EXERCISE: 2 OR 3 DIMENSIONAL HEAT EQUATION

EE2020: PDE AND FUNCTIONS OF A COMPLEX VARIABLE
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Let us consider the two dimensional heat equation,

$$\alpha^2 \nabla^2 u = u_t,$$

where $u = u(x, y, t)$ in a *simply connected open region*[1] S, and u satisfies a Dirichlet boundary condition $u(x, y, t) = f(x, y)$ on ∂S (here, the symbol ∂ is used for denoting the boundary of a region), and an initial condition $u(x, y, 0) = g(x, y)$ in S. Below, we outline how to solve this kind of problems generally.

i. Find a steady-state solution $u^{(p)}(x, y)$ such that $\nabla^2 u^{(p)} = 0$, and $u^{(p)}\big|_{\partial S} = f(x, y)$. Note that we have shown in class that, if the solution exists, the solution is unique.

ii. Let $v(x, y, t) = u(x, y, t) - u^{(p)}(x, y)$. What boundary and initial conditions does $v(x, y, t)$ satisfy?

iii. Now, perform the following separation of space and time; seek $v(x, y, t) = V(x, y)T(t)$. Under this assumption, show that

$$\nabla^2 V = -\kappa^2 V(x, y), \tag{1}$$

and derive the corresponding differential equation for $T(t)$.

iv. Note that Equation (1) is an eigenvalue problem. Generally, we seek to find the set of all possible eigenvalues $\chi = \{\kappa_1, \kappa_2, ...\}$ that correspond to eigenfunctions $\{\phi_1(x, y), \phi_2(x, y), ...\}$, respectively; i.e., assume that we can find $\nabla^2 \phi_j(x, y) = -\kappa_j^2 \phi_j(x, y)$. It would be nice if these eigenfunctions form a complete and orthogonal basis for the space of all well-behaved function in S, so that any such function can be expanded as follows,

$$g(x, y) = \sum_{j=1}^{\infty} \alpha_j \phi_j(x, y). \tag{2}$$

Let us try to prove the orthogonality next.

v. Let v_1 and v_2 be two well-behaved function defined on S, and assume that $v_1 = v_2 = 0$ on ∂S. Use $\nabla \cdot (v_1 \nabla v_2) = (\nabla v_1) \cdot (\nabla v_2) + v_1 \nabla^2 v_2$ to show that

$$\iint_S v_1 \nabla^2 v_2 \; dxdy = -\iint_S \nabla v_1 \cdot \nabla v_2 \; dxdy = \iint_S v_2 \nabla^2 v_1 \; dxdy.$$

[1]definition of "simply connected" can be found in the text book, and we will also cover it soon.

vi. Refer to Text p. 898, or any Linear Algebra textbook, to see that the above equation is an example of *Lagrange identity*; it implies that the eigenvalues here are all real, and $\langle \phi_j, \phi_k \rangle = 0$ if $\kappa_j^2 \neq \kappa_k^2$.

vii. Therefore, indeed if we expand the initial condition $g(x, y)$ by eigenfunctions as in Eq. (2), the coefficients are quite simple to obtain. Write down an expression for the coefficients α_j. To finish, write down the general solution $u(x, y, t)$. Remarks: remember to use the correct initial condition for $v(x, y, t)$, and remember to include $u^{(p)}$.