$w(x,0)=0$.

$$\frac{d}{dt}\int_0^L w^2dx = 2\int_0^L ww_t\,dx = 2\alpha^2\int_0^L \underbrace{w}_{u}\,\underbrace{w_{xx}\,dx}_{dv}$$

$$= 2\alpha^2\left(ww_x\big|_0^L - \int_0^L w_x^2\,dx\right) = 2\alpha^2\left[-\beta w_x^2(L,t) - \int_0^L w_x^2\,dx\right]$$

so

$$\int_0^L w^2(x,t)\,dx - \int_0^L \underbrace{w^2(x,0)}_{0}\,dx = -2\alpha^2\int_0^t\left[\beta w_x^2(x,\tau) + \int_0^L w_x^2(x,\tau)\,dx\right]d\tau \leq 0,$$

so $w(x,t)\equiv 0$.

26. Show that $\sum_1^\infty M_n = Q\sum_1^\infty n e^{-(n\pi\alpha/L)^2 t_0}$ converges, by the ratio test.

$$\lim_{n\to\infty}\left|\frac{M_{n+1}}{M_n}\right| = \lim_{n\to\infty}\frac{Q(n+1)\exp[-[(n+1)\pi\alpha/L]^2 t_0]}{Qn\exp[-(n\pi\alpha/L)^2 t_0]} = \lim_{n\to\infty}\frac{e^{-(n^2+2n+1)(\pi\alpha/L)^2 t_0}}{e^{-n^2(\pi\alpha/L)^2 t_0}}$$

$$= \lim_{n\to\infty} e^{-(2n+1)(\pi\alpha/L)^2 t_0} = 0,$$ which is <1. Thus, convergent.

27. The only difference is in the final step – satisfaction of the initial condition. For brevity, we will focus just on that last step.

(b) $u(x,0) = 10 = 10 - 5x + \sum_{1,3,\dots}^{\infty} D_n \sin\frac{n\pi x}{4} \quad (0<x<2)$

or,

$$5x = \sum_{1,3,\dots}^{\infty} D_n \sin\frac{n\pi x}{4} \quad (0<x<2)$$

The St.-Liou. problem is $X''+k^2X=0$ on $(0<x<2)$, with $X(0)=0$, $X'(2)=0$.
The weight function (for all cases in this exercise) is 1, so

$$D_n = \frac{\langle 5x, \sin n\pi x/4\rangle}{\langle \sin n\pi x/4, \sin n\pi x/4\rangle} = \frac{\int_0^2 5x \sin n\pi x/4 \, dx}{\int_0^2 \sin^2 n\pi x/4 \, dx} = \text{as in Exercise 6(b).}$$

(h) $u(x,0) = f(x) = A + \sum_1^{\infty} C_n \cos\frac{nx}{3} \quad (0<x<3\pi)$

The Sturm-Liouville problem is $X''+k^2X=0 \quad (0<x<3\pi)$
$X'(0)=0,\ X'(3\pi)=0$

The eigenfunctions are 1 and $\cos\frac{nx}{3}$ $(n=1,2,\dots)$ so

$$A = \frac{\langle f(x), 1\rangle}{\langle 1, 1\rangle} = \frac{\int_0^{3\pi} f\,dx}{\int_0^{3\pi} 1\,dx} = \frac{1}{3\pi}\int_0^{3\pi} f\,dx = \text{as in Exercise 6(h)}$$

(k) $u(x,0) = \sin\pi x - 37\sin\frac{\pi x}{5} + 6\sin\frac{9\pi x}{5} = \sum_1^{\infty} D_n \sin\frac{n\pi x}{5} \quad (0<x<5)$

The St.-Liou. problem is $X''+k^2X=0 \quad (0<x<5)$
$X(0)=0,\ X(5)=0$

The eigenfunctions are $\sin n\pi x/5$ $(n=1,2,\dots)$ so

$$D_n = \frac{\langle \sin\pi x - 37\sin\frac{\pi x}{5} + 6\sin\frac{9\pi x}{5}, \sin\frac{n\pi x}{5}\rangle}{\langle \sin\frac{\pi x}{5}, \sin\frac{\pi x}{5}\rangle} = \text{as in Exercise 6(k)}$$

(m) $u(x,0) = 0 = 2x + \sum_{1,3,\dots}^{\infty} C_n \cos\frac{n\pi x}{12}$,

$$-2x = \sum_{1,3,\dots}^{\infty} C_n \cos\frac{n\pi x}{12} \quad (0<x<6)$$

The St-Liou. problem is $X''+k^2X=0 \quad (0<x<6)$
$X'(0)=0,\ X(6)=0$

The eigenfunctions are $\cos n\pi x/12$ $(n=1,3,\dots)$ so

$$C_n = \frac{\langle -2x, \cos n\pi x/12\rangle}{\langle \cos n\pi x/12, \cos n\pi x/12\rangle} = \frac{\int_0^6 (-2x)\cos n\pi x/12 \, dx}{\int_0^6 \cos^2 n\pi x/12 \, dx}$$

28. $\alpha^2(u_{\rho\rho} + \frac{2}{\rho}u_\rho) = u_t \quad (0<\rho<a,\ 0<t<\infty)$
$u(a,t)=0,\ u(\rho,0)=f(\rho),\ u(0,t) = \text{bounded}.$

$u(\rho,t) = R(\rho)T(t)$ gives $\frac{R''+\frac{2}{\rho}R'}{R} = \frac{1}{\alpha^2}\frac{T'}{T} = -k^2$

$R''+\frac{2}{\rho}R'+k^2R=0$, or, $\rho^2R''+2\rho R'+k^2\rho^2R=0$, or, $(\rho^2R')'+k^2\rho^2R=0$.
Use (46) on page 238: $a=2,\ b=k^2,\ c=2$, so $\alpha=2/2=1$ and $\nu=-1/2$.

Then (50) on pg 239 gives $R(\rho) = \rho^{-1/2} Z_{1/2}(\kappa\rho) = \rho^{-1/2}(AJ_{1/2}(\kappa\rho) + BJ_{-1/2}(\kappa\rho))$

$$= \frac{1}{\sqrt{\rho}}\left(A\sqrt{\frac{2}{\pi\kappa\rho}}\sin\kappa\rho + B\sqrt{\frac{2}{\pi\kappa\rho}}\cos\kappa\rho\right)$$

$$= C\frac{\sin\kappa\rho}{\rho} + D\frac{\cos\kappa\rho}{\rho}.$$

The latter is the general solution if $\kappa \neq 0$, but if $\kappa = 0$ we lose the $\sin\kappa\rho/\rho$ solution. For $\kappa = 0$ the ODE is $R'' + \frac{2}{\rho}R' = 0$ with solution $E + F/\rho$. Thus,

$$R = \begin{cases} C\sin\kappa\rho/\rho + D\cos\kappa\rho/\rho, & \kappa \neq 0 \\ E + F/\rho, & \kappa = 0 \end{cases} \qquad T = \begin{cases} Ge^{-\kappa^2\alpha^2 t}, & \kappa \neq 0 \\ H, & \kappa = 0 \end{cases}$$

so

$$u(\rho,t) = \left(E + \frac{F}{\rho}\right)H + \left(C\frac{\sin\kappa\rho}{\rho} + D\frac{\cos\kappa\rho}{\rho}\right)\left(Ge^{-\kappa^2\alpha^2 t}\right)$$

$$= E' + \frac{F'}{\rho} + \left(C'\frac{\sin\kappa\rho}{\rho} + D'\frac{\cos\kappa\rho}{\rho}\right)e^{-\kappa^2\alpha^2 t}$$

$u(0,t)$ bounded $\to F' = D' = 0$ (but the $\sin\kappa\rho/\rho$ term is bounded as $\rho \to 0$), so

$$u(\rho,t) = E' + C'\frac{\sin\kappa\rho}{\rho}\exp(-\kappa^2\alpha^2 t)$$

$$u(a,t) = 0 = E' + C'\frac{\sin\kappa a}{a}\exp(\text{"}) \to E' = 0,\ \kappa a = n\pi \quad (n = 1, 2, \dots)$$

so

$$u(\rho,t) = \sum_1^\infty C_n' \frac{\sin\frac{n\pi\rho}{a}}{\rho}\exp[-(n\pi\alpha/a)^2 t] \qquad ①$$

Finally, $u(\rho,0) = f(\rho) = \sum_1^\infty C_n' \frac{\sin\frac{n\pi\rho}{a}}{\rho} \quad (0 < \rho < a)$ ②

To guide us with the latter expansion, note that the St.-Liou. problem is

$$(\rho^2 R')' + \kappa^2\rho^2 R = 0 \quad (0 < \rho < a)$$
$$R(0) \text{ bounded},\ R(a) = 0 \qquad ③$$

with eigenfunctions $\sin\frac{n\pi\rho}{a}/\rho$ and weight function ρ^2. Thus, the C_n''s in ② are computed as

$$C_n' = \frac{\langle f(\rho), \sin\frac{n\pi\rho}{a}/\rho\rangle}{\langle \sin\frac{n\pi\rho}{a}/\rho, \sin\frac{n\pi\rho}{a}/\rho\rangle} = \frac{\int_0^a f(\rho)(\sin\frac{n\pi\rho}{a}/\rho)\rho^2\,d\rho}{\int_0^a (\sin\frac{n\pi\rho}{a}/\rho)^2\rho^2\,d\rho} = \frac{\int_0^a \rho f(\rho)\sin\frac{n\pi\rho}{a}\,d\rho}{\int_0^a \sin^2\frac{n\pi\rho}{a}\,d\rho}$$

$$= \frac{2}{a}\int_0^a \rho f(\rho)\sin\frac{n\pi\rho}{a}\,d\rho. \qquad ④$$

The solution is given by ① and ④.
NOTE: We used the St.-Liou. problem ③ that is "built in" to assure us that the equality ② is indeed possible and then to show us how to compute the C_n''s. In this example we could have proceeded differently. Namely, multiply ② through by ρ and identify it as a half-range sine series. Then it follows that C_n' is given by ④, as before.

29. $\alpha^2 u_{xx} = u_t$

$u(0,t) = u(L,t)$, $u_x(0,t) = u_x(L,t)$, $u(x,0) = f(x)$

$u(x,t) = X(x)T(t)$ gives

$$u(x,t) = A + Bx + (C\cos kx + D\sin kx)e^{-k^2\alpha^2 t} \qquad ①$$

$$u(0,t) - u(L,t) = 0 = \cancel{A} + C\exp(-k^2\alpha^2 t) - [\cancel{A} + BL + (C\cos kL + D\sin kL)\exp(-k^2\alpha^2 t)]$$

$$u_x(0,t) - u_x(L,t) = 0 = \cancel{B} + kD\exp(\text{"}) - [\cancel{B} + (-kC\sin kL + kD\cos kL)\exp(\quad \text{"})]$$

or,

$$-BL + [(1-\cos kL)C - (\sin kL)D]\exp(-k^2\alpha^2 t) = 0$$

$$[(k\sin kL)C + k(1-\cos kL)D]\exp(\text{"}) = 0$$

so $B=0$ and

$$\begin{pmatrix} 1-\cos kL & -\sin kL \\ k\sin kL & k(1-\cos kL) \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix} = \underset{\sim}{0} \qquad ②$$

If we are to avoid the outcome $C=D=0$, we must set

$$\begin{vmatrix} 1-\cos kL & -\sin kL \\ k\sin kL & k(1-\cos kL) \end{vmatrix} = k[(1-\cos kL)^2 + \sin^2 kL] = 0 \qquad ③$$

Solving ③, we disallow the root $k=0$ because $k \neq 0$ in ①, the $A+Bx$ terms already accounting for the $k=0$ case. ③ → $1-\cos kL = 0$ and $\sin kL = 0$. The roots of the first are $kL = 2\pi, 4\pi, \dots$ and the roots of the second are $\pi, 2\pi, 3\pi, \dots$ so the roots of both are $kL = 2n\pi$ $(n=1,2,\dots)$. With that choice $(k = 2n\pi/L)$, (2) becomes $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix} = \underset{\sim}{0}$, so the solution for C and D is C = arbitrary and D = arbitrary. Thus,

$$u(x,t) = A + \sum_1^\infty \left(C_n \cos\frac{n\pi x}{2L} + D_n \sin\frac{n\pi x}{2L}\right)e^{-(n\pi\alpha/2L)^2 t}. \qquad ④$$

Finally,

$$u(x,0) = f(x) = A + \sum_1^\infty \left(C_n \cos\frac{n\pi x}{2L} + D_n \sin\frac{n\pi x}{2L}\right) \qquad (0<x<L)$$

which is an eigenfunction expansion of $f(x)$ in terms of the eigenfunctions $1, \cos\pi x/2L, \sin\pi x/2L, \cos 2\pi x/2L, \sin 2\pi x/2L, \dots$ of the (singular) St.-Liou. problem

$$X'' + k^2 X = 0 \quad (0<x<L)$$

$$X(0) - X(L) = 0, \quad X'(0) - X'(L) = 0.$$

Thus,

$$A = \frac{\langle f, 1\rangle}{\langle 1, 1\rangle} = \frac{\int_0^L f(x)\,dx}{\int_0^L 1\,dx} = \frac{1}{L}\int_0^L f(x)\,dx \qquad ⑤$$

$$C_n = \frac{\langle f, \cos n\pi x/2L\rangle}{\langle \cos\frac{n\pi x}{2L}, \cos\frac{n\pi x}{2L}\rangle} = \frac{\int_0^L f(x)\cos\frac{n\pi x}{2L}\,dx}{\int_0^L \cos^2\frac{n\pi x}{2L}\,dx} = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{2L}\,dx \qquad ⑥$$

$$D_n = \frac{\langle f, \sin n\pi x/2L\rangle}{\langle \sin n\pi x/2L, \sin n\pi x/2L\rangle} = \frac{\int_0^L f(x)\sin n\pi x/2L\,dx}{\int_0^L \sin^2\frac{n\pi x}{2L}\,dx} = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{2L}\,dx \qquad ⑦$$

so the solution is given by ④-⑦

30. (a) Using the "three-tier" solutions given in the exercise,

$$u(x,t) = (E' + F'e^{2x}) + e^x(C' + D'x)e^{-t} + e^x(A'\cos\omega x + B'\sin\omega x)e^{-k^2 t}$$

where E' is EI, F' is FI, and so on.

$u(0,t)=50=E'+F'+C'e^{-t}+A'e^{-k^2t} \longrightarrow E'+F'=50,\ C'=0,\ A'=0$ so

$u(x,t)=E'+(50-E')e^{2x}+D'xe^{x}e^{-t}+B'e^{x}\sin\omega x\ e^{-k^2t}$

$u(L,t)=50=E'+(50-E')e^{2L}+D'Le^{L}e^{-t}+B'e^{L}\sin\omega L\ e^{-k^2t} \longrightarrow E'=50, D'=0, \omega L=n\pi$ $(n=1,2,\dots)$, so

$$u(x,t)=50+\sum_1^\infty B_n'\ e^{x}\sin\frac{n\pi x}{L}\ e^{-[(n\pi/L)^2+1]t} \qquad ①$$

Finally,

$$u(x,0)=0=50+\sum_1^\infty B_n'\ e^{x}\sin\frac{n\pi x}{L}$$

or,

$$-50=\sum_1^\infty B_n' e^{x}\sin\frac{n\pi x}{L}, \quad (0<x<L) \qquad ②$$

which is an eigenfunction expansion of -50 in terms of the eigenfunctions $e^{x}\sin\frac{n\pi x}{L}$ of the St.-Liou. problem

$$X''-2X'+k^2X=0 \quad (0<x<L) \qquad ③$$
$$X(0)=0,\ X(L)=0.$$

To evaluate the B_n''s we need to determine the weight function. Write

$$\sigma X''-2\sigma X'+k^2\sigma X=0$$

where $-2\sigma=\sigma'$, so $\sigma(x)=e^{-2x}$. Thus the ODE can be written in the standard St.-Liou. form $\quad (e^{-2x}X')'+k^2e^{-2x}X=0, \qquad ④$

so the weight function is e^{-2x}. Then

$$B_n'=\frac{\langle -50,\ e^{x}\sin\frac{n\pi x}{L}\rangle}{\langle e^{x}\sin\frac{n\pi x}{L},\ e^{x}\sin\frac{n\pi x}{L}\rangle}=\frac{\int_0^L -50e^{x}\sin\frac{n\pi x}{L}\ e^{-2x}dx}{\int_0^L e^{2x}\sin^2\frac{n\pi x}{L}\ e^{-2x}dx}=$$
$$=\frac{2}{L}(-50)\int_0^L e^{-x}\sin\frac{n\pi x}{L}dx = \text{etc.} \qquad ⑤$$

The solution is given by ①, where the B_n''s are given by ⑤.

NOTE: Alternatively to using the St.-Liou. theory to solve for the B_n''s in ②, we could multiply ② by e^{-x} and identify the result,

$$-50e^{-x}=\sum_1^\infty B_n'\sin\frac{n\pi x}{L}$$

as a half-range sine expansion. In that case,

$$B_n'=\frac{2}{L}\int_0^L(-50e^{-x})\sin\frac{n\pi x}{L}dx,$$

as in ⑤.

31(a) $(xJ_0')'+xJ_0=0, \quad \int_0^{z_n}(xJ_0')'dx+\int_0^{z_n}xJ_0\,dx=0,$

$$xJ_0'(x)\Big|_0^{z_n}+\int_0^{z_n}xJ_0(x)\,dx=0,$$
$$\int_0^{z_n}xJ_0(x)\,dx=0-z_nJ_0'(z_n)$$
$$=z_nJ_1(z_n)$$

(b) $$P_n=-\frac{200}{c^2J_1^2(z_n)}\int_0^c J_0(z_n\tfrac{r}{c})r\,dr=-\frac{200}{c^2J_1^2(z_n)}\int_0^{z_n}J_0(\mu)\left(\frac{c}{z_n}\right)^2\mu\,d\mu$$
$$=-\frac{200}{z_n^2J_1^2(z_n)}z_nJ_1(z_n)=-\frac{200}{z_nJ_1(z_n)} \quad \text{verifies (83).}$$

Section 18.4

1. $u(x,t) = \frac{1}{2\alpha\sqrt{\pi t}} \int_{-\infty}^{\infty} F e^{-(x-\xi)^2/4\alpha^2 t}\, d\xi \overset{\mu=(\xi-x)/2\alpha\sqrt{t}}{=} \frac{F}{2\alpha\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\mu^2}\, 2\alpha\sqrt{t}\, d\mu$

$= \frac{F}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu^2} d\mu = \frac{F\sqrt{\pi}}{\sqrt{\pi}} = F.$

2.(a) $u(x,t) = \frac{1}{2\alpha\sqrt{\pi t}} \int_0^{\infty} F e^{-(x-\xi)^2/4\alpha^2 t}\, d\xi \overset{\mu=(\xi-x)/2\alpha\sqrt{t}}{=} \frac{F}{2\alpha\sqrt{\pi t}} \int_{-x/2\alpha\sqrt{t}}^{\infty} e^{-\mu^2}\, 2\alpha\sqrt{t}\, d\mu$

$= \frac{F}{\sqrt{\pi}}\left(\int_{-x/2\alpha\sqrt{t}}^{0} e^{-\mu^2} d\mu + \int_0^{\infty} e^{-\mu^2} d\mu \right) = \frac{F}{\sqrt{\pi}}\left(\int_0^{x/2\alpha\sqrt{t}} e^{-\mu^2} d\mu + \frac{\sqrt{\pi}}{2} \right)$

$= \frac{F}{2}\left(\operatorname{erf}(x/2\alpha\sqrt{t}) + 1 \right)$, as in (14).

(b) $u(x,t) = \frac{F}{2}\left(1 + \operatorname{erf}\left(\frac{x}{2\alpha\sqrt{t}}\right)\right)$ where $\operatorname{erf}\varphi = \frac{2}{\sqrt{\pi}} \int_0^{\varphi} e^{-\xi^2} d\xi$

$u_x = \frac{F}{2} \operatorname{erf}'\left(\frac{x}{2\alpha\sqrt{t}}\right) \frac{\partial}{\partial x}\left(\frac{x}{2\alpha\sqrt{t}}\right) = \frac{F}{2} \frac{2}{\sqrt{\pi}} e^{-(x/2\alpha\sqrt{t})^2} \frac{1}{2\alpha\sqrt{t}} = \frac{F}{2\alpha\sqrt{\pi t}} e^{-x^2/4\alpha^2 t}$

$\alpha^2 u_{xx} = \frac{\alpha^2 F}{2\alpha\sqrt{\pi t}} e^{-x^2/4\alpha^2 t} \left(\frac{-2x}{4\alpha^2 t}\right) = -\frac{Fx}{4\alpha\sqrt{\pi}\, t^{3/2}} e^{-x^2/4\alpha^2 t}$

$u_t = \frac{F}{2} \operatorname{erf}'\left(\frac{x}{2\alpha\sqrt{t}}\right) \frac{\partial}{\partial t}\left(\frac{x}{2\alpha\sqrt{t}}\right) = \frac{F}{2} \frac{2}{\sqrt{\pi}} e^{-x^2/4\alpha^2 t} \frac{x}{2\alpha}\left(-\frac{1}{2}\right) t^{-3/2} = -\frac{Fx}{4\alpha\sqrt{\pi}\, t^{3/2}} e^{-x^2/4\alpha^2 t}$

so $\alpha^2 u_{xx}$ does $= u_t$. Next, $u(x,0) = \frac{F}{2}(1 + \operatorname{erf}(\infty)) = \frac{F}{2}(1+1) = F.$ ✓

3. $\int_{-\infty}^{\infty} K(\xi - x; t)\, d\xi = \int_{-\infty}^{\infty} \frac{e^{-(x-\xi)^2/4\alpha^2 t}}{2\alpha\sqrt{\pi t}}\, d\xi \underset{\mu=(\xi-x)/2\alpha\sqrt{t}}{=} \int_{-\infty}^{\infty} \frac{e^{-\mu^2}}{2\alpha\sqrt{\pi t}}\, 2\alpha\sqrt{t}\, d\mu$

$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu^2} d\mu = 1.$ ✓

4. $\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2} d\xi$, $\operatorname{erf}(-x) = \frac{2}{\sqrt{\pi}} \int_0^{-x} e^{-\xi^2} d\xi \overset{\xi=-t}{=} \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}(-dt) = -\operatorname{erf}(x).$

5. $\alpha^2 u_{xx} = u_t$, $u(x,0) = f(x)$ on $-\infty < x < \infty$.

Laplace: $\alpha^2 \int_0^{\infty} \frac{\partial^2 u}{\partial x^2} e^{-st} dt = s\bar{u}(x,s) - u(x,0)$

$\alpha^2 \frac{d^2}{dx^2} \int_0^{\infty} u(x,t) e^{-st} dt = s\bar{u} - f(x)$

$\alpha^2 \frac{d^2}{dx^2} \bar{u}(x,s) - s\bar{u}(x,s) = -f(x)$

$\bar{u}_{xx} - \frac{s}{\alpha^2}\bar{u} = -\frac{1}{\alpha^2} f(x)$

NOTE: Observe that the latter is a nonhomogeneous ODE, whereas the Fourier transform gave us the homogeneous ODE $\frac{d\hat{u}}{dt} + \alpha^2 \omega^2 \hat{u} = 0$. Thus, although the Laplace transform will work, here, it is less convenient than the Fourier transform.

7. (a) $u(x+\tau,t)=\int_{-\infty}^{\infty} f(\xi)K(\xi-(x+\tau);t)\,d\xi \underset{\mu=\xi-\tau}{=} \int_{-\infty}^{\infty} f(\mu+\tau)K(\mu-x;t)\,d\mu$

$= \int_{-\infty}^{\infty} f(\mu)K(\mu-x;t)\,d\mu$ (because f is τ-periodic) $= u(x,t)$, so if f is τ-periodic then so is $u(x,t)$ a τ-periodic function of x.

(b) $u(-x,t)=\int_{-\infty}^{\infty} f(\xi)K(\xi+x;t)\,d\xi \underset{\mu=-\xi}{=} \int_{\infty}^{-\infty} f(-\mu)K(-\mu+x;t)(-d\mu)$

$= \int_{-\infty}^{\infty} -f(\mu)\,K(\mu-x;t)\,d\mu$ because f is odd and K is an even function of its first argument

$= -u(x,t)$, so if f is odd then $u(x,t)$ is an odd function of x.

(c) Same as in (b) but this sign is +.

8. (a) Add these equations:

$$\begin{array}{ll} \alpha^2 v_{xx} - v_t = 0 & v(x,0)=f(x) \\ \alpha^2 w_{xx} - w_t = -F(x,t) & w(x,0)=0 \\ \hline \alpha^2(v_{xx}+w_{xx}) - (v_t+w_t) = -F(x,t), & v(x,0)+w(x,0)=f(x) \end{array}$$

or, if $u(x,t)=v(x,t)+w(x,t)$,

$$\alpha^2 u_{xx} - u_t = -F(x,t), \checkmark \qquad u(x,0)=f(x). \checkmark$$

(b) See Answers to Selected Exercises.

(c) $\alpha^2 w_{xx} - w_t = -F(x)$, $w(x,0)=0$

Fourier transforming, $\alpha^2(i\omega)^2\hat{w} - \hat{w}_t = -\hat{F}(\omega)$,

$$\hat{w}_t + \alpha^2\omega^2\hat{w} = \hat{F}(\omega).$$

The latter differential equation is with respect to t, so $\hat{F}(\omega)$ is merely a constant. Thus,

$$\hat{w}(\omega,t) = Ae^{-\alpha^2\omega^2 t} + \frac{\hat{F}(\omega)}{\alpha^2\omega^2}.$$

Fourier transform of initial condition gives $\hat{w}(\omega,0)=0$, so

$$\hat{w}(\omega,0)=0=A+\hat{F}(\omega)/\alpha^2\omega^2, \quad A=-\hat{F}(\omega)/\alpha^2\omega^2,$$

so

$$\hat{w}(\omega,t) = \hat{F}(\omega)\,\frac{1-e^{-\alpha^2\omega^2 t}}{\alpha^2\omega^2}$$

and, by convolution,

$$w(x,t) = F(x) * F^{-1}\left\{\frac{1-e^{-\alpha^2\omega^2 t}}{\alpha^2\omega^2}\right\}$$

(d) To complete the solution we need the inverse of $(1-e^{-\alpha^2\omega^2 t})/\alpha^2\omega^2$. Define

$$\hat{g} \equiv (1-e^{-\alpha^2\omega^2 t})/\alpha^2\omega^2$$

and observe that d/dt gives a substantial simplification:

$$\hat{g}_t = e^{-\alpha^2\omega^2 t},$$

or,

$$\widehat{g_t} = e^{-\alpha^2\omega^2 t}, \qquad (1)$$

since

$$\hat{g}_t(\omega,t) = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} g(x,t)e^{-i\omega x}\,dx = \int_{-\infty}^{\infty} \frac{\partial g}{\partial t}(x,t)e^{-i\omega x}\,dx = \widehat{g_t}.$$

Inverting ① by entry 6 in appendix D gives

$$g_t = \frac{e^{-x^2/4\alpha^2 t}}{2\alpha\sqrt{\pi t}} \qquad ②$$

Further, $g(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{g}(\omega,t)e^{i\omega x}\,d\omega$, so

$$g(x,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \cancelto{0}{\hat{g}(\omega,0)}\, e^{i\omega x}\,d\omega = 0$$

so we can append to the differential equation ② the initial condition

$$g|_{t=0} = 0. \qquad ③$$

Thus, integrating ② from 0 to t and using ③ gives

$$g(x,t) = \int_0^t \frac{e^{-x^2/4\alpha^2\tau}}{2\alpha\sqrt{\pi\tau}}\,d\tau$$

Let $x^2/(4\alpha^2\tau) = \mu^2$. Then $\tau = x^2/(4\alpha^2\mu^2)$, $d\tau = -\frac{2x^2}{4\alpha^2}\mu^{-3}d\mu$

so

$$g(x,t) = \int_{\infty}^{x/2\alpha\sqrt{t}} \frac{e^{-\mu^2}}{2\alpha\sqrt{\pi}}\frac{2\alpha\mu}{x}\left(-\frac{2x^2}{4\alpha^2}\mu^{-3}d\mu\right) = \frac{x}{2\alpha^2\sqrt{\pi}}\int_{x/2\alpha\sqrt{t}}^{\infty} \underbrace{e^{-\mu^2}}_{u}\,\underbrace{\frac{d\mu}{\mu^2}}_{dv}$$

$$= \frac{x}{2\alpha^2\sqrt{\pi}}\left\{-\frac{1}{\mu}e^{-\mu^2}\Big|_{x/2\alpha\sqrt{t}}^{\infty} - \int_{x/2\alpha\sqrt{t}}^{\infty}\left(-\frac{1}{\mu}\right)(-2\mu)e^{-\mu^2}d\mu\right\}$$

$$= \frac{x}{2\alpha^2\sqrt{\pi}}\left\{\frac{2\alpha\sqrt{t}}{x}e^{-x^2/4\alpha^2 t} - 2\int_{x/2\alpha\sqrt{t}}^{\infty} e^{-\mu^2}d\mu\right\}$$

$$= \frac{1}{\alpha}\sqrt{\frac{t}{\pi}}\,e^{-x^2/4\alpha^2 t} - \frac{x}{2\alpha^2}\operatorname{erfc}(x/2\alpha\sqrt{t}).$$

Finally, $w(x,t) = F(x) * g(x,t) = \frac{1}{\alpha}\int_{-\infty}^{\infty} F(x-\xi)\Big[\sqrt{\frac{t}{\pi}}\,e^{-\xi^2/4\alpha^2 t}$

$$- \frac{x}{2\alpha}\operatorname{erfc}(\xi/2\alpha\sqrt{t})\Big]d\xi.$$

9. (a) The solution to $f_1(x) = -100 + 200[H(x) - H(x-L)]$ is

$$u_1(x,t) = -100 + 200\left[\frac{1}{2}\left(1+\operatorname{erf}\frac{x}{2\alpha\sqrt{t}}\right) - \frac{1}{2}\left(1+\operatorname{erf}\frac{x-L}{2\alpha\sqrt{t}}\right)\right]$$

$$= 100\left[\operatorname{erf}\frac{x}{2\alpha\sqrt{t}} - \operatorname{erf}\frac{x-L}{2\alpha\sqrt{t}} - 1\right],$$

the solution to $f_2(x) = 200[H(x+2L) - H(x+L) + H(x-2L) - H(x-3L)]$ is

$$u_2(x,t) = 200\left[\frac{1}{2}\left(1+\operatorname{erf}\frac{x+2L}{2\alpha\sqrt{t}}\right) - \frac{1}{2}\left(1+\operatorname{erf}\frac{x+L}{2\alpha\sqrt{t}}\right) + \frac{1}{2}\left(1+\operatorname{erf}\frac{x-2L}{2\alpha\sqrt{t}}\right) - \frac{1}{2}\left(1+\operatorname{erf}\frac{x-3L}{2\alpha\sqrt{t}}\right)\right]$$

$$= 100\left[\operatorname{erf}\frac{x+2L}{2\alpha\sqrt{t}} - \operatorname{erf}\frac{x+L}{2\alpha\sqrt{t}} + \operatorname{erf}\frac{x-2L}{2\alpha\sqrt{t}} - \operatorname{erf}\frac{x-3L}{2\alpha\sqrt{t}}\right], \quad \text{and so on.}$$

(b) From their graphs on page 990, observe that $f_1(x)$ agrees exactly with $f_{ext}(x)$ over $-L<x<2L$. The discrepancy occurs only over $x>2L$ and over $x<-L$, which regions are "far away" from the physical rod interval of $0<x<L$. Since it will take time for that misinformation to diffuse into $0<x<L$, it follows that for small t the solution to the f_1 problem should be quite accurate. Even more so for the f_1+f_2 problem since f_1+f_2 agrees with f_{ext} over $-3L<x<4L$, even more so for the $f_1+f_2+f_3$ problem, and so on.

(c) (9.1) gives $u(1,.1)=\frac{400}{\pi}\sum_1^\infty \frac{1}{2n-1}\sin\frac{(2n-1)\pi}{10}\, e^{-[(2n-1)\pi/10]^2(0.114)}$

To sum just the first term use the Maple commands

```
S:= sum( sin((2*i-1)*Pi/10) * exp(-.00114 *(2*i-1)^2 *Pi^2)/
  (2*i-1), i=1..1):
evalf(400*S/Pi);
```

and obtain	38.905
With $i=1..5$, obtain	99.391
With $i=1..10$, obtain	96.3460
With $i=1..20$, obtain	96.3764
With $i=1..30$, obtain	96.3764

So, for the results to settle down to 6 significant figures, say, we need around 20 terms of the series (9.1).

(9.5) gives $u(1,.1)=u_1(1,.1)+u_2(1,.1)+\cdots$

$$\approx u_1(1,.1) = 100\left[\operatorname{erf}\left(\frac{1}{2\sqrt{.114}}\right)-\operatorname{erf}\left(\frac{-9}{2\sqrt{.114}}\right)-1\right]$$

and the Maple command

```
evalf(100*(erf(1/(2*sqrt(.114))) - erf(-9/(2*sqrt(.114))) -1 ));
```

gives 96.3764

so even just one term of (9.5) gives excellent accuracy.

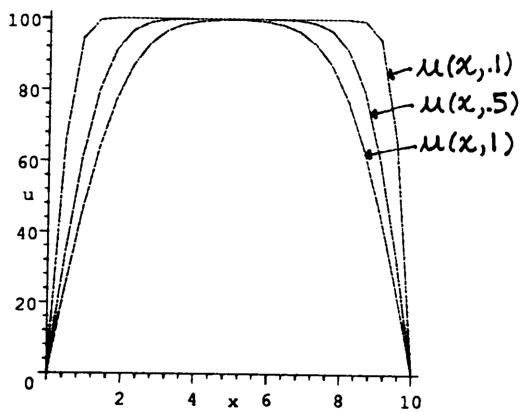
(d) $u_2, u_3,\ldots$ become negligible corrections to u_1, in (9.5), as $t\to 0$ because in that limit the arguments of the erfs in (9.5) $\to\infty$ and the erfs all approach the limiting value $\operatorname{erf}(\infty)=1$, in which case the erf pairs in (9.4) virtually cancel to zero. Thus, as a rule of thumb, let us ask that $L/(2\alpha\sqrt{t})\gg 1$. That is,

$$t \ll (L/2\alpha)^2,$$

which inequality is easily satisfied in the present case, where $L=10$, $\alpha=1.07$, and $t=0.1$. It would even be satisfied for $t=1$, say, which fact we use in part (e), where we use the approximate solution $u(x,t)\approx u_1(x,t)$ to generate some computer plots of the solution at $t=0.1$, 0.5, and 1.

(e) Maple:

```
> with(plots):
> p(x):=100*(erf(x/(2*sqrt(1.14)*sqrt(.1)))-erf((x-10)/(2*sqrt(1.14)
  *sqrt(.1)))-1):
> q(x):=100*(erf(x/(2*sqrt(1.14)*sqrt(.5)))-erf((x-10)/(2*sqrt(1.14)
  *sqrt(.5)))-1):
> r(x):=100*(erf(x/(2*sqrt(1.14)*sqrt(1.)))-erf((x-10)/(2*sqrt(1.14)
  *sqrt(1.)))-1):
> implicitplot({u=p(x),u=q(x),u=r(x)},x=0..10,u=0..100);
```



10. $\alpha^2 u_{xx} = u_t + V u_x \quad (-\infty < x < \infty,\ 0 < t < \infty)$

$u(x,0) = f(x)$

Fourier transform: $\alpha^2 (i\omega)^2 \hat{u} = \hat{u}_t + i\omega V \hat{u}$,

$\hat{u}_t + (\alpha^2\omega^2 + i\omega V)\hat{u} = 0$,

$\hat{u}(\omega,t) = A e^{-(\alpha^2\omega^2 + i\omega V)t}$

$\hat{u}(\omega,0) = \hat{f}(\omega) = A$

so $\hat{u}(\omega,t) = \hat{f}(\omega) e^{-(\alpha^2\omega^2 + i\omega V)t}$

From Appendix D,

Entry 6: $e^{-\alpha^2\omega^2 t} \to \dfrac{1}{2(\alpha\sqrt{t})\sqrt{\pi}} e^{-x^2/4\alpha^2 t}$

Entry 11 with $a=1$ and $b=-Vt$:

$$e^{-\alpha^2\omega^2 t} e^{-iVt\omega} \to \frac{1}{2(\alpha\sqrt{t})\sqrt{\pi}} e^{-(x-Vt)^2/4\alpha^2 t}$$

so $u(x,t) = f(x) * \dfrac{1}{2\alpha\sqrt{\pi t}} e^{-(x-Vt)^2/4\alpha^2 t}$

$$= \frac{1}{2\alpha\sqrt{\pi t}} \int_{-\infty}^{\infty} f(\xi) \exp[-(x-\xi-Vt)^2/4\alpha^2 t]\, d\xi,$$

which does indeed reduce to (18) if $V=0$.

11. Then (27) becomes $L\{u_t\} = s\bar{u} - u_0$, so (28) becomes

$$\alpha^2 \bar{u}_{xx} - s\,\bar{u} = -u_0,$$
$$\bar{u} = Ae^{\sqrt{s}\,x/\alpha} + Be^{-\sqrt{s}\,x/\alpha} + u_0/s$$

u bdd as $x \to \infty \Rightarrow A = 0$ so

$$\bar{u}(x,s) = Be^{-\sqrt{s}\,x/\alpha} + u_0/s.$$

$u(0,t) = u_1 \to \bar{u}(0,s) = u_1/s$, so

$$\bar{u}(0,s) = u_1/s = B + u_0/s \to B = (u_1 - u_0)/s$$

and
$$\bar{u}(x,s) = (u_1 - u_0)\frac{e^{-\sqrt{s}\,x/\alpha}}{s} + u_0/s$$

From Appendix C, entry 21 gives $e^{-\sqrt{s}\,x/\alpha} \to \frac{x/2\alpha}{\sqrt{\pi}}\,\frac{e^{-x^2/4\alpha^2 t}}{t^{3/2}}$

entry 1 gives $1/s \to 1$,

so
$$u(x,t) = (u_1 - u_0)\,1 * \frac{x}{2\alpha\sqrt{\pi}}\,\frac{e^{-x^2/4\alpha^2 t}}{t^{3/2}} + u_0 = u_0 + (u_1 - u_0)\frac{x}{2\alpha\sqrt{\pi}}\int_0^t \frac{e^{-x^2/4\alpha^2\tau}}{\tau^{3/2}}\,d\tau$$
$$= u_0 + (u_1 - u_0)\,\mathrm{erfc}\left(\frac{x}{2\alpha\sqrt{t}}\right).$$

12. $\alpha^2 u_{xx} = u_t \quad (0 < x < \infty,\ 0 < t < \infty)$

$u(0,t) = u_0 \cos\omega t$, $u \to 0$ as $x \to \infty$.

We could include an initial condition but won't need one since we are after the steady-state response, as $t \to \infty$. Following the hint, consider

$$\alpha^2 v_{xx} = v_t \quad (0 < x < \infty,\ 0 < t < \infty)$$
$$v(0,t) = u_0 e^{i\omega t}, \quad v \to 0 \text{ as } x \to \infty.$$

Seeking $v(x,t) = X(x)\,e^{i\omega t}$ obtain $\alpha^2 X''\cancel{e^{i\omega t}} = i\omega X\cancel{e^{i\omega t}}$

$$X'' - \frac{i\omega}{\alpha^2}X = 0,$$

and since $\sqrt{i} = \pm(1+i)/\sqrt{2}$,

$$X(x) = Ae^{\frac{1+i}{\sqrt{2}}\frac{\sqrt{\omega}}{\alpha}x} + Be^{-\frac{1+i}{\sqrt{2}}\frac{\sqrt{\omega}}{\alpha}x}$$

$v \to 0$ as $x \to \infty$ implies that $X(x) \to 0$ as $x \to \infty$ implies that $A = 0$, so

$$v(x,t) = X(x)\,e^{i\omega t} = Be^{-\frac{1+i}{\sqrt{2}}\frac{\sqrt{\omega}}{\alpha}x}\,e^{i\omega t}$$
$$v(0,t) = u_0 e^{i\omega t} = Be^{i\omega t} \to B = u_0.$$

Then,
$$u(x,t) = \mathrm{Re}\,v(x,t) = \mathrm{Re}\left\{u_0 e^{-\sqrt{\frac{\omega}{2}}\frac{x}{\alpha}}\,e^{i\left(\omega t - \sqrt{\frac{\omega}{2}}\frac{x}{\alpha}\right)}\right\}$$
$$= u_0 e^{-rx}\cos(\omega t - rx)$$

where $r = \sqrt{\omega/2}/\alpha$.

13. (a) $u(x,t) = \frac{100x}{2\alpha\sqrt{\pi}} \int_0^t \frac{e^{-x^2/4\alpha^2\tau}}{\tau^{3/2}} d\tau$ Let $x^2/4\alpha^2\tau = \mu^2$, $\tau = \frac{x^2}{4\alpha^2}\frac{1}{\mu^2}$

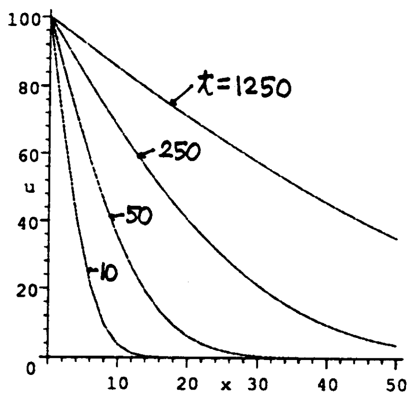
$$= \frac{100x}{2\alpha\sqrt{\pi}} \int_{\infty}^{x/2\alpha\sqrt{t}} \frac{e^{-\mu^2}}{\frac{x^3}{8\alpha^3}\frac{1}{\mu^3}} (-2)\frac{x^2}{4\alpha^2}\mu^{-3}\, d\mu = +\frac{200}{\sqrt{\pi}} \int_{x/2\alpha\sqrt{t}}^{\infty} e^{-\mu^2} d\mu$$

$$= 100 \operatorname{erfc} \frac{x}{2\alpha\sqrt{t}}$$

from (17).

(b) Better yet, let us obtain a plot of $u(x,t)$ versus x at representative t's, as given in Fig. 8. For example, over $0<x<50$, at $t = 10, 50, 250, 1250$.
Maple:

```
> with(plots):
> implicitplot({u=100*erfc(x/(2*sqrt(1.14*10))),u=100*erfc(x/(2*sqrt
  (1.14*50))),u=100*erfc(x/(2*sqrt(1.14*250))),u=100*erfc(x/(2*sqrt(
  1.14*1250)))},x=0..50,u=0..100);
```



14. $\alpha^2 u_{xx} = u_t$ $(0<x<\infty,\ 0<t<\infty)$

$u(x,0)=0$, $u_x(0,t) = -Q$, $u \to 0$ as $x \to \infty$.

(a) Laplace: $\alpha^2 \bar{u}_{xx} = s\bar{u} - 0$

$$\bar{u}_{xx} - \frac{s}{\alpha^2}\bar{u} = 0$$

$$\bar{u}(x,s) = Ae^{\sqrt{s}\,x/\alpha} + Be^{-\sqrt{s}\,x/\alpha}$$

$u \to 0$ as $x \to \infty$ implies $\bar{u} \to 0$ as $x \to \infty$, so we need $A=0$. Thus,

$$\bar{u}(x,s) = Be^{-\sqrt{s}\,x/\alpha}.$$

Finally, $u_x(0,t) = -Q$ gives $\bar{u}_x(0,s) = -Q/s = B(-\sqrt{s}/\alpha)e^0$, so $B = \frac{\alpha Q}{s^{3/2}}$ and

$$\bar{u}(x,s) = \alpha Q \frac{e^{-\sqrt{s}\,x/\alpha}}{s^{3/2}} = \alpha Q \frac{1}{s} \frac{e^{-\sqrt{s}\,x/\alpha}}{\sqrt{s}}$$

Appendix C:
Entry 1: $1/s \to 1$
Entry 20: $\frac{e^{-\sqrt{s}\,x/\alpha}}{\sqrt{s}} \to \frac{e^{-x^2/4\alpha^2 t}}{\sqrt{\pi t}}$

so $u(x,t) = \alpha Q\, 1 * e^{-x^2/4\alpha^2 t}/\sqrt{\pi t} = \frac{\alpha Q}{\sqrt{\pi}} \int_0^t \frac{e^{-x^2/4\alpha^2\tau}}{\sqrt{\tau}} d\tau.$

(b) Let $x^2/4\alpha^2\tau = \mu^2$, $\tau = x^2/4\alpha^2\mu^2$. Then

$$u(x,t) = \frac{\alpha Q}{\sqrt{\pi}} \int_{\infty}^{x/2\alpha\sqrt{t}} \frac{e^{-\mu^2}}{\left(\frac{x}{2\alpha\mu}\right)} \left(-\frac{2x^2}{4\alpha^2\mu^3}\right) d\mu = \frac{Qx}{\sqrt{\pi}} \int_{x/2\alpha\sqrt{t}}^{\infty} \underbrace{e^{-\mu^2}}_{u} \underbrace{\frac{d\mu}{\mu^2}}_{dv}$$

$$= \frac{Qx}{\sqrt{\pi}} \left\{ -\frac{e^{-\mu^2}}{\mu} \Big|_{x/2\alpha\sqrt{t}}^{\infty} - \int_{x/2\alpha\sqrt{t}}^{\infty} \left(-\frac{1}{\mu}\right)(-2\mu) e^{-\mu^2} d\mu \right\}$$

$$= \frac{Qx}{\sqrt{\pi}} \left\{ 0 + \frac{e^{-x^2/4\alpha^2 t}}{x/2\alpha\sqrt{t}} - 2\int_{x/2\alpha\sqrt{t}}^{\infty} e^{-\mu^2} d\mu \right\} = 2\alpha Q \sqrt{\frac{t}{\pi}}\, e^{-x^2/4\alpha^2 t} - Qx \operatorname{erfc}\left(\frac{x}{2\alpha\sqrt{t}}\right)$$

Section 18.5

1. $$u(x,t) = \frac{1}{2\alpha\sqrt{\pi t}} \int_0^{\infty} 100 \left(e^{-(\xi - x)^2/4\alpha^2 t} - e^{-(\xi + x)^2/4\alpha^2 t}\right) d\xi$$

Let $(\xi - x)/2\alpha\sqrt{t} = \mu$, let $(\xi + x)/2\alpha\sqrt{t} = \nu$

$$= \frac{50}{\alpha\sqrt{\pi t}} \left[\int_{-x/2\alpha\sqrt{t}}^{\infty} e^{-\mu^2} 2\alpha\sqrt{t}\, d\mu - \int_{x/2\alpha\sqrt{t}}^{\infty} e^{-\nu^2} 2\alpha\sqrt{t}\, d\nu \right]$$

$$= \frac{100}{\sqrt{\pi}} \int_{-x/2\alpha\sqrt{t}}^{x/2\alpha\sqrt{t}} e^{-\mu^2} d\mu = \frac{200}{\sqrt{\pi}} \int_0^{x/2\alpha\sqrt{t}} e^{-\mu^2} d\mu = 100 \operatorname{erf}\left(\frac{x}{2\alpha\sqrt{t}}\right)$$

2. (a) $$u(x,t) = \int_{-\infty}^{0} f_{ext}(\xi) K(\xi - x; t)\, d\xi + \int_0^{\infty} f_{ext}(\xi) K(\xi - x; t)\, d\xi$$

Let $\xi = -\mu$

$$= \int_{\infty}^{0} f_{ext}(-\mu) K(-\mu - x; t)(-d\mu) + \quad ''$$

$$= \int_0^{\infty} f_{ext}(\mu) K(\mu + x; t)\, d\mu \quad + \quad ''$$ because we are now using an even extension, and also K is an even function of its 1st argument

now let $\mu = \xi$

$$= \int_0^{\infty} f_{ext}(\xi) [K(\xi + x; t) + K(\xi - x, t)]\, d\xi$$

$$= \int_0^{\infty} f(\xi) \left(\frac{e^{-(\xi + x)^2/4\alpha^2 t} + e^{-(\xi - x)^2/4\alpha^2 t}}{2\alpha\sqrt{\pi t}} \right) d\xi$$ since $f_{ext} = f$ on $(0, \infty)$.

(c) $$u(x,t) = \frac{100}{2\alpha\sqrt{\pi t}} \left\{ \int_{x/2\alpha\sqrt{t}}^{\infty} e^{-\nu^2} 2\alpha\sqrt{t}\, d\nu + \int_{-x/2\alpha\sqrt{t}}^{\infty} e^{-\mu^2} 2\alpha\sqrt{t}\, d\mu \right\}$$

$$= \frac{100}{\sqrt{\pi}}\, 2\int_0^{\infty} e^{-\nu^2} d\nu = \frac{100}{\sqrt{\pi}}\, 2\, \frac{\sqrt{\pi}}{2} = 100,$$

which makes perfect sense: if $u(x,0) = 100$ then $u(x,t) = 100$.

3. Since (11) holds for all x, it also holds with x changed to $-x$. Thus,

$$E_1(x)+O_1(x) = E_2(x)+O_2(x) \quad ①$$
$$E_1(-x)+O_1(-x) = E_2(-x)+O_2(-x) \quad ②$$
or, $E_1(x)-O_1(x) = E_2(x)-O_2(-x) \quad ③$

Adding ① and ③ gives $2E_1(x)=2E_2(x)$, so $E_1(x)=E_2(x)$, and subtracting them gives $2O_1(x)=2O_2(x)$, so $O_1(x)=O_2(x)$.

4. $F'(0)=\lim_{\Delta x\to 0}\dfrac{F(0+\Delta x)-F(0)}{\Delta x}=\lim_{\Delta x\to 0}\dfrac{F(\Delta x)-F(0)}{\Delta x}$

Also, $F'(0)=\lim_{\Delta x\to 0}\dfrac{F(0-\Delta x)-F(0)}{-\Delta x}=\lim_{\Delta x\to 0}\dfrac{F(\Delta x)-F(0)}{-\Delta x}$

Thus, $F'(0)=-F'(0)$, $2F'(0)=0$, $F'(0)=0$.

7. (b) $F_t(x,t)=\lim_{\Delta t\to 0}\dfrac{F(x,t+\Delta t)-F(x,t)}{\Delta t}$

$F_t(-x,t)=\lim_{\Delta t\to 0}\dfrac{F(-x,t+\Delta t)-F(x,t)}{\Delta t}=\lim_{\Delta t\to 0}\dfrac{F(x,t+\Delta t)-F(x,t)}{\Delta t}=F_t(x,t)$, so the latter is an even function of x.

8. (a) Yes (b) No, e^{-x} is not even (c) No, the coefficient of u_{xt}, namely 1, is not odd; further, the coefficient of u_x, namely 1, is not odd
(d) No, it is not linear, due to the u^2 term
(e) Yes (f) No, the coefficient $\cos x$ of u_x is not odd
(g) Yes (h) Yes (i) Yes (j) Yes; note that $\sin t$ <u>is</u> an even function of x, namely, a constant (k) Yes (l) Yes
(m) $(x^3u_x)_x-u_{tt}+u = x^3u_{xx}+3x^2u_x-u_{tt}+u$ hence, no, because the coefficient x^3 of u_{xx} is not even, nor is the coefficient $3x^2$ odd.

Section 18.6

1. (a) $$u_{xx}=(u_x)_x\approx\frac{u_x(x+\Delta x,t)-u_x(x,t)}{\Delta x}\approx\frac{\dfrac{u(x+2\Delta x,t)-u(x+\Delta x,t)}{\Delta x}-\dfrac{u(x+\Delta x,t)-u(x,t)}{\Delta x}}{\Delta x}$$
$$=\frac{u(x+2\Delta x,t)-2u(x+\Delta x,t)+u(x,t)}{(\Delta x)^2}$$

and $u_t\approx\dfrac{u(x,t+\Delta t)-u(x,t)}{\Delta t}$

so $\alpha^2u_{xx}=u_t$ gives $\alpha^2\dfrac{U_{j+2,k}-2U_{j+1,k}+U_{jk}}{(\Delta x)^2}=\dfrac{U_{j,k+1}-U_{jk}}{\Delta t}$

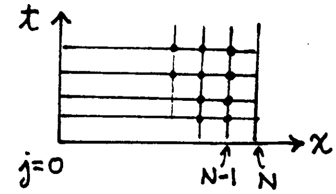
so $U_{j,k+1}=(1+r)U_{jk}-2rU_{j+1,k}+rU_{j+2,k}$

(b) $$u_{xx}=(u_x)_x\approx\frac{u_x(x,t)-u_x(x-\Delta x,t)}{\Delta x}\approx\frac{\dfrac{u(x,t)-u(x-\Delta x,t)}{\Delta x}-\dfrac{u(x-\Delta x,t)-u(x-2\Delta x,t)}{\Delta x}}{\Delta x}$$
$$=\frac{u(x,t)-2u(x-\Delta x,t)+u(x-2\Delta x,t)}{(\Delta x)^2}$$

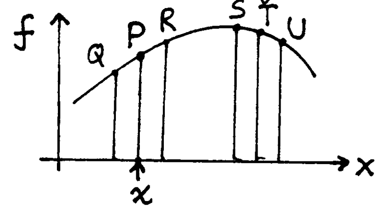
so $\alpha^2 \dfrac{u(x,t)-2u(x-\Delta x,t)+u(x-2\Delta x,t)}{(\Delta x)^2} \approx \dfrac{u(x,t+\Delta t)-u(x,t)}{\Delta t}$

so $U_{j,k+1} = (1+r)U_{jk} - 2rU_{j-1,k} + rU_{j-2,k}$.

(c) Two drawbacks come to mind. First, if we use (1.1), then at the grid points next to the right end ($j=N-1$) the $U_{j+2,k}$ is meaningless, in (1.1), since $j+2=N+1$ and there are not points at $N+1$.



Similarly, if we use (1.2) then the $U_{j-2,k}$ term is meaningless when $j=1$, for $U_{1-2,k}$ is $U_{-1,k}$ is not defined. Also, we can expect the "double forward" formula (1.1) and the "double backward" formula (1.2) to be less accurate than the centered formula (8). Why? Look at it in this intuitive way: Suppose we seek $f''(x)$ (see sketch) knowing only the values of f at Q,P,R. We can fit a parabola through those 3 points and then take d^2/dx^2 of that parabolic funtion to evaluate f'' at x, approximately. We could, alternatively, fit a parabola through S,T,U, say, as an approximation of f, and then take d^2/dx^2 to evaluate f'' at x, but surely we expect less accuracy using S,T,U than using Q,P,R, which are centered at the point x. Well, in using the "double-forward" formula the points S,T,U are not shifted as much as in the figure, but they are indeed shifted so as not to be centered at x. The foregoing argument has been intuitive; a rigorous case can be made using Taylor series.



2. $U_{j,k+1} = 0.16U_{j-1,k} + 0.68U_{jk} + 0.16U_{j+1,k}$

so

$U_{13} = 0.16(12) + 0.68(7.3) + 0.16(10.5) = 8.564$

$U_{23} = 0.16(7.3) + 0.68(10.5) + 0.16(21.4) = 11.732$

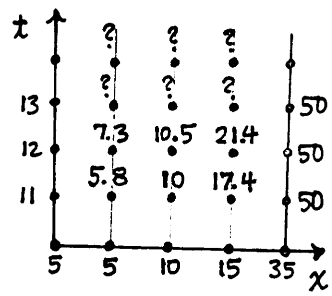
$U_{33} = 0.16(10.5) + 0.68(21.4) + 0.16(50) = 24.232$

and

$U_{14} = 0.16(13) + 0.68(8.564) + 0.16(11.732) = 9.7806$

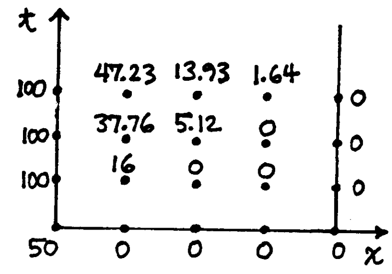
$U_{24} = 0.16(8.564) + 0.68(11.732) + 0.16(24.232) = 13.2251$

$U_{34} = 0.16(11.732) + 0.68(24.232) + 0.16(50) = 26.3549$



3. $r = \alpha^2 \Delta t/(\Delta x)^2 = 2/(2.5)^2 = 0.32$

$U_{j,k+1} = 0.32U_{j-1,k} + 0.36U_{jk} + 0.32U_{j+1,k}$



$U_{11} = .32(50) + 0 + 0 = 16$
$U_{21} = 0+0+0 = 0$
$U_{31} = 0+0+0 = 0$

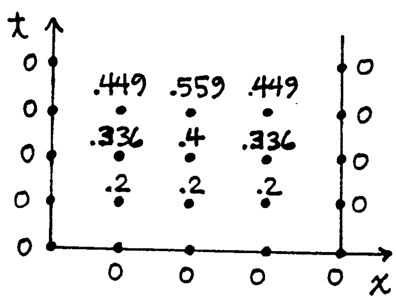
$U_{12} = .32(100) + .36(16) + 0 = 37.76$
$U_{22} = .32(16) + 0 + 0 = 5.12$
$U_{32} = 0+0+0 = 0$

$U_{13} = .32(100) + .36(37.76) + .32(5.12) = 47.232$
$U_{23} = .32(37.76) + .36(5.12) + 0 = 13.926$
$U_{33} = .32(5.12) + 0 + 0 = 1.638$

4. $\alpha^2 \dfrac{U_{j-1,k} - 2U_{jk} + U_{j+1,k}}{(\Delta x)^2} = \dfrac{U_{j,k+1} - U_{jk}}{\Delta t} + HU_{jk} - F_{jk}$

multiplying by Δt, $r(U_{j-1,k} - 2U_{jk} + U_{j+1,k}) = U_{j,k+1} - U_{jk} + H\Delta t\, U_{jk} - F_{jk}\Delta t$
so

$$U_{j,k+1} = rU_{j-1,k} + (1-2r-H\Delta t)U_{jk} + rU_{j+1,k} + F_{jk}\Delta t$$

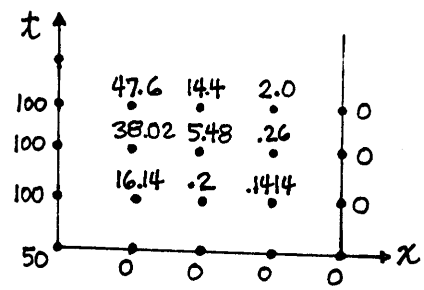


5. $r = (1)(0.02)/(0.25)^2 = 0.32$, $H = 0$, $F(x,t) = 10$, so
$U_{j,k+1} = .32U_{j-1,k} + .36U_{jk} + .32U_{j+1,k} + .2$

$U_{11} = 0+0+0+.2 = .2$
$U_{21} = 0+0+0+.2 = .2$
$U_{31} = 0+0+0+.2 = .2$

$U_{12} = .32(0) + .36(.2) + .32(.2) + .2 = .336$
$U_{22} = .32(.2) + .36(.2) + .32(.2) + .2 = .4$
$U_{32} = .32(.2) + .36(.2) + .32(0) + .2 = .336$ (Yes, there is symmetry about $x=0.5$.)

$U_{13} = .32(0) + .36(.336) + .32(.4) + .2 = .44896$
$U_{23} = .32(.336) + .36(.4) + .32(.336) + .2 = .55904$
$U_{33} = .32(.4) + .36(.336) + 0 + .2 = .44896$



6. $r = 0.32$, $H = 0$, $u(0,t) = 100$, $u(x,0) = u(1,t) = 0$,
$F(x,t) = 10\sin \pi x$, so

$$U_{j,k+1} = .32U_{j-1,k} + .36U_{jk} + .32U_{j+1,k} + .2\sin(j\pi/4)$$

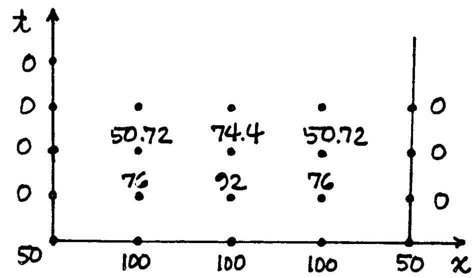
$U_{11} = .32(50)+0+0+(.7071)(.2) = 16.1414$
$U_{21} = 0+0+0+(1)(.2) = 0.2000$
$U_{31} = 0+0+0+(.7071)(.2) = 0.1414$

$U_{12} = .32(100)+.36(16.1414)+.32(.2)+(.7071)(.2) = 38.0163$
$U_{22} = .32(16.1414)+.36(.2)+.32(.1414)+(1)(.2) = 5.4825$
$U_{32} = .32(.2)+.36(.1414)+.32(0)+(.7071)(.2) = 0.2563$

$U_{13} = .32(100)+.36(38.0163)+.32(5.4825)+.2(.7071) = 47.5817$
$U_{23} = .32(38.0163)+.36(5.4825)+.32(.2563)+.2(1) = 14.4209$
$U_{33} = .32(5.4825)+.36(.2563)+0+.2(.7071) = 1.9881$

7. $r=0.32$, $H=4$, $u(0,t)=u(1,t)=0$, $u(x,0)=100$, $F(x,t)=0$, so

$$U_{j,k+1} = .32U_{j-1,k}+.28U_{jk}+.32U_{j+1,k}$$

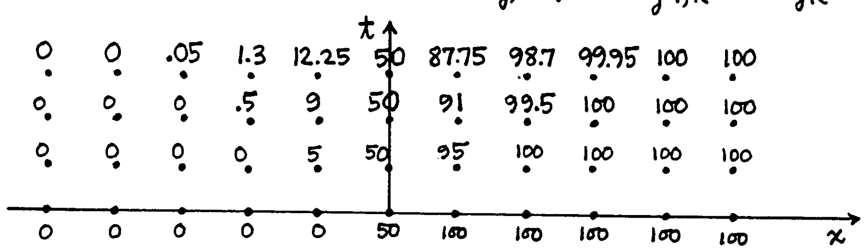


$U_{11} = .32(50)+.28(100)+.32(100) = 76$
$U_{21} = .32(100)+.28(100)+.32(100) = 92$
$U_{31} = .32(100)+.28(100)+.32(50) = 76$

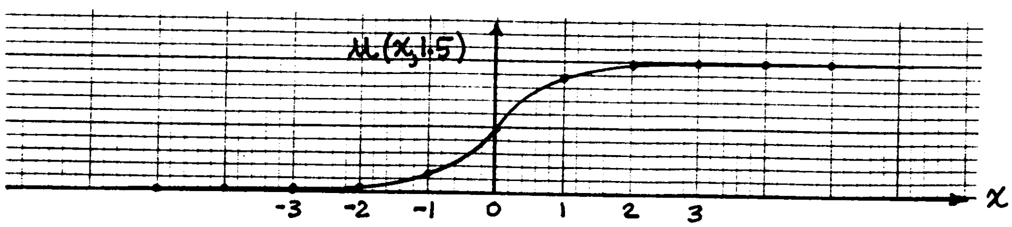
$U_{12} = .32(0)+.28(76)+.32(92) = 50.72$
$U_{22} = .32(76)+.28(92)+.32(76) = 74.4$
$U_{32} = .32(92)+.28(76)+.32(0) = 50.72$

$U_{13} = .32(0)+.28(50.72)+.32(74.4) = 38.0096$
$U_{23} = .32(50.72)+.28(74.4)+.32(50.72) = 53.2928$
$U_{33} = .32(74.4)+.28(50.72)+.32(0) = 38.0096$

8. $r=\alpha^2\Delta t/(\Delta x)^2 = 0.2(0.5)/(1)^2 = 0.1$, so $U_{j,k+1} = .1U_{j-1,k}+.8U_{jk}+.1U_{j+1,k}$



Let's plot the last one, at $t = 1.5$



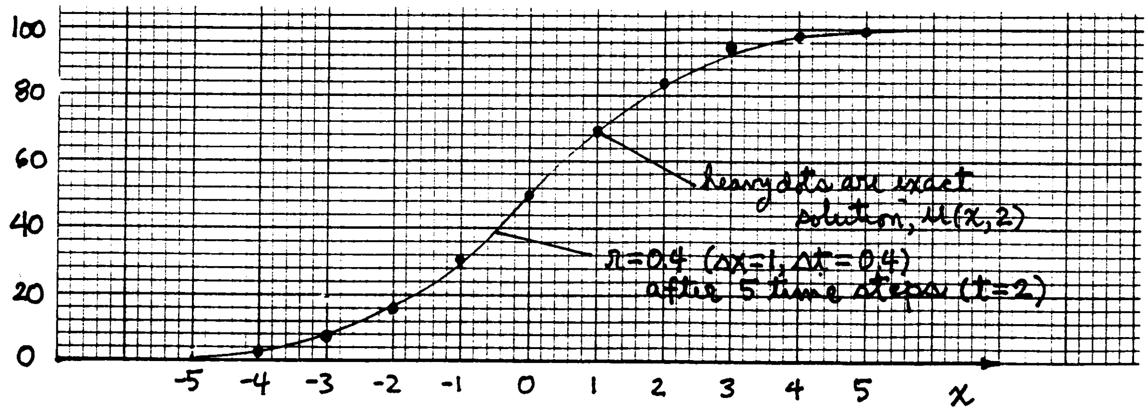
9. Let us do only the cases $\Delta t = 0.4$ (so $r = \alpha^2 \Delta t/(\Delta x)^2 = .4$) and $\Delta t = 0.6$ (so $r = .6$).
For $r = .4$, $U_{j,k+1} = .4U_{j-1,k} + .2U_{jk} + .4U_{j+1,k}$
For $r = .6$, $U_{j,k+1} = .6U_{j-1,k} - .2U_{jk} + .6U_{j+1,k}$

Here are the results for these two cases: upper numbers for $r = 0.4$, lower for 0.6.

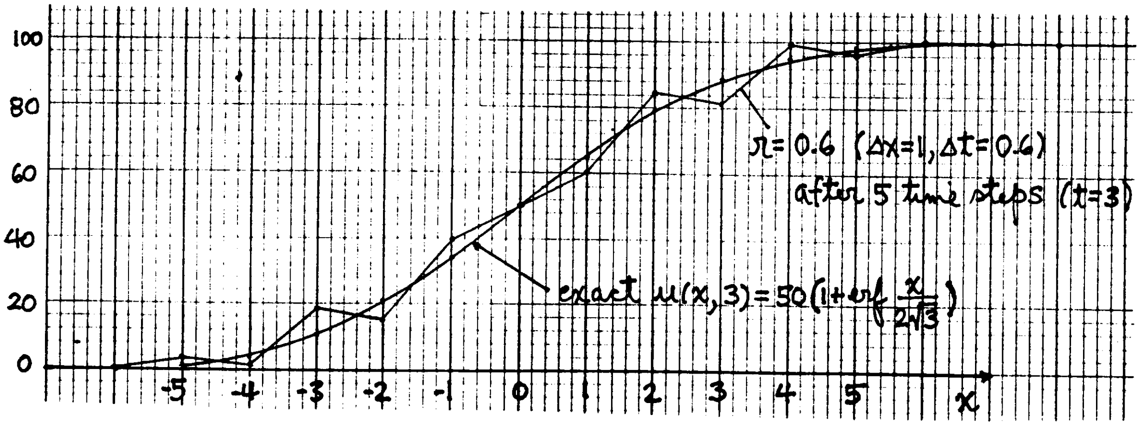
t \ x	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
	0 0	0.51 3.89	2.30 1.30	7.42 18.58	17.02 14.98	28.96 39.70	50 50	68.10 60.3	82.98 85.02	92.58 81.42	97.70 98.70	99.49 96.11	100 100
	0 0	0 0	1.28 6.48	5.12 4.32	14.72 25.92	30.08 29.28	50 50	69.92 70.72	85.28 74.08	94.88 95.68	98.72 93.52	100 100	100 100
	0 0	0 0	0 0	3.2 10.8	11.2 10.8	28 36	50 50	72 64	88.8 89.2	96.8 89.2	100 100	100 100	100 100
	0 0	0 0	0 0	0 0	8 18	24 24	50 50	76 76	92 82	100 100	100 100	100 100	100 100
	0 0	0 0	0 0	0 0	0 0	20 30	50 50	80 70	100 100	100 100	100 100	100 100	100 100
	0	0	0	0	0	0	50	100	100	100	100	100	100

Let us plot the final results (at $t = 5 \times 0.4 = 2$ for $r = .4$ and at $t = 5 \times .6 = 3$ for $r = .6$) together with the exact solution, namely, $u(x,t) = 50\left(1 + \operatorname{erf} \frac{x}{2\alpha\sqrt{t}}\right)$.

For $r = .4$:



For $r = .6$:



Of these two cases, the $r=0.6$ results reveal the anticipated instability. The $r=0.4$ results are stable but not very accurate since the grid is coarse. How can we tell it is coarse? Because most of the variation in u occurs over $-5<x<5$, so letting $\Delta x=1$ gives merely 10 subdivisions of that interval. Here are the comparisons:

x:	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
$r=.4$ eval. of u:	0.51	2.30	7.42	17.02	28.96	50	68.10	82.98	92.58	97.70	99.49	100
Exact u:	0.62	2.28	6.68	15.87	30.85	50	69.15	84.13	93.32	97.72	99.38	100

10. $\Delta x=2$, $\Delta t=.5$

Left $(0<x<6)$: $r=1.8(.5)/4=0.225$ so $U_{j,k+1}=.225U_{j-1,k}+.55U_{jk}+.225U_{j+1,k}$ ①

Right $(3<x<12)$: $r=.2(.5)/4=0.025$ so $U_{j,k+1}=.025U_{j-1,k}+.95U_{jk}+.025U_{j+1,k}$ ②

But the latter finite-difference formulas do not hold at grid points at $x=6$. There, proceed as suggested in the exercise:

$$K_L\,\frac{U_{3k}-U_{2k}}{\Delta x}=K_R\,\frac{U_{4k}-U_{3k}}{\Delta x}$$

gives

$$U_{3k}=\frac{K_LU_{2k}+K_RU_{4k}}{K_L+K_R}=.893U_{2k}+.107U_{4k} \qquad ③$$

so the idea is to use ① to compute U_{1k} and U_{2k} using ①, and U_{4k} and U_{5k} using ②, then U_{3k} using ③.

$k=1$: $U_{11}=.225(50)+0+0=11.25$, $U_{21}=\cdots=U_{51}=0$

$k=2$: $U_{12}=.225(100)+.55(11.25)+0=28.69$,

$U_{22}=.225(11.25)+0+0=2.53$

$U_{42}=0+0+0=0$, $U_{52}=0+0+0=0$,

$U_{32}=.893(2.53)+.107(0)=2.26$

$k=3$: $U_{13}=.225(100)+.55(28.69)+.225(2.53)=38.85$,

$U_{23}=.225(28.69)+.55(2.53)+.225(2.26)=8.36$,

$U_{43}=.025(2.26)+0+0=.06$, $U_{53}=0+0+0=0$,

$U_{33}=.893(8.36)+.107(.06)=7.47$

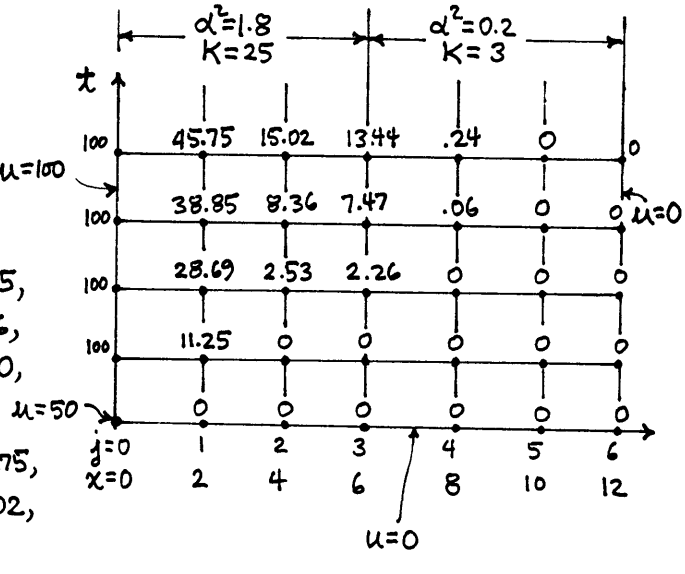
$k=4$: $U_{14}=.225(100)+.55(38.85)+.225(8.36)=45.75$,

$U_{24}=.225(38.85)+.55(8.36)+.225(7.47)=15.02$,

$U_{44}=.025(7.47)+.95(.06)+.025(0)=0.24$,

$U_{54}=0.25(.06)+0+0=0.00$,

$U_{34}=.893(15.02)+.107(.24)=13.44$



12. $u(x+\Delta x,t) = u(x,t) + u_x(x,t)\Delta x + \frac{1}{2}u_{xx}(x,t)(\Delta x)^2 + \frac{1}{6}u_{xxx}(x,t)(\Delta x)^3 + \cdots$

$u(x-\Delta x,t) = u(x,t) - u_x(x,t)\Delta x + \frac{1}{2}u_{xx}(x,t)(\Delta x)^2 - \frac{1}{6}u_{xxx}(x,t)(\Delta x)^3 + \cdots$

Addition gives $u(x+\Delta x,t) + u(x-\Delta x,t) = 2u(x,t) + u_{xx}(x,t)(\Delta x)^2 + O(\Delta x)^4$

Neglecting the $O(\Delta x)^4$ terms and solving for u_{xx} gives

$$u_{xx}(x,t) \approx \frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{(\Delta x)^2}$$

13. The final vector in (13.1) comes from the boundary conditions. That is, writing out (8) for an entire "time line":

$$\begin{aligned}
U_{1,k+1} &= rU_{0k} + (1-2r)U_{1k} + rU_{2k}\\
U_{2,k+1} &= rU_{1k} + (1-2r)U_{2k} + rU_{3k}\\
&\ \ \vdots\\
U_{N-2,k+1} &= rU_{N-3,k} + (1-2r)U_{N-2,k} + rU_{N-1,k}\\
U_{N-1,k+1} &= rU_{N-2,k} + (1-2r)U_{N-1,k} + rU_{Nk}
\end{aligned}$$

or,

$$\begin{array}{llr}
U_{1,k+1} = (1-2r)U_{1k} + rU_{2k} & & \text{Known from b.c.'s}\\
 & & +\,rU_{0k}\\
U_{2,k+1} = rU_{1k} + (1-2r)U_{2k} + rU_{2k} & & \\
\ \ \vdots & & \\
U_{N-2,k+1} = & rU_{N-3,k} + (1-2r)U_{N-2,k} + rU_{N-1,k} & \\
U_{N-1,k+1} = & rU_{N-2,k} + (1-2r)U_{N-1,k} & +\,rU_{Nk}
\end{array}$$

which, in matrix form, gives (13.1). Continuing as suggested, arrive at (13.4):

$$\mathbf{e}_{k+1} = \mathbf{A}\mathbf{e}_k + \mathbf{b}_k.$$

Thus,

$$\begin{aligned}
\mathbf{e}_1 &= \mathbf{A}\mathbf{e}_0 + \mathbf{b}_0 && \mathbf{0} \text{ since } \mathbf{b}_k = \mathbf{c}_k - \mathbf{c}_k^* \neq \mathbf{0} \text{ only for } k=0\\
\mathbf{e}_2 &= \mathbf{A}\mathbf{e}_1 + \mathbf{b}_1 = \mathbf{A}(\mathbf{A}\mathbf{e}_0 + \mathbf{b}_0) + \mathbf{b}_1 = \mathbf{A}^2\mathbf{e}_0 + \mathbf{A}\mathbf{b}_0 + \cancel{\mathbf{b}_1} && \mathbf{0}\\
\mathbf{e}_3 &= \mathbf{A}\mathbf{e}_2 + \mathbf{b}_2 = \mathbf{A}(\mathbf{A}^2\mathbf{e}_0 + \mathbf{A}\mathbf{b}_0) + \mathbf{b}_2 = \mathbf{A}^3\mathbf{e}_0 + \mathbf{A}^2\mathbf{b}_0 + \cancel{\mathbf{b}_3}
\end{aligned}$$

and so on, which gives (13.5). Now, the $(N-1)\times(N-1)$ matrix $\mathbf{A}$ is symmetric so it gives $N-1$ orthogonal — and hence LI — eigenvectors, which eigenvectors therefore comprise a basis. (We will not use the orthogonality but only need to be assured that they are LI and therefore do give a basis.) Putting (13.6,7) into (13.5) easily gives (13.8):

$$\begin{aligned}
\mathbf{e}_k &= (\alpha_1\lambda_1^k + \beta_1\lambda_1^{k-1})\boldsymbol{\Phi}_1 + \cdots + (\alpha_{N-1}\lambda_{N-1}^k + \beta_{N-1}\lambda_{N-1}^{k-1})\boldsymbol{\Phi}_{N-1},\\
&= \lambda_1^{k-1}(\alpha_1\lambda_1 + \beta_1)\boldsymbol{\Phi}_1 + \cdots + \lambda_{N-1}^{k-1}(\alpha_{N-1}\lambda_{N-1} + \beta_{N-1})\boldsymbol{\Phi}_{N-1}, \qquad (13.8)
\end{aligned}$$

where the α's and β's, from (13.6,7) can be considered as known and arbitrary. If all of the λ's are less than or $=1$ in magnitude then (13.8) shows that $\mathbf{e}_k \to \mathbf{0}$ as $k\to\infty$. (That is not to say that the total roundoff error $\to 0$ since $\mathbf{e}_k$ is only the roundoff error resulting from a single line of roundoffs, at $k=0$. Such "initial roundoffs" are actually being injected at each time line.) If any λ is greater than 1 in magnitude then $|\mathbf{e}_k| \to \infty$ as $k\to\infty$ and we have instability.

Thus, for stability set $|\lambda_n| \le 1$ for each $n = 1, 2, \dots, N-1$. Or, using (13.11),

$$-1 \le 1 - 2r + 2r\cos\frac{n\pi}{N} \le 1$$

gives $2r\left(\cos\frac{n\pi}{N} - 1\right) \ge -2$

$r\left(1 - \cos\frac{n\pi}{N}\right) \le 1$

$$r \le \frac{1}{1 - \cos\frac{n\pi}{N}}$$

$1 - 2r + 2r\cos\frac{n\pi}{N} \le 1$,

$2r\left(\cos\frac{n\pi}{N} - 1\right) \le 0$, which is always satisfied (since $r > 0$) and which is, therefore, uninformative

for $n = 1, 2, \dots, N-1$. Of these $N-1$ inequalities, the one with the smallest right-hand side supercedes all the others. The smallest right-hand side occurs when $n = N-1$, in which case we have

$$r \le \frac{1}{1 - \cos\left(\frac{N-1}{N}\pi\right)}.$$

14. $u_t = -Au \quad (A > 0)$

$$\rightarrow \frac{U_{k+1} - U_k}{\Delta t} = -(1-\theta)AU_k - \theta A U_{k+1},$$

$$U_{k+1} = \frac{1 - (1-\theta)A\Delta t}{1 + \theta A \Delta t} U_k \qquad ①$$

With an initial roundoff error $U_0 - U_0^* \equiv e_0 \ne 0$, and without any subsequent roundoff error (i.e., using a perfect computer thereafter) we have

$$U_{k+1}^* = \frac{1 - (1-\theta)A\Delta t}{1 + \theta A \Delta t} U_k^* \qquad ②$$

and subtracting ② from ① gives

$$e_{k+1} = K e_k, \qquad K \equiv \frac{1 - A(1-\theta)\Delta t}{1 + A\theta\Delta t}.$$

Thus,

$$e_1 = Ke_0,\ e_2 = Ke_1 = K^2 e_0,\ \dots,\ e_k = K^k e_0$$

so the scheme is stable if and only if $|K| \le 1$; i.e.,

$$-1 \le \frac{1 - A(1-\theta)\Delta t}{1 + A\theta\Delta t} \le 1$$

gives

$-1 - A\theta\Delta t \le 1 - A(1-\theta)\Delta t$

or, $[A(1-\theta) - A\theta]\Delta t \le 2$

or, $A(1-2\theta)A\Delta t \le 2$. ✓

gives $1 - A(1-\theta)\Delta t \le 1 + A\theta\Delta t$

or, $-A\Delta t \le 0$, which is always satisfied and where, therefore, is uninformative

15. Crank-Nicolson scheme: $-rU_{j-1,k+1} + 2(1+r)U_{j,k+1} - rU_{j+1,k+1} = rU_{j-1,k} + 2(1-r)U_{jk} + rU_{j+1,k}$

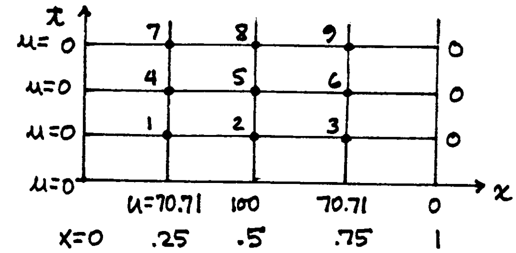
for $u_{xx} = u_t \quad (0 < x < 1)$

$u(0,t) = u(1,t) = 0,\ u(x,0) = 100\sin\pi x$

with $r = \alpha^2\Delta t/(\Delta x)^2 = (1)(.1)/(.25)^2 = 1.6$.

With $r=1.6$ the method is given by

$$-1.6U_{j-1,k+1}+5.2U_{j,k+1}-1.6U_{j+1,k+1}$$
$$=1.6U_{j-1,k}-1.2U_{jk}+1.6U_{j+1,k}$$



For this hand calculation it will be simpler to denote the grid points as 1…9. Thus,

$$0+5.2U_1-1.6U_2=0-1.2(70.71)+1.6(100)=75.15 \qquad (1)$$
$$-1.6U_1+5.2U_2-1.6U_3=1.6(70.71)-1.2(100)+1.6(70.71)=106.27 \qquad (2)$$
$$-1.6U_2+5.2U_3-0=1.6(100)-1.2(70.71)+0=75.15 \qquad (3)$$
$$0+5.2U_4-1.6U_5=0-1.2U_1+1.6U_2 \qquad (4)$$
$$-1.6U_4+5.2U_5-1.6U_6=1.6U_1-1.2U_2+1.6U_3 \qquad (5)$$
$$-1.6U_5+5.2U_6-0=1.6U_2-1.2U_3+0 \qquad (6)$$
$$0+5.2U_7-1.6U_8=0-1.2U_4+1.6U_5 \qquad (7)$$
$$-1.6U_7+5.2U_8-1.6U_9=1.6U_4-1.2U_5+1.6U_6 \qquad (8)$$
$$-1.6U_8+5.2U_9-0=1.6U_5-1.2U_6+0 \qquad (9)$$

We can solve these one line at a time. That is, we can solve (1)-(3) for U_1, U_2, U_3. Then put those values into the RHS's (right-hand sides) of (4)-(6) and solve (4)-(6) for U_4, U_5, U_6. Put those values into the RHS's of (7)-(9) and solve those for U_7, U_8, U_9. Alternatively, we could solve (1)-(9) as a linear system for the unknowns $U_1, \ldots, U_9$. Let us do that, using the Maple linsolve command:

```
> with(linalg):
Warning, new definition for norm
Warning, new definition for trace
> linsolve(array([[5.2,-1.6,0,0,0,0,0,0,0],[-1.6,5.2,-1.6,0,0,0,0,0,
  0],[0,-1.6,5.2,0,0,0,0,0,0],[1.2,-1.6,0,5.2,-1.6,0,0,0,0],[-1.6,1.
  2,-1.6,-1.6,5.2,-1.6,0,0,0],[0,-1.6,1.2,0,-1.6,5.2,0,0,0],[0,0,0,1
  .2,-1.6,0,5.2,-1.6,0],[0,0,0,-1.6,1.2,-1.6,-1.6,5.2,-1.6],[0,0,0,0
  ,-1.6,1.2,0,-1.6,5.2]]),array([75.15,106.27,75.15,0,0,0,0,0,0]));
[25.58448905, 36.18083939, 25.58448905, 9.256512057, 13.09119158, 9.256512057,
  3.349285247, 4.736369514, 3.349285247]
```

u_1 u_9

Compare these results with the exact solution, which is $u(x,t)=100\sin\pi x\, e^{-\pi^2 t}$.

$u_1=u(.25,.1)=100\sin(\pi/4)\exp(-.9869)\approx 26.35$

$u_2=u(.5,.1)\approx 37.27$, $u_3=u(.75,.1)=26.35$, $u_4=u(.25,.2)\approx 9.82$,

$u_5=u(.5,.2)\approx 13.89$, $u_6=u(.75,.2)\approx 9.82$, $u_7=u(.25,.3)\approx 3.66$

$u_8=u(.5,.3)\approx 5.18$, $u_9=u(.75,.3)\approx 3.66$

so the Crank-Nicolson results look okay – considering how coarse the grid is.

NOTE: By symmetry about $x=0.5$, it is evident that $U_3=U_1$, $U_6=U_4$, and $U_9=U_7$. We could have used this fact to work with six equations instead of nine, but we chose to do the nine equations and use the symmetry of the results as a partial check on those results.

16. **SOR:** $U_{11}^{(0)} = 37.5$
$U_{21}^{(0)} = 50$
$U_{31}^{(0)} = 15$

Tentative G-S step: $U_{11}^{(1)} = 50$ so $\Delta U_{11}^{(0)} = 50 - 37.5 = 12.5$
$U_{21}^{(1)} = 66.25$ so $\Delta U_{21}^{(0)} = 66.25 - 50 = 16.25$
$U_{31}^{(1)} = 31.56$ so $\Delta U_{31}^{(0)} = 31.56 - 15 = 16.56$

Now an SOR step: $U_{11}^{(1)} = U_{11}^{(0)} + \omega \Delta U_{11}^{(0)} = 37.5 + 1.03(12.5) = 50.38$
$U_{21}^{(1)} = U_{21}^{(0)} + \omega \Delta U_{21}^{(0)} = 50 + 1.03(16.25) = \boxed{66.74}$
$U_{31}^{(1)} = U_{31}^{(0)} + \omega \Delta U_{31}^{(0)} = 15 + 1.03(16.56) = 32.06$

Tentative G-S step: $U_{11}^{(2)} = \frac{1}{4}(U_{21}^{(1)} + 150) = \frac{1}{4}(66.74 + 150) = 54.19$
so $\Delta U_{11}^{(1)} = 54.19 - 50.38 = 3.81$
$U_{21}^{(2)} = \frac{1}{4}(U_{11}^{(2)} + U_{31}^{(1)} + 200) = \frac{1}{4}(54.19 + 32.06 + 200) = 71.56$
so $\Delta U_{21}^{(1)} = 71.56 - 66.74 = 4.82$
$U_{31}^{(2)} = \frac{1}{4}(U_{31}^{(2)} + 60) = \frac{1}{4}(71.56 + 60) = 32.89$
so $\Delta U_{31}^{(1)} = 32.89 - 32.06 = 0.83$

Now an SOR step: $U_{11}^{(2)} = U_{11}^{(1)} + \omega \Delta U_{11}^{(1)} = 50.38 + 1.03(3.81) = 54.30$
$U_{21}^{(2)} = U_{21}^{(1)} + \omega \Delta U_{21}^{(1)} = 66.74 + 1.03(4.82) = \boxed{71.70}$
$U_{31}^{(2)} = U_{31}^{(1)} + \omega \Delta U_{31}^{(1)} = 32.06 + 1.03(0.83) = 32.91$

Jacoby: $U_{11}^{(2)} = \frac{1}{4}(63.13 + 150) = 53.28$
$U_{21}^{(2)} = \frac{1}{4}(U_{11}^{(1)} + U_{31}^{(1)} + 200) = \frac{1}{4}(50 + 27.5 + 200) = \boxed{69.38}$
$U_{31}^{(2)} = \frac{1}{4}(U_{21}^{(1)} + 60) = \frac{1}{4}(63.13 + 60) = 30.78$
Further,
$U_{21}^{(3)} = \frac{1}{4}(U_{11}^{(2)} + U_{31}^{(2)} + 200) = \frac{1}{4}(53.28 + 30.78 + 200) = 71.02$

Gauss-Seidel: $U_{11}^{(2)} = \frac{1}{4}(66.25 + 150) = 54.06$
$U_{21}^{(2)} = \frac{1}{4}(54.06 + 31.56 + 200) = \boxed{71.41}$
$U_{31}^{(2)} = \frac{1}{4}(71.41 + 60) = 32.85$
Further,
$U_{11}^{(3)} = \frac{1}{4}(71.41 + 150) = 55.35$
$U_{21}^{(3)} = \frac{1}{4}(55.35 + 32.85 + 200) = 72.05$

18. $\underset{\sim}{c} = \sum_{1}^{N-1} c_j \underset{\sim}{\Phi}_j$

$\underset{\sim}{U}_{k+1}^{(0)} = \beta \sum c_j \underset{\sim}{\Phi}_j$ from (31).

$$\underset{\sim}{U}_{k+1}^{(1)} = \beta \underset{\sim}{c} - \beta \underset{\sim}{A}' \underset{\sim}{U}_{k+1}^{(0)} = \beta \sum c_j \underset{\sim}{\Phi}_j - \beta^2 \sum \lambda_j c_j \underset{\sim}{\Phi}_j = \beta \sum (1-\beta\lambda_j) c_j \underset{\sim}{\Phi}_j$$

$$\begin{aligned} \underset{\sim}{U}_{k+1}^{(2)} = \beta \underset{\sim}{c} - \beta \underset{\sim}{A}' \underset{\sim}{U}_{k+1}^{(1)} &= \beta \sum c_j \underset{\sim}{\Phi}_j - \beta \underset{\sim}{A}' \beta \sum (1-\beta\lambda_j) c_j \underset{\sim}{\Phi}_j \\ &= \quad '' \quad - \beta^2 \sum (1-\beta\lambda_j) c_j \lambda_j \underset{\sim}{\Phi}_j \\ &= \beta \sum (1-\beta\lambda_j + \beta^2\lambda_j^2) c_j \underset{\sim}{\Phi}_j \end{aligned}$$

$\vdots$

$$\underset{\sim}{U}_{k+1}^{(p)} = \beta \sum_{1}^{N-1} \underbrace{(1-\beta\lambda_j + \beta^2\lambda_j^2 - \cdots + (-1)^p \beta^p \lambda_j^p)}_{*} c_j \underset{\sim}{\Phi}_j$$

As $p \to \infty$, $*$ becomes a geometric series, which converges to $1/(1+\beta\lambda_j)$ if $|\beta\lambda_j| < 1$ and diverges otherwise. Since λ_j's are the eigenvalues of $\underset{\sim}{A}'$ and $\underset{\sim}{A}'$ is of the type in Exercise 7 of Section 11.2, with "a" = "c" = $-r$ and "b" = 0, then

$$\lambda_j = \text{``}b + 2\sqrt{ac}\cos\frac{j\pi}{n+1}\text{''} = 2r\cos\frac{j\pi}{N}$$

so

$$|\beta\lambda_j| = \frac{2r}{2(1+r)}\left|\cos\frac{j\pi}{N}\right| < \frac{r}{1+r} < 1$$

for each $j = 1, \ldots, N-1$ and for every positive value of r. Thus,

$$\lim_{p\to\infty} \underset{\sim}{U}_{k+1}^{(p)} = \beta \sum_{1}^{N-1} \frac{1}{1+\beta\lambda_j} c_j \underset{\sim}{\Phi}_j$$

and we need to show that the latter satisfies (26). Recalling that $\underset{\sim}{A} = 2(1+r)\underset{\sim}{I} + \underset{\sim}{A}'$ we have

$$\begin{aligned} \left(2(1+r)\underset{\sim}{I} + \underset{\sim}{A}'\right) \beta \sum_{1}^{N-1} \frac{1}{1+\beta\lambda_j} c_j \underset{\sim}{\Phi}_j &= \sum_{1}^{N-1} \frac{c_j}{1+\beta\lambda_j} \underset{\sim}{\Phi}_j + \sum_{1}^{N-1} \frac{\beta c_j \lambda_j}{1+\beta\lambda_j} \underset{\sim}{\Phi}_j \\ &= \sum_{1}^{N-1} \frac{1+\beta\lambda_j}{1+\beta\lambda_j} c_j \underset{\sim}{\Phi}_j = \sum_{1}^{N-1} c_j \underset{\sim}{\Phi}_j = \underset{\sim}{c}. \checkmark \end{aligned}$$