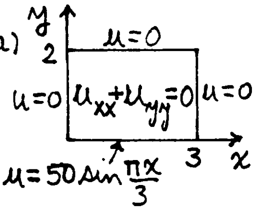
CHAPTER 20

Section 20.2

1. (a)



Note: For the sake of space, we don't always include a picture like the one at the left, but we always urge the student to begin with a simple picture or sketch whenever there is one that is relevant. Also for brevity, our solutions often omit steps and details that the student should include, such as the separation process and derivation of the product solution forms.

$$u(x,y) = (A+Bx)(C+Dy) + (E\sin kx + F\cos kx)(G\sinh ky + H\cosh ky)$$

$$u(0,y)=0= A(\ \text{"}\ ) + F(\ \text{"}\quad \text{"}\ ) \Rightarrow A=F=0, \text{ so}$$

$$u(x,y) = x(C'+D'y) + \sin kx\,(G'\sinh ky + H'\cosh ky)$$

$$u(3,y)=0 = 3(C'+D'y) + \sin 3k(\ \text{"}\quad \text{"}\ ) \Rightarrow C'=D'=0,\ 3k=n\pi\ (n=1,2,\dots)$$

so $$u(x,y) = \sum_1^\infty \sin\frac{n\pi x}{3}\left(G_n'\sinh\frac{n\pi y}{3} + H_n'\cosh\frac{n\pi y}{3}\right)$$

$$u(x,0) = 50\sin\frac{\pi x}{3} = \sum_1^\infty H_n'\sin\frac{n\pi x}{3} \implies H_1'=50,\ \text{others}=0$$

so

$$u(x,y) = 50\sin\frac{\pi x}{3}\cosh\frac{\pi y}{3} + \sum_1^\infty G_n'\sin\frac{n\pi x}{3}\sinh\frac{n\pi y}{3}$$

$$u(x,2)=0 = 50\sin\frac{\pi x}{3}\cosh\frac{2\pi}{3} + \sum_1^\infty G_n'\sinh\frac{2n\pi}{3}\sin\frac{n\pi x}{3}$$

$$-50\cosh\frac{2\pi}{3}\sin\frac{\pi x}{3} = \sum_1^\infty G_n'\sinh\frac{2n\pi}{3}\sin\frac{n\pi x}{3} \implies -50\cosh\frac{2\pi}{3} = G_1'\sinh\frac{2\pi}{3}$$

so $G_1' = -50\coth\frac{2\pi}{3}$, others $=0$

so $$u(x,y) = 50\sin\frac{\pi x}{3}\left(\cosh\frac{\pi y}{3} - \coth\frac{2\pi}{3}\sinh\frac{\pi y}{3}\right)$$

NOTE: The latter can be expressed more cogently by using the identity $\sinh(A-B) = \sinh A\cosh B - \cosh A\sinh B$.

$$u(x,y) = 50\sin\frac{\pi x}{3}\,\frac{\cosh\pi y/3\,\sinh 2\pi/3 - \cosh 2\pi/3\,\sinh\pi y/3}{\sinh 2\pi/3}$$

$$= 50\sin\frac{\pi x}{3}\,\frac{\sinh\frac{\pi}{3}(2-y)}{\sinh\frac{2\pi}{3}}.$$ Normally we will not carry out such rearrangement, but it can be <u>important</u>: see solution to Exercise 2d, below.

(b)
$$u(x,y) = (A+Bx)(C+Dy) + (E\sin kx + F\cos kx)(G\sinh ky + H\cosh ky)$$

$$u(0,y) = A(\ \text{"}\ ) + F(\ \text{"}\quad \text{"}\ ) \rightarrow A=F=0$$

$$u(x,y) = x(C'+D'y) + \sin kx\,(G'\sinh ky + H'\cosh ky)$$

$$u(x,0)=0 = C'x + \sin kx\,(H') \rightarrow C'=H'=0$$

$$u(x,y) = D'xy + G'\sin kx\sinh ky$$

$u(3,y)=0= 3D'y + G'\sin 3k \sinh ky \rightarrow D'=0,\ 3k=n\pi\ (n=1,2,\dots)$

$u(x,y)= \sum_1^\infty G_n' \sin\frac{n\pi x}{3}\sinh\frac{n\pi y}{3}$

$u(x,2) = 10\sin(\pi x/3) - 4\sin\pi x = \sum_1^\infty G_n' \sinh\frac{2n\pi}{3}\sin\frac{n\pi x}{3}$

so $G_1'\sinh\frac{2\pi}{3} = 10,\ G_3'\sinh 2\pi = -4$

$$u(x,y) = \frac{10}{\sinh\frac{2\pi}{3}}\sin\frac{\pi x}{3}\sinh\frac{\pi y}{3} - \frac{4}{\sinh 2\pi}\sin\pi x \sinh\pi y$$

(c) This time choose $X''/X = -Y''/Y = +k^2$ since the expansion will be in y. Thus,

$u(x,y) = (A+Bx)(C+Dy) + (E\sinh kx + F\cosh kx)(G\sin ky + H\cos ky)$

$u(x,0)=0=(A+Bx)C + (\quad " \quad)H \rightarrow C=H=0$

$u(x,y) = (A'+B'x)y + (E'\sinh kx + F'\cosh kx)\sin ky$

$u(x,2) = (\quad " \quad)2 + (\quad " \quad)\sin 2k \rightarrow A'=B'=0,\ 2k=n\pi,$

$u(x,y) = \sum_1^\infty (E_n'\sinh\frac{n\pi x}{2} + F_n'\cosh\frac{n\pi x}{2})\sin\frac{n\pi y}{2}$

$u(0,y) = 5\sin\pi y + 4\sin 2\pi y - \sin 3\pi y = \sum_1^\infty F_n'\sin\frac{n\pi y}{2}$

so $F_2'=5,\ F_4'=4,\ F_6'=-1$, others $=0$, so

$u(3,y)=0=\sum_1^\infty (E_n'\sinh\frac{3n\pi}{2} + F_n'\cosh\frac{3n\pi}{2})\sin\frac{n\pi y}{2}$

gives $E_n' = -F_n'\coth(3n\pi/2)$ for all n,

so

$$u(x,y) = \sum_1^\infty F_n'\left(\cosh\frac{n\pi x}{2} - \coth\frac{3n\pi}{2}\sinh\frac{n\pi x}{2}\right)\sin\frac{n\pi y}{2}$$

$$= 5(\cosh\pi x - \coth 3\pi\sinh\pi x)\sin\pi y + 4(\cosh 2\pi x - \coth 6\pi \sinh 2\pi x)\sin 2\pi y - (\cosh 3\pi x - \coth 9\pi\sinh 3\pi x)\sin 3\pi y$$

(e) Expansion will be on x, so use $-k^2$.

$u(x,y) = (A+Bx)(C+Dy) + (E\sin kx + F\cos kx)(G\sinh ky + H\cosh ky)$

$u_x(0,y) = 0 = B(\quad " \quad) + \quad kE \quad (\quad " \quad) \rightarrow B=E=0$

$u(x,y) = C'+D'y + \cos kx\,(G'\sinh ky + H'\cosh ky)$

$u(3,y)=0 = C'+D'y + \cos 3k\,(\quad " \quad) \rightarrow C'=D'=0,\ 3k=n\pi/2\ (n \text{ odd})$

$u(x,y) = \sum_{1,3,\dots} \cos\frac{n\pi x}{6}(G_n'\sinh\frac{n\pi y}{6} + H_n'\cosh\frac{n\pi y}{6})$ ①

$u(x,0) = 50H(x-2) = \sum_{1,3,\dots}^\infty H_n'\cos\frac{n\pi x}{6}$

QRC: $H_n' = \frac{2}{3}\int_0^3 50H(x-2)\cos\frac{n\pi x}{6}\,dx = \frac{200}{n\pi}\left(\sin\frac{n\pi}{2} - \sin\frac{n\pi}{3}\right)$ ②

$u(x,2) = 0 = \sum_{1,3,\dots}^\infty (G_n'\sinh\frac{n\pi}{3} + H_n'\cosh\frac{n\pi}{3})\cos\frac{n\pi x}{3}$

so $G_n' \sinh \frac{n\pi}{3} + H_n' \cosh \frac{n\pi}{3} = 0$ ③

The solution is given by ①, where H_n' is given by ② and then G_n' by ③.

(f) Expansion will be on x, so use $-k^2$.

$u(x,y) = (A+Bx)(C+Dy) + (E\sin kx + F\cos kx)(G\sinh ky + H\cosh ky)$

$u(0,y) = 0 = A(\ \text{"}\ ) + \qquad F\ (\ \text{"} \qquad \text{"}\ ) \rightarrow A=F=0$

$u(x,y) = x(C'+D'y) + \sin kx\,(G'\sinh ky + H'\cosh ky)$

$u_x(3,y) = 0 = C'+D'y + k\cos 3k\,(\ \text{"} \qquad \text{"}\ ) \rightarrow C'=D'=0,\ 3k=\frac{n\pi}{2}$ (n odd)

$u(x,y) = \sum_{1,3,\dots} \sin\frac{n\pi x}{6}\left(G_n'\sinh\frac{n\pi y}{6} + H_n'\cosh\frac{n\pi y}{6}\right)$ ①

$u(x,0) = 50H(x-2) = \sum_{1,3,\dots}^{\infty} H_n' \sin\frac{n\pi x}{6}$

QRS: $H_n' = \frac{2}{3}\int_0^3 50H(x-2)\sin\frac{n\pi x}{6}\,dx = \frac{200}{n\pi}\left(\cos\frac{n\pi}{3} - \cos\frac{n\pi}{2}\right)$ ②

$u(x,2) = 0 = \sum_{1,3,\dots}^{\infty}\left(G_n'\sinh\frac{n\pi}{3} + H_n'\cosh\frac{n\pi}{2}\right)\sin\frac{n\pi x}{6}$

gives $\qquad G_n'\sinh\frac{n\pi}{3} + H_n'\cosh\frac{n\pi}{2} = 0$ ③

so the solution is given by ①, with H_n' and G_n' given by ② and ③.

(g) Expansion will be on x, so use $-k^2$.

$u(x,y) = (A+Bx)(C+Dy) + (E\sin kx + F\cos kx)(G\sinh ky + H\cosh ky)$

$u_x(0,y) = 0 = B(\ \text{"}\ ) \qquad + kE\ (\ \text{"} \qquad \text{"}\ ) \rightarrow B=E=0$

$u(x,y) = C'+D'y + \cos kx\,(G'\sinh ky + H'\cosh ky)$

$u_x(3,y) = 0 = \quad -k\sin 3k\,(\ \text{"} \qquad \text{"}\ ) \rightarrow 3k = n\pi \quad (n=1,2,\dots)$

$u(x,y) = C'+D'y + \sum_1^{\infty}\cos\frac{n\pi x}{3}\left(G_n'\sinh\frac{n\pi y}{3} + H_n'\cosh\frac{n\pi y}{3}\right)$ ①

$u(x,2) = 0 = C'+D'2 + \sum_1^{\infty}\left(G_n'\sinh\frac{2n\pi}{3} + H_n'\cosh\frac{2n\pi}{3}\right)\cos\frac{n\pi x}{3}$

HRC: $C'+2D'=0$, ②

$G_n'\sinh\frac{2n\pi}{3} + H_n'\cosh\frac{2n\pi}{3} = 0$ ③

$u(x,0) = 50H(x-2) = C' + \sum_1^{\infty} H_n'\cos\frac{n\pi x}{3}$

HRC: $C' = \frac{1}{3}\int_0^3 50H(x-2)\,dx = 50/3$ ④

$H_n' = \frac{2}{3}\int_0^3 50H(x-2)\cos\frac{n\pi x}{3}\,dx = -\frac{100}{n\pi}\sin\frac{2n\pi}{3}$ ⑤

so $u(x,y)$ is given by ①, with C', D', G_n', H_n' given by ②-⑤.

2(d) In Exercise 1(e) we obtained this solution:

$$u(0,y) = \sum_{1,3,\dots} \left[\left(-\coth\frac{n\pi}{3}\right)\sinh\frac{n\pi y}{6} + \cosh\frac{n\pi y}{6}\right]\frac{200}{n\pi}\left(\sin\frac{n\pi}{2} - \sin\frac{n\pi}{3}\right)\cancel{\cos\frac{n\pi x}{6}}\,1$$

If, for a particular value of y we sum 10 terms, 20 terms, 30, etc., we find that the results fail to settle down to a limit. Why?? Observe that as $n\to\infty$, $\coth\frac{n\pi}{3}\to 1$ so $[\;]\sim\left[-\sinh\frac{n\pi y}{6} + \cosh\frac{n\pi y}{6}\right]$

$$= \frac{1}{2}\left\{-\cancel{e^{n\pi y/6}} + e^{-n\pi y/6} + \cancel{e^{n\pi y/6}} + e^{-n\pi y/6}\right\} = e^{-n\pi y/6} \to 0.$$

However, the approach to zero is only by virtue of the cancellation of almost-equal oppositely signed large numbers (observe the $e^{n\pi y/6}$'s within the $-\sinh n\pi y/6$ and the $+\cosh n\pi y/6$). Carrying only a limited number of decimal places, Maple is unable to handle this calculation — as it stands.* BUT, as discussed above (See the NOTE in Exercise 1(a)), we can express the solution in the alternative form

$$u(0,y) = \sum_{1,3,\dots} \frac{200}{n\pi}\,\frac{\sin\frac{n\pi}{2} - \sin\frac{n\pi}{3}}{\sinh\frac{n\pi}{3}}\,\sinh\frac{n\pi(2-y)}{6}, \qquad \text{♀}$$

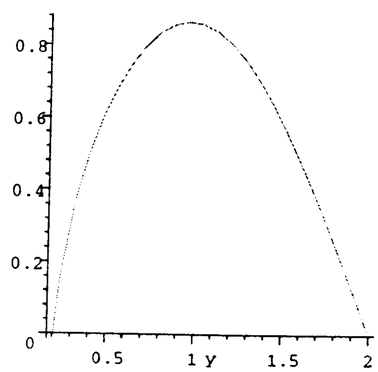
which contains a ratio of large numbers (the sinh's) rather than a difference of large numbers. Applying the Maple sum command to ♀ at $y=0.25$ gives

```
> sum((200/(2*i-1)/Pi)*(sin((2*i-1)*Pi/2)-sin((2*i-1)*Pi/3))*sinh((2
  *i-1)*Pi*(2-.25)/6)/sinh((2*i-1)*Pi/3),i=1..25);
                          .350771873
```

Similarly for $y = .5, .75, 1, 1.25, 1.5, 1.75, 2$ we obtain (settled down to three significant figures) .638, .816, .862, .778, .584, .312, 0. But it is easier to plot directly per

```
> with(plots):
> implicitplot(u=sum((200/(2*i-1)/Pi)*(sin((2*i-1)*Pi/2)-sin((2*i-1)
  *Pi/3))*sinh((2*i-1)*Pi*(2-y)/6)/sinh((2*i-1)*Pi/3),i=1..10),y=0..
  2,u=0..3);
```

$u(0,y)$



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* To clarify this point let us focus on the Maple calculation of the left- and right-hand sides of the identity

$$\sinh\frac{n\pi(2-y)}{6} = \sinh\frac{n\pi}{3}\cosh\frac{n\pi y}{6} - \sinh\frac{n\pi y}{6}\cosh\frac{n\pi y}{3}$$

with $y=1$, say. Increasing n we obtain

	$n=1$	$n=10$	$n=20$	$n=21$	$n=22$
LHS =	.5478534741	93.955	17655.95	29804.87	50313.36
RHS =	.5478534728	93.955	20000	30000	0

of which the LHS values are trustworthy and the RHS values are not; they are obtained as the difference of large numbers.

3\. (a) From the bdy conditions there seems to be a bleak future for the $k=0$ terms so let us try omitting them. (Go ahead and do this if you can see your way clearly; if not, play it safe.)

$$u(x,y) = (A\sin kx + B\cos kx)(C\sinh ky + D\cosh ky)$$

$$u(0,y) = 0 = B(\quad '' \quad '' \quad) \rightarrow B=0$$

$$u(x,y) = \sin kx\,(C'\sinh ky + D'\cosh ky)$$

$$u(2,y) = 0 = \sin 2k\,(\quad '' \quad '' \quad) \rightarrow 2k = n\pi \quad (n=1,2,\dots)$$

$$u(x,y) = \sum_1^\infty \sin\frac{n\pi x}{2}\left(C_n'\sinh\frac{n\pi y}{2} + D_n'\cosh\frac{n\pi y}{2}\right)$$

$$u(x,0) = 100\sin\frac{\pi x}{2} = \sum_1^\infty D_n'\sin\frac{n\pi x}{2} \rightarrow D_1' = 100,\ \text{all others} = 0$$

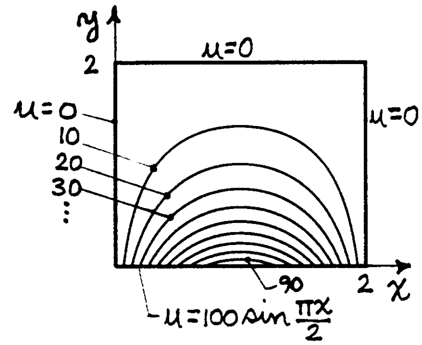
and $u(x,2)=0$ then gives the C_n''s, but it is more convenient to satisfy $u(x,2)=0$ first by expressing

$$u(x,y) = \sum_1^\infty E_n \sin\frac{n\pi x}{2}\sinh\frac{n\pi}{2}(2-y).$$

Then $u(x,0) = 100\sin\frac{\pi x}{2} = \sum E_n \sin\frac{n\pi x}{2}\sinh n\pi$

so $E_1\sinh\pi = 100$, others $=0$. Thus,

$$u(x,y) = 100\sin\frac{\pi x}{2}\,\frac{\sinh\frac{\pi}{2}(2-y)}{\sinh\pi}$$



(b) Let's jump in with

$$u(x,y) = \sin\frac{\pi x}{2}\left(C'\sinh\frac{\pi y}{2} + D'\cosh\frac{\pi y}{2}\right)$$

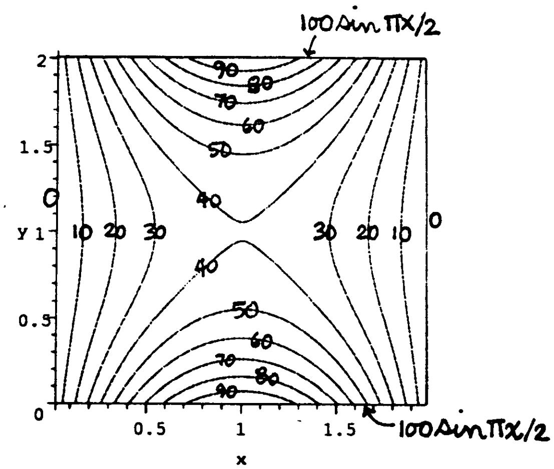
$$u(x,0) = 100\sin\frac{\pi x}{2} = D'\sin\frac{\pi x}{2} \rightarrow D' = 100.$$

$$u(x,2) = 100\sin\frac{\pi x}{2} = \sin\frac{\pi x}{2}\left(C'\sinh\pi + 100\cosh\pi\right)$$

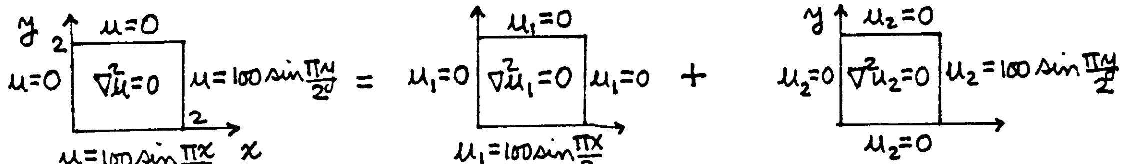
so $C' = 100(1-\cosh\pi)/\sinh\pi$ and

$$u(x,y) = 100\sin\frac{\pi x}{2}\left(\frac{1-\cosh\pi}{\sinh\pi}\sinh\frac{\pi y}{2} + \cosh\frac{\pi y}{2}\right)$$

```
> with(plots):
> u:=100*sin(Pi*x/2)*((1-cosh(Pi))*sinh(Pi*y/2)+sinh(Pi)*cosh(Pi*y/2
  ))/sinh(Pi):
> implicitplot({u=10,u=20,u=30,u=40,u=50,u=60,u=70,u=80,u=90},x=0..2
  ,y=0..2,grid=[100,100]);
```



(c)

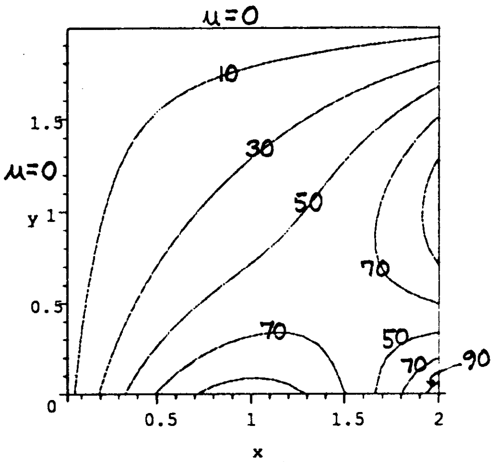


From (a), $u_1(x,y) = 100 \sin\frac{\pi x}{2} \, \frac{\sinh \frac{\pi}{2}(2-y)}{\sinh \pi}$

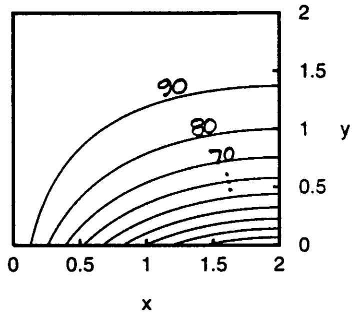
Similarly, $u_2(x,y) = 100 \sin\frac{\pi y}{2} \, \frac{\sinh \frac{\pi}{2}x}{\sinh \pi}$

Adding these,

$$u(x,y) = \frac{100}{\sinh \pi}\left(\sin\frac{\pi x}{2} \sinh\frac{\pi}{2}(2-y) + \sin\frac{\pi y}{2}\sinh\frac{\pi x}{2}\right)$$

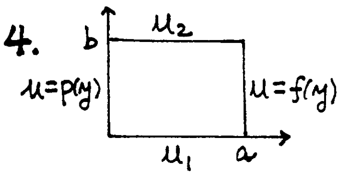


(d)



$$u(x,y) = 100 \sin\frac{\pi x}{4}\left(\cosh\left(\frac{\pi y}{4}\right) - \frac{\sinh(\pi y/4)}{\tanh(\pi/2)}\right)$$

4.



$u(x,y) = (A+Bx)(C+Dy) + (E\cosh kx + F\sinh kx)(G\cos ky + H\sin ky)$

$u(x,0) = u_1 = ($" $)C + ($ " " $)G \rightarrow B=0, AC=u_1, G=0$

$u(x,y) = u_1 + D'y + (E'\cosh kx + F'\sinh kx)\sin ky$

$u(x,b) = u_2 = u_1 + D'b + ($ " " $)\sin kb$

so $D' = (u_2-u_1)/b$, $kb = n\pi$ $(n=1,2,\dots)$

$$u(x,y) = u_1 + (u_2-u_1)\frac{y}{b} + \sum_1^\infty \left(E_n'\cosh\frac{n\pi x}{b} + F_n'\sinh\frac{n\pi x}{b}\right)\sin\frac{n\pi y}{b} \qquad ①$$

$$u(0,y) = p(y) = u_1 + (u_2-u_1)\frac{y}{b} + \sum_1^\infty E_n'\sin\frac{n\pi y}{b}$$

or,

$$p(y) - u_1 - (u_2-u_1)\frac{y}{b} = \sum_1^\infty E_n'\sin\frac{n\pi y}{b} \qquad (0<y<b)$$

HRS: $E_n' = \frac{2}{b}\int_0^b \left[p(y) - u_1 - (u_2-u_1)\frac{y}{b}\right]\sin\frac{n\pi y}{b}\,dy \qquad ②$

Finally,

$$u(a,y) = f(y) = u_1 + (u_2-u_1)\frac{y}{b} + \sum_1^\infty \left(E_n'\cosh\frac{n\pi a}{b} + F_n'\sinh\frac{n\pi a}{b}\right)\sin\frac{n\pi y}{b}$$

HRS:

$$E_n'\cosh\frac{n\pi a}{b} + F_n'\sinh\frac{n\pi a}{b} = \frac{2}{b}\int_0^b \left[f(y) - u_1 - (u_2-u_1)\frac{y}{b}\right]\sin\frac{n\pi y}{b}\,dy \qquad ③$$

The solution is given by ①-③.

5. (a) $u(x,y) = (A+Bx)(C+Dy) + (E\cos kx + F\sin kx)(G\cosh ky + H\sinh ky)$

$u(0,y) = u_1 = A(C+Dy) + E(\quad " \quad " \quad) \to AC = u_1, D=0, E=0$

$u(x,y) = u_1 + B'x + \sin kx(G'\cosh ky + H'\sinh ky)$

$u(a,y) = u_2 = u_1 + B'a + \sin ka(\quad " \quad " \quad) \to B' = (u_2-u_1)/a,\ ka = n\pi$

$u(x,y) = u_1 + (u_2-u_1)\frac{x}{a} + \sum_1^\infty \sin\frac{n\pi x}{a}\left(G_n'\cosh\frac{n\pi y}{a} + H_n'\sinh\frac{n\pi y}{a}\right)$ ①

$u(x,0) = f(x) = u_1 + (u_2-u_1)\frac{x}{a} + \sum_1^\infty G_n'\sin\frac{n\pi x}{a}$

HRS:

$$G_n' = \frac{2}{a}\int_0^a \left[f(x) - u_1 - (u_2-u_1)\frac{x}{a}\right]\sin\frac{n\pi x}{a}\,dx \quad ②$$

Then

$u(x,b) = p(x) = u_1 + (u_2-u_1)\frac{x}{a} + \sum_1^\infty \left(G_n'\cosh\frac{n\pi b}{a} + H_n'\sinh\frac{n\pi b}{a}\right)\sin\frac{n\pi x}{a}$

HRS:

$$G_n'\cosh\frac{n\pi b}{a} + H_n'\sinh\frac{n\pi b}{a} = \frac{2}{a}\int_0^a \left[p(x) - u_1 - (u_2-u_1)\frac{x}{a}\right]\sin\frac{n\pi x}{a}\,dx \quad ③$$

Solution given by ①-③.

(b) This time we anticipate having to expand $p(y)$ and $f(y)$, so choose the sign of k^2 so as to obtain sines and cosines of y, not x.

$u(x,y) = (A+Bx)(C+Dy) + (E\cosh kx + F\sinh kx)(G\cos ky + H\sin ky)$

$u(x,0) = u_2 = (A+Bx)C + (\quad " \quad " \quad)G \to AC = u_2, B=0, G=0$

$u(x,y) = u_2 + D'y + (E'\cosh kx + F'\sinh kx)\sin ky$

$u(x,b) = u_1 = u_2 + D'b + (\quad " \quad " \quad)\sin kb \to D' = (u_1-u_2)/b,\ kb = n\pi$

$u(x,y) = u_2 + (u_1-u_2)\frac{y}{b} + \sum_1^\infty \left(E_n'\cosh\frac{n\pi x}{b} + F_n'\sinh\frac{n\pi x}{b}\right)\sin\frac{n\pi y}{b}$ ①

$u_x(0,y) = p(y) = 0 + \sum_1^\infty \frac{n\pi}{b}F_n'\sin\frac{n\pi y}{b} \quad (0<y<b)$

HRS:

$$\frac{n\pi}{b}F_n' = \frac{2}{b}\int_0^b p(y)\sin\frac{n\pi y}{b}\,dy, \quad F_n' = \frac{2}{n\pi}\int_0^b p(y)\sin\frac{n\pi y}{b}\,dy \quad ②$$

$u(a,y) = f(y) = u_2 + (u_1-u_2)\frac{y}{b} + \sum_1^\infty \left(E_n'\cosh\frac{n\pi a}{b} + F_n'\sinh\frac{n\pi a}{b}\right)\sin\frac{n\pi y}{b} \quad (0<y<b)$

HRS:

$$E_n'\cosh\frac{n\pi a}{b} + F_n'\sinh\frac{n\pi a}{b} = \frac{2}{b}\int_0^b \left[f(y) - u_2 - (u_1-u_2)\frac{y}{b}\right]\sin\frac{n\pi y}{b}\,dy \quad ③$$

Solution is given by ①-③.

(c) As in (b), $u(x,y) = u_2 + (u_1-u_2)\frac{y}{b} + \sum_1^\infty \left(E_n'\cosh\frac{n\pi x}{b} + F_n'\sinh\frac{n\pi x}{b}\right)\sin\frac{n\pi y}{b}$ ①

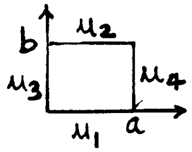
$u_x(0,y) = p(y)$ gives, as in (b),

$$F_n' = \frac{2}{n\pi}\int_0^b p(y)\sin\frac{n\pi y}{b}\,dy \quad ②$$

Then $u_x(a,y) = f(y)$ gives

$$E_n'\sinh\frac{n\pi a}{b} + F_n'\cosh\frac{n\pi a}{b} = \frac{2}{n\pi}\int_0^b \left[f(y) - u_2 - (u_1-u_2)\frac{y}{b}\right]\sin\frac{n\pi y}{b}\,dy \quad ③$$

and the solution is given by ①-③.

6. 

$\frac{X''}{X} = -\frac{Y''}{Y} = +k^2$ leads to

$$u(x,y) = u_1 + (u_2 - u_1)\frac{y}{b} + \sum_1^\infty \left(A_n \cosh\frac{n\pi x}{b} + B_n \sinh\frac{n\pi x}{b}\right)\sin\frac{n\pi y}{b}$$

$$A_n = \frac{2}{b}\int_0^b \left[u_3 - u_1 - (u_2 - u_1)\frac{y}{b}\right]\sin\frac{n\pi y}{b}\,dy$$

$$A_n \cosh\frac{n\pi a}{b} + B_n \sinh\frac{n\pi a}{b} = \frac{2}{b}\int_0^b \left[u_4 - u_1 - (u_2 - u_1)\frac{y}{b}\right]\sin\frac{n\pi y}{b}\,dy$$

alternatively,

$\frac{X''}{X} = -\frac{Y''}{Y} = -k^2$ leads to

$$u(x,y) = u_3 + (u_4 - u_3)\frac{x}{a} + \sum_1^\infty \sin\frac{n\pi x}{a}\left(A_n \cosh\frac{n\pi y}{a} + B_n \sinh\frac{n\pi y}{a}\right)$$

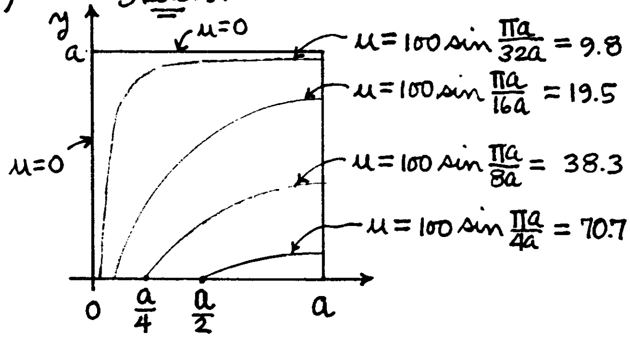
$$A_n = \frac{2}{a}\int_0^a \left[u_1 - u_3 - (u_4 - u_3)\frac{x}{a}\right]\sin\frac{n\pi x}{a}\,dx$$

$$A_n \cosh\frac{n\pi b}{a} + B_n \sinh\frac{n\pi b}{a} = \frac{2}{a}\int_0^a \left[u_2 - u_3 - (u_4 - u_3)\frac{x}{a}\right]\sin\frac{n\pi x}{a}\,dx$$

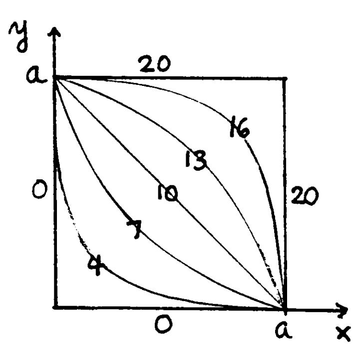
7. If $b = a$ and $f(y) = g(x) = p(y) = q(x) = 100$, then (see Fig. 4) it is clear that $u_1 = u_2 = u_3 = u_4$ at the center, $(a/2, a/2)$. Also clear is that u at the center (in fact everywhere in the rectangle) is 100. Thus, $u_1 + u_2 + u_3 + u_4 = 100$ or, since $u_1 = u_2 = u_3 = u_4$ at the center, $4u_1 = 100$, $u_1 = 25$ (at the center).

8. (a) Same as in Exercise 3(a), with the "2"'s changed to "a"'s.

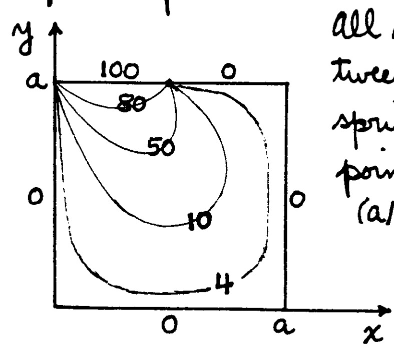
(b) Sketch:



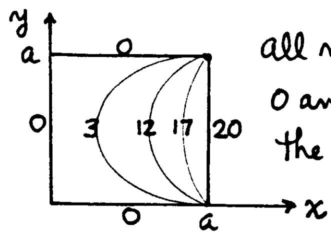
The sketch should be "topologically correct" – i.e., in its key features. In particular, each isotherm must be horizontal at the $x = a$ edge because $u_x(a,y) = 0$.

(c) 

(d) Remember, these are rough sketches, not computer plots of actual solutions.



All values between 0 and 100 spring from the points $(0,a)$ and $(a/2, a)$.

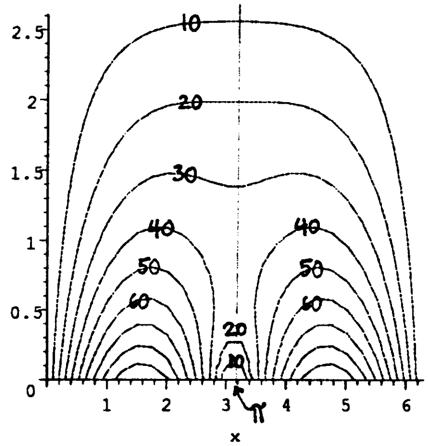
(e) 

All values between 0 and 20 spring from the corners $(a,0)$, (a,a).

NOTE: The preceding sketches of level curves have been fairly straightforward. In other cases the topography of these curves may be trickier. To illustrate, consider the problem $\nabla^2 u = 0$ in $0 < x < \pi, 0 < y < \pi$ with the boundary conditions $u(0,y) = u(x,\pi) = u_x(\pi,y) = 0, u(x,0) = 100\sin x$. In particular, note that the "plate", say, is insulated at the right edge $x = \pi$, so the isotherms will have to be horizontal at that edge. Without deriving the solution, here is the Maple plotting of the isotherms, where we have plotted over the extended region $0 < x < 2\pi$ simply because that picture seems to make it easier to see the patterns.

```
> u:=sum(-(800/Pi)*(sin((2*i-1)*Pi/2)/(sinh((2*i-1)*Pi/2)*((2*i-1)^2
  -4)))*sin((2*i-1)*x/2)*sinh((2*i-1)*(Pi-y)/2),i=1..10):
> with(plots):
> implicitplot({u=90,u=80,u=70,u=60,u=50,u=40,u=30,u=20,u=10},x=0..2
  *Pi,y=0..Pi);
```

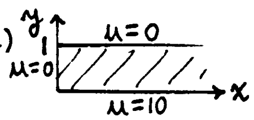
The tricky topological feature of the level curve pattern is the way the lower region, with "3 emanations" of curves gives way, above, to a region of single curves. Exploration of the details of that transition might make a nice computer project. To capture those details we'd surely need to include more terms in the sum than the 10 used above.



9. (11) and (13) give $u(x,y) = \sum_1^\infty \left(\frac{2}{b\sinh\frac{n\pi a}{b}} \int_0^b f(y)\sin\frac{n\pi y}{b}\,dy \right) \sinh\frac{n\pi x}{b} \sin\frac{n\pi y}{b}$

$$= \int_0^b \Big(\underbrace{\sum_1^\infty \frac{2}{b\sinh\frac{n\pi a}{b}} \sinh\frac{n\pi x}{b} \sin\frac{n\pi y}{b} \sin\frac{n\pi \eta}{b}}_{K(\eta;\,x,y)} \Big) f(\eta)\,d\eta$$

10. (a)



Anticipating the Fourier series expansion to be on the finite edge $x=0$, write $\frac{X''}{X} = -\frac{Y''}{Y} = +k^2$. Then

$u(x,y) = (A+Bx)(C+Dy) + (Ee^{kx} + Fe^{-kx})(G\cos ky + H\sin ky)$

u bdd as $x \to \infty \Rightarrow B = E = 0$, so

$u(x,y) = C' + D'y + e^{-kx}(G'\cos ky + H'\sin ky)$

$u(x,0) = 10 = C' + e^{-kx}G' \rightarrow C' = 10,\ G' = 0$ so

$u(x,y) = 10 + D'y + H'e^{-kx}\sin ky$

$u_y(x,1) = 0 = D' + kH'\cos k\, e^{-kx} \rightarrow D' = 0, \cos k = 0$ so $k = n\pi/2$ (n odd)

$u(x,y) = 10 + \sum_{1,3,..} H_n' e^{-n\pi x/2} \sin n\pi y/2,$

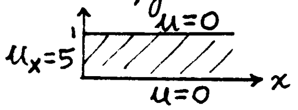
$u(0,y) = 0 = 10 + \sum_{1,3,..} H_n' \sin n\pi y/2,\ -10 = \sum_{1,3,..} H_n' \sin\frac{n\pi y}{2}\ (0<y<1)$

QRS: $H_n' = 2\int_0^1 (-10)\sin(n\pi y/2)\,dy = -40/n\pi$, so

$u(x,y) = 10 - \frac{40}{\pi} \sum_{1,3,..} \frac{1}{n} \sin\frac{n\pi y}{2} e^{-n\pi x/2}$

(b) Proceeding essentially as in (a) we will arrive at $u(x,y) = 100$ (everywhere), which result could have been seen by inspection.

(c) $u(x,y) = (A+Bx)(C+Dy) + (Ee^{kx} + Fe^{-kx})(G\cos ky + H\sin ky)$



Sequence of application of the 4 boundary conditions: We must do the $y=0$ and $y=1$ b.c.'s before the $x=0$ one so as to get ready for the Fourier series expansion that will take place at the $x=0$ edge. But I advise applying any boundedness conditions (in this case at $x=\infty$) first since they knock terms out and simplify the solution form. Thus,

Boundedness as $x\to\infty \Rightarrow B=0$ and $E=0$, so

$u(x,y) = C' + D'y + e^{-kx}(G'\cos ky + H'\sin ky)$

$u(x,0) = 0 = C' + e^{-kx}G' \to G'=0$ and $C'=0$, so

$u(x,y) = D'y + H'e^{-kx}\sin ky$

$u(x,1) = 0 = D' + H'e^{-kx}\sin k \to D'=0$, $\sin k = 0$ so $k = n\pi$ $(n=1,2,\dots)$

$u(x,y) = \sum_1^\infty H_n' e^{-n\pi x}\sin n\pi y$

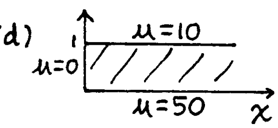
$u_x(0,y) = 5 = \sum_1^\infty -n\pi H_n' \sin n\pi y \quad (0<y<1)$

HRS:

$-n\pi H_n' = \frac{2}{1}\int_0^1 5\sin n\pi y\, dy$ so $H_n' = \frac{10}{n^2\pi^2}(\cos n\pi - 1) = \begin{cases} -20/n^2\pi^2, & n \text{ odd} \\ 0, & n \text{ even}\end{cases}$

so $u(x,y) = -\frac{20}{\pi^2}\sum_{1,3,\dots}^\infty \frac{1}{n^2} e^{-n\pi x}\sin n\pi y.$

(d)



$u(x,y) = (A+Bx)(C+Dy) + (Ee^{kx} + Fe^{-kx})(G\cos ky + H\sin ky)$

u bounded as $x\to\infty \Rightarrow B=E=0$ so

$u(x,y) = C' + D'y + e^{-kx}(G'\cos ky + H'\sin ky)$

$u(x,0) = 50 = C' + e^{-kx}G' \to C'=50$, $G'=0$ so

$u(x,y) = 50 + D'y + H'e^{-kx}\sin ky$

$u(x,1) = 10 = 50 + D' + H'e^{-kx}\sin k \to D' = -40$, $k = n\pi$ $(n=1,2,\dots)$

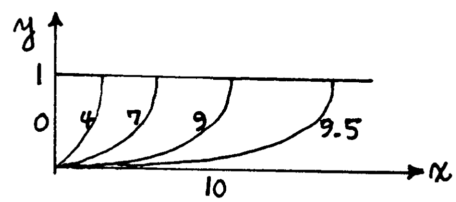
$u(x,y) = 50 - 40y + \sum_1^\infty H_n' e^{-n\pi x}\sin n\pi y$

$u(0,y) = 0 = 50 - 40y + \sum H_n' \sin n\pi y$

$40y - 50 = \sum_1^\infty H_n' \sin n\pi y$

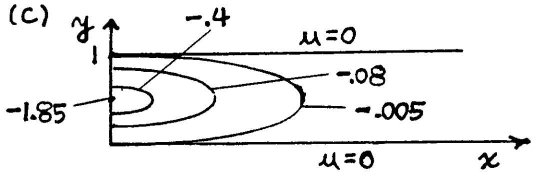
HRS: $H_n' = \frac{2}{1}\int_0^1 (40y - 50)\sin n\pi y\, dy = \frac{20}{n\pi}[(-1)^n - 5]$

11. (a) The $u_y(x,1) = 0$ condition implies that the isotherms are vertical at $y=1$. Also, the $y=0$ and $y=1$ b.c.'s show that $u(x,y) \sim 10$ as $x\to\infty$. Thus:



(b) $u = 100$ everywhere.

(c)

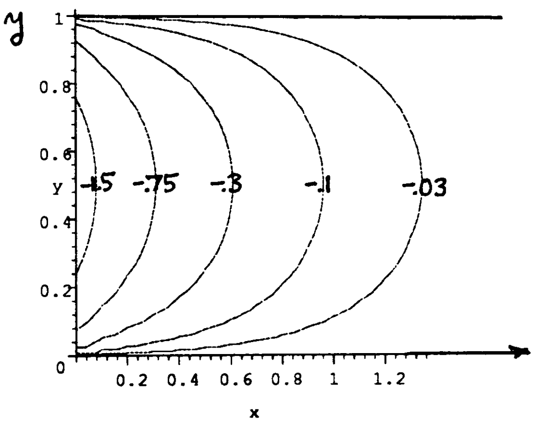


u will be negative at each (x,y) in the interior of the domain. Its largest negative value will be at $(0, 0.5)$, and the isotherm values will increase and approach zero as the isotherms penetrate more deeply into the strip. It's difficult to estimate the actual values so I used the solution

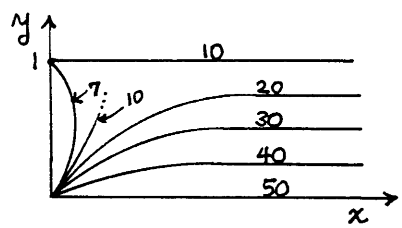
$$u(0,.5) = -\frac{20}{\pi^2} \sum_{1,3,..} \frac{1}{n^2} \sin\frac{n\pi}{2}$$

from Exercise 10(c) to determine that $u(0,.5) \approx -1.85$. Thus, at least qualitatively, the isothermal values will be somewhat as noted in the sketch, above. Let us check with a computer plot using Maple.

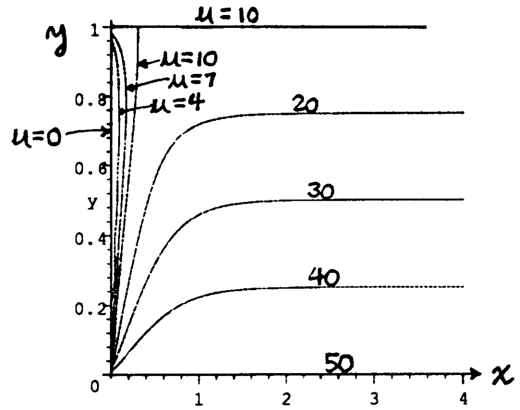
```
> u:=-(20/Pi^2)*sum(exp(-(2*i-1)*Pi*x)*sin((2*i-1)*Pi*y)/(2*i-1)^2,i
  =1..20):
> implicitplot({u=-1.5,u=-.75,u=-.3,u=-.1,u=-.03},x=0..2.5,y=0..1);
```



(d) This one is trickier. Isotherms between 0 and 10 extend from (0,0) to (0,1) and those between 10 and 50 asymptote (as $x \to \infty$) to a linear variation in y (see sketch), but as we move to the left along $y = 0.8$, say, u falls from 20 to 0. Thus, it must pass through 10, so there must be a $u = 10$ isotherm in the interior that starts at the origin and "heads north". Does that isotherm intersect the line $y = 1$ or does it come in to the corner (0,1)? Attempting some sketches the former seems more reasonable, but to be sure let us do a Maple plot:




```
> with(plots):
> u:=50-40*y+sum((20/(i*Pi))*((-1)^i-5)*exp(-i*Pi*x)*sin(i*Pi*y),i=1
  ..20):
> implicitplot({u=4,u=7,u=10,u=20,u=30,u=40},x=0..4,y=0..1,grid=[400
  ,100]);
```



Thus, we see that the $u=10$ isotherm does intersect the line $y=1$ rather than coming in to the corner $(0,1)$. NOTE: Aside from the latter occurrence the isotherms begin/end at the corners $(0,0)$ and $(0,1)$. They don't quite do that in the figure but that is because our calculation sums only the first 20 terms of the series.

12. Observe that the $e^{-(x/10b)^2}$ factor is a slowly varying function of x. For instance, it diminishes from 1 to $e^{-1} = 0.368$ only after x increases from 0 to $10b$ (i.e. ten widths, the width b being the natural length scale). Then, approximately, we can neglect the u_{xx} term in the PDE, which becomes

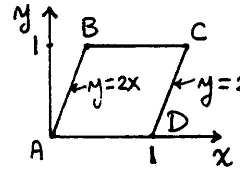
$$u_{yy} \approx 0 \rightarrow u = Ay + B,$$

where A, B can be slowly-varying functions of x.

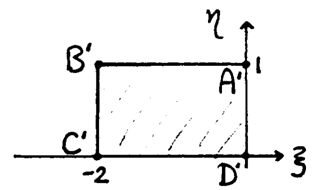
$$\left.\begin{aligned} u(x,0) &= 0 = 0 + B \\ u(x,b) &= 50e^{-(x/10b)^2} = Ab + B \end{aligned}\right\} \rightarrow \begin{aligned} &B = 0, \\ &A = 50e^{-(x/10b)^2}/b, \end{aligned}$$

so $u(x,y) \approx 50\, e^{-(x/10b)^2}(y/b)$.

14. (a)



AB: $y=2x$, so $\xi=0, 0<\eta<1$
BC: $y=1$, so $\eta=1, -2<\xi<0$
CD: $y=2x-2$, so $\xi=-2, 0<\eta<1$
AD: $y=0$, so $\eta=0, -2<\xi<0$
$\rightarrow$



(b) $\dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial \xi}\dfrac{\partial \xi}{\partial x} + \dfrac{\partial}{\partial \eta}\dfrac{\partial \eta}{\partial x} = -2\dfrac{\partial}{\partial \xi}$, $\dfrac{\partial}{\partial y} = \dfrac{\partial}{\partial \xi}\dfrac{\partial \xi}{\partial y} + \dfrac{\partial}{\partial \eta}\dfrac{\partial \eta}{\partial y} = \dfrac{\partial}{\partial \xi} + \dfrac{\partial}{\partial \eta}$

so $u_{xx} + u_{yy} = (-2\frac{\partial}{\partial \xi})(-2\frac{\partial}{\partial \xi})u + (\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta})(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta})u = 4u_{\xi\xi} + (\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta})(u_\xi + u_\eta)$

$$= 5u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} = 0$$

Then, $u = X(\xi)Y(\eta)$ gives

$$5\frac{X''}{X} + 2\frac{X'}{X}\frac{Y'}{Y} + \frac{Y''}{Y} = 0$$

and because of this "mixed" term we are unable to complete the separation.

15. (a) With $u = \frac{f}{2}x^2 + U$, $u_{xx} + u_{yy} = f + U_{xx} + U_{yy} = f$ gives $U_{xx} + U_{yy} = 0$

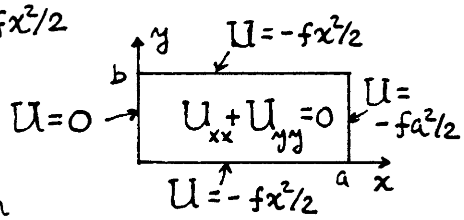
Then, $u(0,y) = 0 = 0 + U(0,y)$ gives $U(0,y) = 0$,

$u(a,y) = 0 = fa^2/2 + U(a,y)$ gives $U(a,y) = -fa^2/2$,

$u(x,0) = 0 = fx^2/2 + U(x,0)$ gives $U(x,0) = -fx^2/2$,

and $u(x,b) = 0 = fx^2/2 + U(x,b)$ gives $U(x,b) = -fx^2/2$

so the U problem is as summarized at the right. Of the b.c.'s the N (north) and S (south) are functions and the E and W are constants, so we will need Fourier series in $\underline{x}$. Thus, with $U(x,y) = X(x)Y(y)$, set $\frac{X''}{X} = -\frac{Y''}{Y} = -k^2$, so



$$U(x,y) = (A+Bx)(C+Dy) + (E\cos kx + F\sin kx)(G\cosh ky + H\sinh ky)$$
$$U(0,y) = 0 = A(\;\;\text{"}\;\;) + \quad E(\;\;\text{"}\;\;\;\;\text{"}\;\;) \to A = E = 0$$
$$U(x,y) = x(C' + D'y) + \sin kx\,(G'\cosh ky + H'\sinh ky)$$
$$U(a,y) = -fa^2/2 = a(C'+D'y) + \sin ka\,(\;\;\text{"}\;\;\;\;\text{"}\;\;) \to C' = -fa/2,\ D' = 0,$$
$$k = n\pi/a \ (n = 1,2,\dots)$$

$$U(x,y) = -\frac{fax}{2} + \sum_1^\infty \sin\frac{n\pi x}{a}\left(G_n'\cosh\frac{n\pi y}{a} + H_n'\sinh\frac{n\pi y}{a}\right) \qquad ①$$

$$U(x,0) = -f\frac{x^2}{2} = -\frac{fa}{2}x + \sum_1^\infty G_n'\sin\frac{n\pi x}{a} \qquad ②$$

or, $$\frac{f}{2}x(a-x) = \sum_1 G_n'\sin\frac{n\pi x}{a} \qquad (0<x<a)$$

HRS: $$G_n' = \frac{2}{a}\int_0^a \frac{f}{2}x(a-x)\sin\frac{n\pi x}{a}\,dx = \frac{f}{a}\int_0^a (ax - x^2)\sin\frac{n\pi x}{a}\,dx \qquad ③$$

$$U(x,b) = -f\frac{x^2}{2} = -\frac{fa}{2}x + \sum_1^\infty \left(G_n'\cosh\frac{n\pi b}{a} + H_n'\sinh\frac{n\pi b}{a}\right)\sin\frac{n\pi x}{a} \qquad ④$$

Comparing ② and ④ gives $G_n' = G_n'\cosh\frac{n\pi b}{a} + H_n'\sinh\frac{n\pi b}{a}$

so
$$H_n' = \frac{1 - \cosh(n\pi b/a)}{\sinh(n\pi b/a)}G_n' \qquad ⑤$$

and so $U(x,y)$ is given by ①, ③, ⑤.

(b) Putting (15.4) and (15.6) into $u_{xx} + u_{yy} = f$ gives
$$\sum_1^\infty g_n''\sin\frac{n\pi y}{b} + \sum_1^\infty \left(-\frac{n\pi}{b}\right)^2 g_n\sin\frac{n\pi y}{b} = \sum_1^\infty f_n\sin\frac{n\pi y}{b}$$
so equating coefficients of sine terms gives
$$g_n'' - \left(\frac{n\pi}{b}\right)^2 g_n = f_n.$$

Then, $u(0,y) = 0 = \sum_1^\infty g_n(0)\sin\frac{n\pi y}{b} \to g_n(0) = 0$

and $u(a,y) = 0 = \sum_1^\infty g_n(a)\sin\frac{n\pi y}{b} \to g_n(a) = 0$.

(c) If $f(x,y)=xy$ then $f_n(x)=\frac{2}{b}\int_0^b xy\sin\frac{n\pi y}{b}\,dy = -\frac{2b(-1)^n}{n\pi}x$

so

$$g_n''-\left(\frac{n\pi}{b}\right)^2 g_n = -\frac{2b(-1)^n}{n\pi}x, \quad g_n(x)=\frac{2(-1)^n b^3}{n^3\pi^3}x + A\sinh\frac{n\pi x}{b}+B\cosh\frac{n\pi x}{b}$$

$$g_n(0)=0=B, \quad g_n(a)=0=\frac{2(-1)^n b^3}{n^3\pi^3}a + A\sinh\frac{n\pi a}{b} \text{ gives } A=\frac{-2(-1)^n ab^3}{n^3\pi^3\sinh\frac{n\pi a}{b}}$$

so

$$u(x,y)=\frac{2b^3}{\pi^3}\sum_1^\infty \frac{(-1)^n}{n^3}\left[x-a\frac{\sinh(n\pi x/b)}{\sinh(n\pi a/b)}\right]\sin\frac{n\pi y}{b}$$

16. (a) $u(x,y,z)=X(x)Y(y)Z(z)$ gives $\frac{X''}{X}+\frac{Y''}{Y}+\frac{Z''}{Z}=0$. Anticipating an expansion on x and y we seek sines and cosines in x and y. Thus, write $\frac{Y''}{Y}+\frac{Z''}{Z}=-\frac{X''}{X}=\alpha^2$,

and $\frac{Y''}{Y}=-\frac{Z''}{Z}+\alpha^2=-\beta^2$

so (omitting the special cases that give "ramp" terms, since the homogeneous b.c.'s on all faces except $z=c$ will, no doubt, knock out those terms)

$$X=A\cos\alpha x+B\sin\alpha x$$
$$Y=C\cos\beta y+D\sin\beta y$$
$$Z=E\cosh\sqrt{\alpha^2+\beta^2}\,z+F\sinh\sqrt{\alpha^2+\beta^2}\,z.$$

Now, $u(0,y,z)=0\to A=0$
$u(a,y,z)=0\to \alpha=m\pi/a \quad (m=1,2,\dots)$
$u(x,0,z)=0\to C=0$
$u(x,b,z)=0\to \beta=n\pi/b \quad (n=1,2,\dots)$
$u(x,y,0)=0\to E=0$,

so

$$u(x,y,z)=\sum_{m=1}^\infty\sum_{n=1}^\infty G_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sinh\omega_{mn}z$$

where

$$\omega_{mn}=\pi\sqrt{(m/a)^2+(n/b)^2}.$$

Finally,

$$u(x,y,c)=f(x,y)=\sum_{m=1}^\infty\sum_{n=1}^\infty (G_{mn}\sinh\omega_{mn}c)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

gives (by HR sine series on x and y) (16.4) for G_{mn}.

(b)

$$G_{mn}=\frac{400}{a^2\sinh(\pi\sqrt{m^2+n^2})}\int_0^a\int_0^a \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a}\,dx\,dy$$
$$=\frac{1600}{\pi^2}\frac{1}{mn\sinh(\pi\sqrt{m^2+n^2})} \quad \text{if } m,n \text{ are both odd, 0 otherwise}$$

Using Maple on

$$u(a/2,a/2,a/2) = \frac{1600}{\pi^2} \sum_{1,3,..}^{\infty} \sum_{1,3,..}^{\infty} \frac{\sin m\pi/2 \sin n\pi/2}{mn} \frac{\sinh\left(\frac{\pi}{2}\sqrt{m^2+n^2}\right)}{\sinh(\pi\sqrt{m^2+n^2})}$$

```
> u:=(1600/Pi^2)*sum(sum(sin((2*i-1)*Pi/2)*sin((2*j-1)*Pi/2)*sinh(Pi
  *sqrt((2*i-1)^2+(2*j-1)^2)/2)/((2*i-1)*(2*j-1)*sinh(Pi*sqrt((2*i-1
  )^2+(2*j-1)^2))),i=1..8),j=1..8):
> evalf(u);
                              16.66666666
```

The convergence was rapid; summing $i=1..2, j=1..2$ gave 16.6479, $i=1..4, j=1..4$ gave 16.666647, and $i=1..8, j=1..8$ gave

$$u(a/2,a/2,a/2) = 16.66666666$$

which does not change with the inclusion of more terms.

17. $\int_V \nabla^2 u\, dV = \int_V \nabla\cdot(\nabla u)\, dV = \int_S \hat{n}\cdot\nabla u\, dA$ by divergence theorem
$= \int_S \frac{\partial u}{\partial n} dA$ by directional derivative formula,

so $\int_S \frac{\partial u}{\partial n} dA = \int_V f\, dV$

18. (a) Green's 1st identity: $\int_V (\nabla u\cdot\nabla v + u\nabla^2 v)\, dV = \int_S u \frac{\partial v}{\partial n} dA$ ①

$\nabla^2 u_1 = f$ in V
$\nabla^2 u_2 = f$ " "
$\nabla^2 u_1 - \nabla^2 u_2 = f - f$
or, $\nabla^2(u_1-u_2)=0$, or, $\nabla^2 w = 0$ in V ②
$u_1 = g$ on S
$u_2 = g$ on S
$u_1 - u_2 = g - g$ or $w=0$ on S ③

Then, letting "u"="v" $= w$ in ①, and using ② and ③ gives

$$\int_V (\nabla w\cdot\nabla w + w\underbrace{\nabla^2 w}_{0})\, dV = \int_S \underbrace{w}_{0} \frac{\partial w}{\partial n} dA \quad ④$$

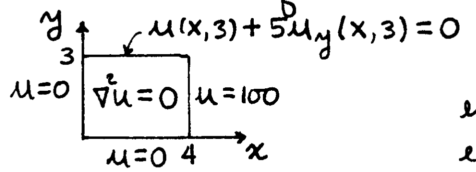
so $\int_V \nabla w\cdot\nabla w\, dV = \int_V (w_x^2 + w_y^2 + w_z^2)\, dV = 0$ ⑤

so $w_x=0, w_y=0, w_z=0$ in V. Thus, $w=$ constant in V, and $w=0$ on S implies that that constant is zero. Thus, $w(x,y,z)\equiv 0$ so $u_1(x,y,z)=u_2(x,y,z)$ and the solution is unique.

(b) Again we arrive at ④ and ⑤, so $w=$ constant in V, but this time we cannot argue that that constant is zero due to w being 0 on S, so all we can conclude is that $w=$ const. That is, two solutions $u_1(x,y,z)$ and $u_2(x,y,z)$ differ by at most a constant.

(c) Again we arrive at ④: $\int_V (\nabla w\cdot\nabla w + w\underbrace{\nabla^2 w}) dV = \int_S w \frac{\partial w}{\partial n} dA = 0$ because w is zero on part of S and $\partial w/\partial n$ is zero on the rest of S. Thus, ⑤ holds again so $w=$ constant. But $w=0$ on part of S, so that constant is zero. Thus, $w \equiv 0$ so $u_1 - u_2 = 0$, $u_1 = u_2$, and the solution is unique.

19.



Anticipating the Fourier series (i.e., the eigenfunction series) expansion on the $x=4$ edge we write

$$\frac{X''}{X} = -\frac{Y''}{Y} = +k^2$$

so $u(x,y) = (A+Bx)(C+Dy) + (E\cosh kx + F\sinh kx)(G\cos ky + H\sin ky)$.

$u(0,y) = 0 = A(\;"\;) + E(\;"\;\;"\;) \rightarrow A = E = 0$

$u(x,y) = x(C'+D'y) + \sinh kx\,(G'\cos ky + H'\sin ky)$

$u(x,0) = 0 = C'x + G'\sinh kx \rightarrow C' = G' = 0,$

$u(x,y) = D'xy + H'\sinh kx \sin ky$

$u(x,3) + 5u_y(x,3) = 0 = 3D'x + H'\sin 3k \sinh kx + 5D'x + 5kH'\cos 3k \sinh kx$

implies $D' = 0$ and the characteristic equation

$$\sin 3k + 5k\cos 3k = 0.$$

Denoting the successive roots as k_n $(n=1,2,\ldots)$, the Maple fsolve command gives $k_1 = 0.6266$, $k_2 = 1.6119$, $k_3 = 2.6432$, $k_4 = 3.6833$, $k_5 = 4.7265$.

H_n' remains arbitrary, so

$$u(x,y) = \sum_1^\infty H_n' \sinh k_n x \sin k_n y$$

Finally,

$$u(4,y) = 100 = \sum_1^\infty H_n' \sinh 4k_n \sin k_n y$$

The St.-Liou. problem is

$$Y'' + k^2 Y = 0 \quad (0<y<3)$$

$$Y(0) = 0, \quad Y(3) + 5Y'(3) = 0$$

with eigens $\lambda_n = k_n^2$ and $\phi_n(y) = \sin k_n y$. Thus,

$$H_n' \sinh 4k_n = \frac{\langle 100, \sin k_n y\rangle}{\langle \sin k_n y, \sin k_n y\rangle} = \frac{100\int_0^3 \sin k_n y\,dy}{\int_0^3 \sin^2 k_n y\,dy}$$

gives the H_n''s: $H_1' = 19.74$, $H_2' = 14.70$, $H_3' = 3.84$, $H_4' = 0.86$, $H_5' = 0.26, \ldots$

so $u(2,1) \simeq 18.62 + 1.40 + 0.07 - 0.01 - \cdots \approx 20.08$.

NOTE: Since 3 of the 4 b.c.'s are homogeneous it is natural to anticipate the expansion to occur on the nonhomogeneous b.c., at $x=4$. However, the "ramp" term from $k=0$ is quite capable of handling both conditions $u(0,y)=0$ and $u(4,y)=0$ so we can expand on the $y=0$ and $y=3$ edges instead. The advantage of doing it this way is that the associated Sturm-Liouville problem will be simpler and, indeed, the expansion will merely be a half-range sine series. Let us go through it. The key point is that—anticipating the expansions to be on the $y=0$ and $y=3$ edges (hence on the x variable)—we write

$$\frac{X''}{X}=-\frac{Y''}{Y}=-k^2$$

and apply the $x=0$ and $x=4$ b.c.'s first. Thus,

$$u(x,y)=(A+Bx)(C+Dy)+(E\cos kx+F\sin kx)(G\cosh ky+H\sinh ky)$$

$$u(0,y)=0=A(\;"\;)+\qquad E(\;"\qquad"\;)\rightarrow A=E=0$$

$$u(x,y)=x(C'+D'y)+\sin kx(G'\cosh ky+H'\sinh ky)$$

$$u(4,y)=100=4C'+4D'y+\sin 4k(\;"\qquad"\;)\rightarrow C'=25,\ D'=0,\ k=n\pi/4$$

$$u(x,y)=25x+\sum_1^\infty \sin\frac{n\pi x}{4}\left(G_n'\cosh\frac{n\pi y}{4}+H_n'\sinh\frac{n\pi y}{4}\right) \qquad ①$$

$$u(x,0)=0=25x+\sum_1^\infty G_n'\sin\frac{n\pi x}{4}$$

or, $$-25x=\sum_1^\infty G_n'\sin\frac{n\pi x}{4},\quad (0<x<4) \qquad ②$$

which is an eigenfunction expansion of $-25x$ in terms of the eigenfunctions $\sin(n\pi x/4)$ of the St.-Liou. problem

$$X''+k^2X=0 \quad (0<x<4)$$

$$X(0)=0,\ X(4)=0$$ (<u>not</u> 100; the 25x "ramp term" in ① handles the 100)

but it is also simply a HR sine series, so

$$G_n'=\frac{2}{4}\int_0^4(-25x)\sin\frac{n\pi x}{4}\,dx \qquad ③$$

Finally,

$$u(x,3)+5u_y(x,3)=0=25x+\sum_1^\infty \sin\frac{n\pi x}{4}\left(G_n'\cosh\frac{3n\pi}{4}+H_n'\sinh\frac{3n\pi}{4}\right)$$
$$+5\sum_1 \sin\frac{n\pi x}{4}\left(\frac{n\pi}{4}\right)\left(G_n'\sinh\frac{3n\pi}{4}+H_n'\cosh\frac{3n\pi}{4}\right)$$

or, $$-25x=\sum_1^\infty \sin\frac{n\pi x}{4}\left[G_n'\left(\cosh\frac{3n\pi}{4}+\frac{5n\pi}{4}\sinh\frac{3n\pi}{4}\right)+H_n'\left(\sinh\frac{3n\pi}{4}+\frac{5n\pi}{4}\cosh\frac{3n\pi}{4}\right)\right], \qquad ④$$

which is of the same form as ②. In fact, comparing ② and ④ we see that

$$G_n'\left(\cosh\frac{3n\pi}{4}+5\frac{n\pi}{4}\sinh\frac{3n\pi}{4}\right)+H_n'\left(\sinh\frac{3n\pi}{4}+\frac{5n\pi}{4}\cosh\frac{3n\pi}{4}\right)=G_n'$$

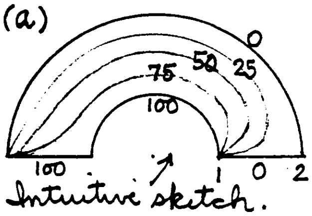
so

$$H_n'=\frac{1-\cosh\frac{3n\pi}{4}-\frac{5n\pi}{4}\sinh\frac{3n\pi}{4}}{\sinh\frac{3n\pi}{4}+\frac{5n\pi}{4}\cosh\frac{3n\pi}{4}} \qquad ⑤$$

and $u(x,y)$ is given by ①–⑤.

Section 20.3

2. (a)



Intuitive sketch.

$u(r,\theta) = (A + B\ln r)(C + D\theta) + (Er^k + Fr^{-k})(G\cos k\theta + H\sin k\theta)$

$u(r,0) = (\ \text{"}\ \ \text{"}\ )C + (\ \text{"}\ \ \text{"}\ )G \to C = G = 0$

$u(r,\theta) = (A' + B'\ln r)\theta + (E'r^k + F'r^{-k})\sin k\theta$

$u(r,\pi) = 100 = (A' + B'\ln r)\pi + (\ \text{"}\ \ \text{"}\ )\sin k\pi \to A' = 100/\pi, B' = 0, k = n$

$u(r,\theta) = 100\theta/\pi + \sum_1^\infty (E_n' r^n + F_n' r^{-n})\sin n\theta$ ①

$u(1,\theta) = 100 = 100\theta/\pi + \sum_1 (E_n' + F_n')\sin n\theta$

$100(\pi - \theta)/\pi = \sum_1^\infty (E_n' + F_n')\sin n\theta$

HRS: $E_n' + F_n' = \frac{2}{\pi}\int_0^\pi \frac{100}{\pi}(\pi - \theta)\sin n\theta\, d\theta$ ②

$u(2,\theta) = 0 = 100\theta/\pi + \sum_1^\infty (2^n E_n' + 2^{-n} F_n')\sin n\theta$

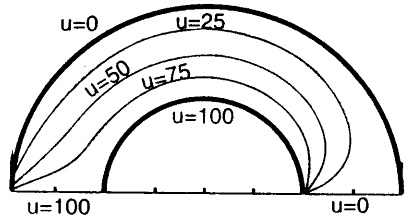
HRS: $2^n E_n' + 2^{-n} F_n' = \frac{2}{\pi}\int_0^\pi (-100\theta/\pi)\sin n\theta\, d\theta$ ③

so $u(r,\theta)$ is given by ①, where E_n' and F_n' are found from ② and ③. These give $E_n' = \frac{200}{n\pi}\frac{(-1)^n - 2^{-n}}{2^n - 2^{-n}}$, $F_n' = \frac{200}{n\pi}\frac{-2^n + (-1)^n}{2^{-n} - 2^n}$

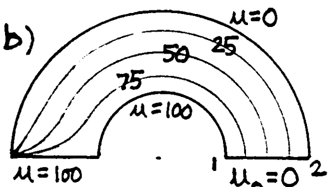
We used these Maple commands:

```
f:= (200/Pi^2)*int((Pi-t)*sin(n*t), t=0..π)
g:= (-200/Pi^2)*int(t*sin(n*t), t=0..π);
with(linalg):
A:= array([[1,1],[2^n,2^(-n)]]);
B:= array([f,g]);
linsolve(A,B);
```

Though not asked for, here is a computer plot:



(b)



Sketch ↑

This time the $\theta = 0$ edge is insulated: $u_\theta(r,0) = 0$.

$u(r,\theta) = (A + B\ln r)(C + D\theta) + (Er^k + Fr^{-k})(G\cos k\theta + H\sin k\theta)$

$u_\theta(r,0) = 0 = (\ \text{"}\ )D + (\ \text{"}\ \ \text{"}\ )kH \to D = H = 0$

$u(r,\theta) = A' + B'\ln r + (E'r^k + F'r^{-k})\cos k\theta$

$u(r,\pi) = 100 = A' + B'\ln r + (\ \text{"}\ \ \text{"}\ )\cos k\pi \to A' = 100, B' = 0, k = n/2$ where $n = 1, 3, \ldots$

$u(r,\theta) = 100 + \sum_{1,3,\ldots}^\infty (E_n' r^{n/2} + F_n' r^{-n/2})\cos(n\theta/2)$

$u(1,\theta) = 100 = 100 + \sum_{1,3,\ldots}^\infty (E_n' + F_n')\cos(n\theta/2)$

or, $0 = \sum_{1,3,\ldots} (E_n' + F_n')\cos(n\theta/2) \longrightarrow F_n' = -E_n'$

NOTE: Since $u_\theta(r,0) = 0$, the isotherms are perpendicular to the edge $\theta = 0$.

so

$u(r,\theta) = 100 + \sum_{1,3,\ldots}^\infty E_n'(r^{n/2} - r^{-n/2})\cos(n\theta/2)$

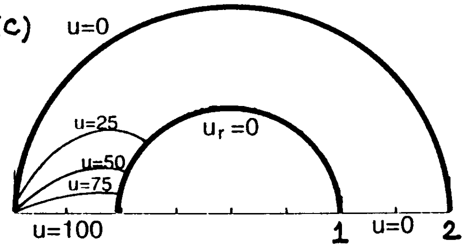
Finally,

$u(2,\theta) = 0 = 100 + \sum_{1,3,\ldots}^\infty E_n'(2^{n/2} - 2^{-n/2})\cos(n\theta/2) \quad (0 < \theta < \pi)$

so, by QRC series, $E_n'(2^{n/2}-2^{-n/2}) = \frac{2}{\pi}\int_0^{\pi}(-100)\cos\frac{n\theta}{2}\,d\theta = -\frac{400}{n\pi}\sin\frac{n\pi}{2}$,

$$u(r,\theta) = 100 - \frac{400}{\pi}\sum_{1,3,\dots}^{\infty} \frac{1}{n}\sin\frac{n\pi}{2}\;\frac{r^{n/2}-r^{-n/2}}{2^{n/2}-2^{-n/2}}\;\cos\frac{n\theta}{2}$$

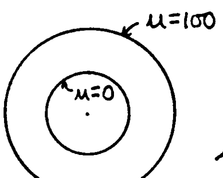
(c)



$$u(r,\theta) = 100\frac{\theta}{\pi} + \frac{200}{\pi}\sum_1^{\infty}\frac{(-1)^n}{n}\;\frac{r^n+r^{-n}}{2^n+2^{-n}}$$

NOTE: Since $u_r=0$ on $r=1$, the isotherms are perpendicular to that circle when they reach $r=1$.

(d)



u is 2π-periodic in θ so

$$u(r,\theta) = A + B\ln r + \sum_1^{\infty}\left[(C_n r^n + D_n r^{-n})\cos n\theta + (E_n r^n + F_n r^{-n})\sin n\theta\right]$$

$$u(1,\theta) = 0 = A + B\ln 1 + \sum_1^{\infty}\left[(C_n+D_n)\cos n\theta + (E_n+F_n)\sin n\theta\right]$$

$$\Rightarrow A=0,\; C_n+D_n=0,\; E_n+F_n=0 \qquad (1)$$

$$u(2,\theta) = 100 = A + B\ln 2 + \sum_1\left[(C_n 2^n + D_n 2^{-n})\cos n\theta + (E_n 2^n + F_n 2^{-n})\sin n\theta\right]$$

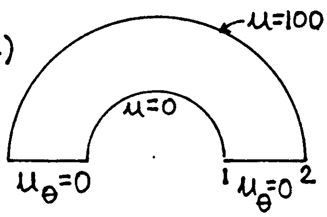
$$\Rightarrow B = 100/\ln 2,\; 2^n C_n + 2^{-n}D_n = 0,\; 2^n E_n + 2^{-n}F_n = 0 \qquad (2)$$

(1) and (2) give $C_n = D_n = E_n = F_n = 0$ so we simply have

$$u(r,\theta) = 100\,\frac{\ln r}{\ln 2}$$

and the $u = 25, 50, 75$ isotherms are the circles $r = 2^{1/4}, 2^{1/2}, 2^{3/4}$, respectively.
NOTE: We can see at the outset that u does not vary with θ so all we need is the $A + B\ln r$ part of the solution form.

(e)



$$u(r,\theta) = (A + B\ln r)(C + D\theta) + (Er^k + Fr^{-k})(G\cos k\theta + H\sin k\theta)$$

$$u_\theta(r,0) = 0 = (\;\text{''}\;)D + (\;\text{''}\;)Hk \to D = H = 0$$

$$u(r,\theta) = A' + B'\ln r + (E'r^k + F'r^{-k})\cos k\theta$$

$$u_\theta(r,\pi) = 0 = (\;\text{''}\;)(-k)\sin k\pi \to k = n$$

$$u(r,\theta) = A' + B'\ln r + \sum_1^{\infty}(E_n' r^n + F_n' r^{-n})\cos n\theta$$

$$u(1,\theta) = 0 = A' + B'\ln 1 + \sum_1^{\infty}(E_n' + F_n')\cos n\theta \to A'=0,\; E_n'+F_n'=0 \qquad (1)$$

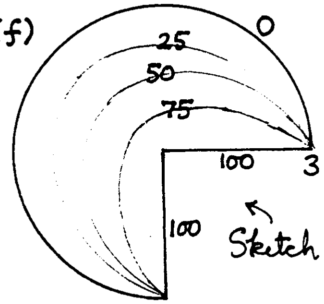
$$u(2,\theta) = 100 = 0 + B'\ln 2 + \sum_1(E_n' 2^n + F_n' 2^{-n})\cos n\theta \to B' = 100/\ln 2,\; E_n' 2^n + F_n' 2^{-n} = 0 \qquad (2)$$

(1) and (2) $\to E_n' = F_n' = 0$, so

$$u(r,\theta) = 100\,\frac{\ln r}{\ln 2},$$

as in (d). In fact, if you've studied the method of images you will see that the problem in (e) is, by that method, equivalent to the one in (d), which had the same simple solution.

(f)



$u(r,\theta) = (A + B\ln r)(C + D\theta) + (Er^{k} + Fr^{-k})(G\cos k\theta + H\sin k\theta)$

u bdd as $r \to 0 \Rightarrow B = F = 0$, so

$u(r,\theta) = C' + D'\theta + r^{k}(G'\cos k\theta + H'\sin k\theta)$

$u(r,0) = 100 = C' + r^{k}G' \to C' = 100,\ G' = 0$

$u(r,\theta) = 100 + D'\theta + H' r^{k}\sin k\theta$

$u(r,3\pi/2) = 100 = 100 + \frac{3\pi}{2}D' + H' r^{k}\sin\frac{3\pi k}{2}$

$\to D' = 0,\ 3\pi k/2 = n\pi \ (n = 1,2,\dots)$

$$u(r,\theta) = 100 + \sum_{1}^{\infty} H_n' r^{2n/3}\sin\frac{2n\theta}{3}$$

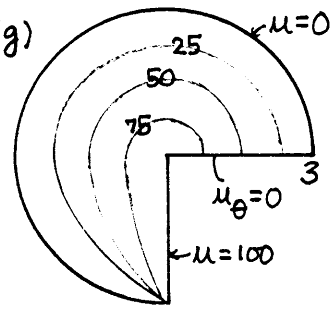
$$u(3,\theta) = 0 = 100 + \sum_{1}^{\infty} H_n' 3^{2n/3}\sin\frac{2n\theta}{3} \qquad (0 < \theta < \tfrac{3\pi}{2})$$

HRS: $$H_n' 3^{2n/3} = \frac{2}{3\pi/2}\int_0^{3\pi/2}(-100)\sin\frac{2n\theta}{3}\,d\theta = \frac{200}{n\pi}(\cos n\pi - 1)$$

$H_n' = -400/(n\pi 3^{2n/3})$ for n odd, 0 for n even, so

$$u(r,\theta) = 100 - \frac{400}{\pi}\sum_{1,3,\dots}^{\infty}\frac{1}{n}\left(\frac{r}{3}\right)^{2n/3}\sin\frac{2n\theta}{3}$$

(g)



$u(r,\theta) = (A + B\ln r)(C + D\theta) + (Er^{k} + Fr^{-k})(G\cos k\theta + H\sin k\theta)$

u bdd as $r \to 0 \Rightarrow B = F = 0$ so

$u(r,\theta) = C' + D'\theta + r^{k}(G'\cos k\theta + H'\sin k\theta)$

$u_\theta(r,0) = 0 = D' + r^{k}kH' \to D' = H' = 0$ so

$u(r,\theta) = C' + G' r^{k}\cos k\theta$

$u(r,3\pi/2) = 100 = C' + G' r^{k}\cos 3\pi k/2 \to C' = 100$, and

$3\pi k/2 = n\pi/2$ (n odd), or, $k = n/3$

$$u(r,\theta) = 100 + \sum_{1,3,\dots}^{\infty} G_n' r^{n/3}\cos\frac{n\theta}{3}$$

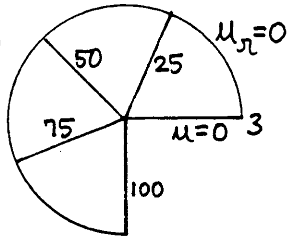
$$u(3,\theta) = 0 = 100 + \sum_{1,3,\dots} G_n' 3^{n/3}\cos\frac{n\theta}{3}$$

NOTE: Since $u_\theta(r,0) = 0$, the isotherms are perpendicular to the edge $\theta = 0$.

QRC: $$G_n' 3^{n/3} = \frac{2}{3\pi/2}\int_0^{3\pi/2}(-100)\cos\frac{n\theta}{3}\,d\theta = -\frac{400}{n\pi}\sin\frac{n\pi}{2}$$

so $$u(r,\theta) = 100 - \frac{400}{\pi}\sum_{1,3,\dots}^{\infty}\frac{1}{n}\sin\frac{n\pi}{2}\left(\frac{r}{3}\right)^{n/3}\cos\frac{n\theta}{3}$$

(h)



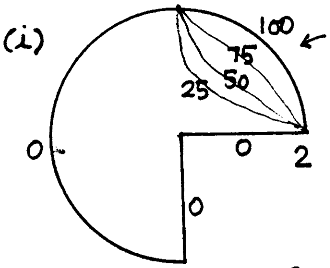
$u(r,\theta) = (A + B\ln r)(C + D\theta) + (Er^{k} + Fr^{-k})(G\cos k\theta + H\sin k\theta)$

u bdd $\to B = F = 0$

$\vdots$

$u(r,\theta) = \frac{200}{3\pi}\theta$ and the isotherms are radial lines.

(i)



← Rough sketch. Applying boundedness, we obtain

$u(r,\theta) = A + B\theta + r^k(C\cos k\theta + D\sin k\theta)$

$u(r,0) = 0 = A + r^k C \to A = C = 0$

$u(r,\theta) = B\theta + D r^k \sin k\theta$

$u(r, 3\pi/2) = 0 = (3\pi/2)B + D r^k \sin(3\pi k/2)$

so $B = 0$ and $3\pi k/2 = n\pi$, so $k = 2n/3$ $(n = 1, 2, \dots)$

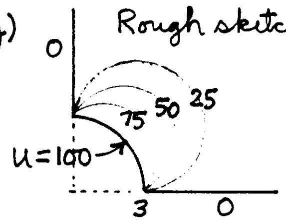
$u(r,\theta) = \sum_1^\infty D_n r^{2n/3} \sin(2n\theta/3)$.

$u(2,\theta) = \sum_1^\infty D_n 2^{2n/3} \sin(2n\theta/3) \quad (0 < \theta < 3\pi/2)$

HRS: $D_n 2^{2n/3} = \frac{2}{3\pi/2}\int_0^{3\pi/2} u(2,\theta)\sin\frac{2n\theta}{3}\,d\theta = \frac{400}{3\pi}\int_0^{\pi/2}\sin\frac{2n\theta}{3}\,d\theta = \frac{200}{n\pi}\left(1-\cos\frac{n\pi}{3}\right)$

so $u(r,\theta) = \frac{200}{\pi}\sum_1^\infty \frac{1-\cos\frac{n\pi}{3}}{n}\left(\frac{r}{2}\right)^{2n/3}\sin\frac{2n\theta}{3}$

(j)



Rough sketch.

$u(r,\theta) = (A + B\ln r)(C + D\theta) + (Er^k + Fr^{-k})(G\cos k\theta + H\sin k\theta)$

u bdd as $r \to \infty \Rightarrow B = E = 0$ so

$u(r,\theta) = C' + D'\theta + r^{-k}(G'\cos k\theta + H'\sin k\theta)$

$u(r,0) = 0 = C' + r^k G' \to C' = G' = 0$

$u(r,\theta) = D'\theta + H' r^{-k}\sin k\theta$

$u(r,\pi/2) = 0 = \pi D'/2 + H' r^{-k}\sin k\pi/2 \to D' = 0,\ k = 2n\ (n = 1, 2, \dots)$

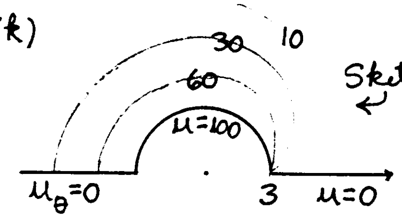
$u(r,\theta) = \sum_1^\infty H_n' r^{-2n}\sin 2n\theta$

$u(3,\theta) = 100 = \sum_1^\infty H_n' 3^{-2n}\sin 2n\theta \quad (0 < \theta < \pi/2)$

HRS: $H_n' 3^{-2n} = \frac{2}{\pi/2}\int_0^{\pi/2} 100\sin 2n\theta\,d\theta = \frac{400}{2n\pi}(1-\cos n\pi)$

so $u(r,\theta) = \frac{400}{\pi}\sum_{1,3,\dots}^\infty \frac{1}{n}\left(\frac{3}{r}\right)^{2n}\sin 2n\theta$

(k)



Sketch. ↙

$u(r,\theta) = (A + B\ln r)(C + D\theta) + (Er^k + Fr^{-k})(G\cos k\theta + H\sin k\theta)$

u bdd as $r \to \infty \Rightarrow B = E = 0$, so

$u(r,\theta) = C' + D'\theta + r^{-k}(G'\cos k\theta + H'\sin k\theta)$

$u(r,0) = 0 = C' + G' r^{-k} \to C' = G' = 0$

$u(r,\theta) = D'\theta + H' r^{-k}\sin k\theta$

$u_\theta(r,\pi) = 0 = D' + kH' r^{-k}\cos k\pi \to D' = 0,\ k\pi = n\pi/2$ (n odd), so

$u(r,\theta) = \sum_{1,3,\dots}^\infty H_n' r^{-n/2}\sin\frac{n\theta}{2}$.

$u(3,\theta) = 100 = \sum_{1,3,\dots}^\infty H_n' 3^{-n/2}\sin\frac{n\theta}{2} \quad (0 < \theta < \pi)$

QRS: $H_n' 3^{-n/2} = \frac{2}{\pi}\int_0^\pi 100\sin\frac{n\theta}{2}\,d\theta = 400/n\pi$

so

$u(r,\theta) = \frac{400}{\pi}\sum_{1,3,\dots}^\infty \frac{1}{n}\left(\frac{3}{r}\right)^{n/2}\sin n\theta.$

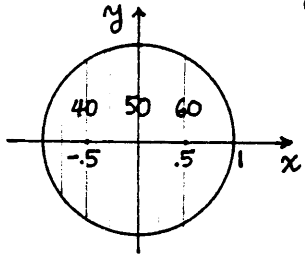
3. In all of these we can use (31) and (33), with $b=1$. Actually, the f's given are simple enough so that it is much easier to evaluate I, P_n, Q_n by matching terms in (32) rather than using (33).

(a) $f(\theta) = 50 + 20\cos\theta = I + \sum_1^\infty (P_n \cos n\theta + Q_n \sin n\theta)$

$\Rightarrow I = 50,\ P_1 = 20$, other P_n's and Q_n's $= 0$.

Thus, $u(r,\theta) = 50 + 20 r\cos\theta$

$= 50 + 20x$.

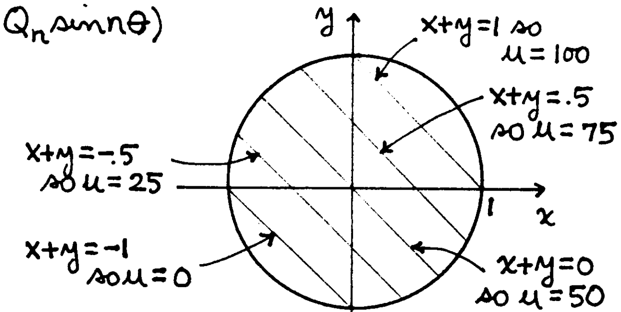


(b) $f(\theta) = 50 + 50(\cos\theta + \sin\theta) = I + \sum_1^\infty (P_n \cos n\theta + Q_n \sin n\theta)$

$\Rightarrow I = 50,\ P_1 = Q_1 = 50$, others $= 0$.

$u(r,\theta) = 50 + 50(r\cos\theta + r\sin\theta)$

$= 50 + 50(x+y)$



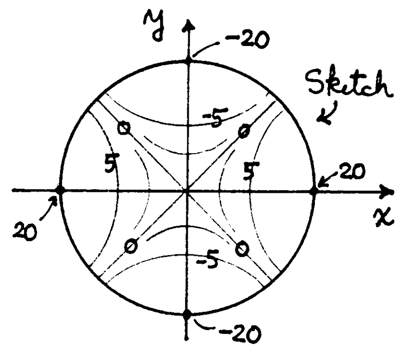
(c) $f(\theta) = 20\cos 2\theta = I + \sum_1^\infty (P_n \cos n\theta + Q_n \sin n\theta)$

$\rightarrow I = 0,\ P_2 = 20$, others $= 0$.

$u(r,\theta) = 20 r^2 \cos 2\theta = 20 r^2 (1 - 2\sin^2\theta)$

$= 20(x^2+y^2) - 40y^2 = 20(x^2 - y^2)$

so the isotherms are a family of hyperbolas, as sketched.



(e) $f(\theta) = 20\cos 3\theta = I + \sum_1^\infty (P_n \cos n\theta + Q_n \sin n\theta)$

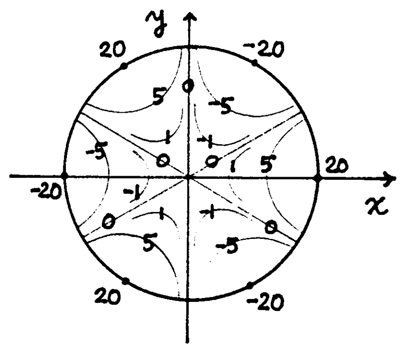
$\rightarrow I = 0,\ P_3 = 20$, others $= 0$.

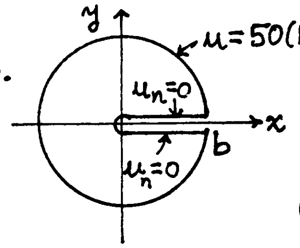
$u(r,\theta) = 20 r^3 \cos 3\theta$

$= 20 r^3 (4\cos^3\theta - 3\cos\theta)$

$= 80x^3 - 60x(x^2+y^2)$

The $\cos 3\theta$ is 0 along the rays $\theta = \pm\frac{\pi}{6}, \pm\frac{\pi}{2}, \pm\frac{5\pi}{6}$, and the isotherms are as sketched at the right.



4. 

On the $\theta=0$ edge $\frac{\partial u}{\partial n} = \nabla u \cdot \hat{n} = \left(\frac{\partial u}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial u}{\partial \theta}\hat{e}_\theta\right)\cdot(-\hat{e}_\theta) = -\frac{1}{r}\frac{\partial u}{\partial \theta}$, so $\partial u/\partial n = 0$ there implies that $\partial u/\partial\theta = 0$ there. Similarly, on the $\theta = 2\pi$ edge $\partial u/\partial n = \nabla u\cdot\hat{n} = \left(\frac{\partial u}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial u}{\partial\theta}\hat{e}_\theta\right)\cdot\hat{e}_\theta = \frac{1}{r}\frac{\partial u}{\partial\theta}$, so $\partial u/\partial n = 0$ there implies that $\partial u/\partial\theta = 0$ there.

$u(r,\theta) = (A+B\ln r)(C+D\theta) + (Er^k + Fr^{-k})(G\cos k\theta + H\sin k\theta)$

u bdd as $r\to 0 \Rightarrow B=F=0$, so

$u(r,\theta) = C' + D'\theta + r^k(G'\cos k\theta + H'\sin k\theta)$

$\frac{\partial u}{\partial\theta}(r,0) = 0 = D' + kr^k(0+H') \to D' = H' = 0$ so

$u(r,\theta) = C' + G' r^k \cos k\theta$

$\frac{\partial u}{\partial\theta}(r,2\pi) = 0 = -kG'r'\sin 2\pi k \to 2\pi k = n\pi$ so $k = n/2$ $(n=1,2,\dots)$

$$u(r,\theta) = C' + \sum_1^\infty G_n' r^{n/2}\cos\frac{n\theta}{2}$$

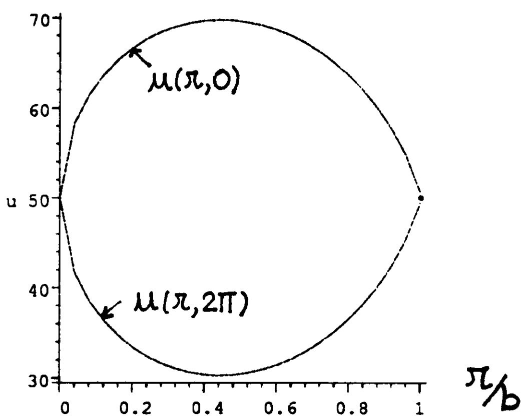
$$u(b,\theta) = 50(1+\sin\theta) = C' + \sum_1^\infty G_n' b^{n/2}\cos\frac{n\theta}{2} \quad (0<\theta<2\pi)$$

HRC: $\to$ $C' = 50$, $G_n' = -\frac{400\,b^{-n/2}}{\pi(n^2-4)}$ for n odd, 0 for n even, so

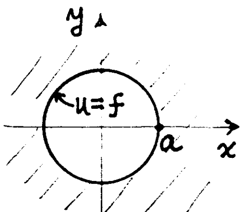
$$u(r,\theta) = 50 - \frac{400}{\pi}\sum_{1,3,\dots}^\infty \frac{1}{n^2-4}\left(\frac{r}{b}\right)^{n/2}\cos\frac{n\theta}{2}$$

$$u(r,0) = 50 - \frac{400}{\pi}\sum_{1,3,\dots}^\infty \frac{1}{n^2-4}\left(\frac{r}{b}\right)^{n/2}$$

$$u(r,2\pi) = 50 + \frac{400}{\pi}\sum_{1,3,\dots}^\infty \frac{1}{n^2-4}\left(\frac{r}{b}\right)^{n/2}$$



No, the field $u(r,\theta)$ is insensitive to the material (steel, brass, ...). The only place the nature of the specific material enters is in the diffusivity α^2 in the diffusion equation $\alpha^2\nabla^2 u = u_t$. The larger the diffusivity the faster u approaches steady state. Once at steady state, however, $u_t \to 0$ so $\alpha^2\nabla^2 u = 0$ and α^2 cancels out. Thus, the steady state temperature fields discussed in this chapter are completely insensitive to the specific material (i.e., to the diffusivity).

5. 

$u(r,\theta) = (A+B\ln r)(C+D\theta) + (Er^k + Fr^{-k})(G\cos k\theta + H\sin k\theta)$

u bdd as $r\to\infty \Rightarrow B=0, E=0$ so

$u(r,\theta) = C' + D'\theta + r^{-k}(G'\cos k\theta + H'\sin k\theta)$

u 2π-periodic in $\theta \Rightarrow D'=0, k=n$, so

$$u(r,\theta) = C' + \sum_1^\infty r^{-n}(G_n'\cos n\theta + H_n'\sin n\theta) \quad ①$$

$$u(a,\theta) = f(\theta) = C' + \sum_1^\infty a^{-n}(G_n'\cos n\theta + H_n'\sin n\theta)$$

so $C' = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$, $a^{-n}G_n' = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\,d\theta$ so $G_n' = \frac{a^n}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\,d\theta$,

$a^{-n}H_n' = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\,d\theta$ so $H_n' = \frac{a^n}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\,d\theta$

As $r\to\infty$, ① gives $u(r,\theta) \sim C' = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$, which is the average value of f.

6.(a) $\Phi(r,\theta) = (A+B\ln r)(C+D\theta) + (Er^k + Fr^{-k})(G\cos k\theta + H\sin k\theta)$

$\Phi_\theta(r,0) = 0 = (\;\cdot\cdot\;)D + (\;\cdot\cdot\;\cdot\cdot\;)Hk \to D = H = 0$,

$\Phi(r,\theta) = A' + B'\ln r + (E'r^k + F'r^{-k})\cos k\theta$

$\Phi_\theta(r,\pi) = 0 = (\;\text{"}\;\;\text{"}\;)(-k\sin k\pi) \to k = n \;(n=1,2,\dots)$

$\Phi(r,\theta) = A' + B'\ln r + \sum_1^\infty (E_n' r^n + F_n' r^{-n})\cos n\theta$

$\Phi_r(a,\theta) = 0 = \frac{B'}{a} + \sum_1^\infty (nE_n' a^{n-1} - nF_n' a^{-n-1})\cos n\theta \to B' = 0,\; F_n' = a^{2n}E_n'$

so

$$\Phi(r,\theta) = A' + \sum_1^\infty E_n'\left(r^n + \frac{a^{2n}}{r^n}\right)\cos n\theta$$

Finally, as $r\to\infty$

$$\Phi(r,\theta) = A' + E_1'\left(r + \frac{a^2}{r}\right)\cos\theta + E_2'\left(r^2 + \frac{a^4}{r^2}\right)\cos 2\theta + \cdots \sim Ur\cos\theta$$

implies A' = arbitrary, $E_1' = U$, $E_2' = E_3' = \cdots = 0$. The reasoning is as follows. Suppose $E_4' = E_5' = \cdots = 0$, say. Then the dominant term in Φ, as $r\to\infty$, is the $E_3'r^3$ term. Then Φ would be $\sim E_3'r^3$, which cannot (by any choice of E_3') be matched with $Ur\cos\theta$. Thus we need $E_3' = 0$. But then Φ would be $\sim E_2'r^2$, which is still too big as $r\to\infty$. Thus we need $E_2' = 0$. Then we have

$$\Phi(r,\theta) = A' + E_1'\left(r + \frac{a^2}{r}\right)\cos\theta \sim E'r\cos\theta \quad \text{as } r\to\infty,$$

for any value of A'. Finally, we can match $E'r\cos\theta$ with $Ur\cos\theta$ by choosing $E_1' = U$. Thus we obtain

$$\Phi(r,\theta) = A' + U\left(r + \frac{a^2}{r}\right)\cos\theta .$$

The arbitrary constant A' can be set $= 0$ without loss since it will drop out anyway when we take the gradient of Φ to obtain the velocity field. It is easily verified that (6.1) does indeed satisfy all the requirements in (38) of Section 16.10.

(b) $\Phi = U\left(r + \frac{a^2}{r}\right)\cos\theta = Ux + Ua^2x/(x^2+y^2)$

$\Psi_x = -\Phi_y = -Ua^2x\frac{(-1)2y}{(x^2+y^2)^2}$ so $\Psi = 2Ua^2\int\frac{xy\,\partial x}{(x^2+y^2)^2} = -\frac{Ua^2y}{x^2+y^2} + A(y)$ (arb.)

Then, $\Psi_y = \Phi_x$ gives

$$-\frac{Ua^2}{x^2+y^2}-\frac{Ua^2y(-1)(2y)}{(x^2+y^2)^2}+A'(y)=U+\frac{Ua^2}{x^2+y^2}+\frac{Ua^2x(-1)(2x)}{(x^2+y^2)^2}$$

or, $-Ua^2(x^2+y^2)+2Ua^2y^2+A'(y)(x^2+y^2)^2=U(x^2+y^2)^2+Ua^2(x^2+y^2)-2Ua^2x^2$

or, after cancellations, $A'(y)=U$, so $A(y)=Uy+\text{const}$. Thus,

$$\Psi(x,y)=-\frac{Ua^2y}{x^2+y^2}+Uy+\cancel{\text{const}}^{\,0}$$

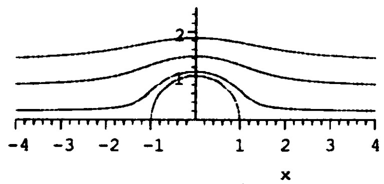
$\Psi(-4,.2)=0.1875$, $\Psi(-4,.8)=0.7519$, $\Psi(-4,1.4)=1.3220$, $\Psi(-4,2)=1.9$.

Maple:
```
> with(plots):
> p:=y-y/(x^2+y^2);
```

$$p:=y-\frac{y}{x^2+y^2}$$

```
> implicitplot({p=.1875,p=.7519,p=1.322,p=1.9,x^2+y^2=1},x=-4..4,y=0
  ..2,view=[-4..4,0..8]);
```

gives



NOTE: If we don't include the view option then the view will be $-4<x<4$ on x, but only $0<y\lesssim 2$ since the uppermost streamline reaches only $y\approx 2$. Since the print will be square then the x axis will appear compressed and the y axis elongated; e.g., the circle $r=1$ will be a tall and narrow ellipse. To keep the same x,y scales we need to force the printed y-interval to be $0<y<8$ (although I cut off the upper part by hand), which was accomplished by the view option.

7. (a) $\Phi(r,\theta)=(A+B\ln r)(C+D\theta)+(Er^k+Fr^{-k})(G\cos k\theta+H\sin k\theta)$

$\Phi_r(a,\theta)=0=\frac{B}{a}(C+D\theta)+k(Ea^{k-1}-Fa^{-k-1})(\quad "\quad\quad "\quad)\to B=0$ and $F=a^{2k}E$, so

$\Phi(r,\theta)=C'+D'\theta+(r^k+a^{2k}r^{-k})(G'\cos k\theta+H'\sin k\theta)$

$$\left.\begin{aligned}\Phi(r,2\pi)-\Phi(r,0)&=-\Gamma=2\pi D'+(r^k+a^{2k}r^{-k})(G'\cos 2\pi k+H'\sin 2\pi k-G')\\ \Phi_\theta(r,2\pi)-\Phi_\theta(r,0)&=0=(r^k+a^{2k}r^{-k})(-kG'\sin 2\pi k+kH'\cos 2\pi k-kH')\end{aligned}\right\}\ \text{♀}$$

♀ gives $D'=-\Gamma/2\pi$ and $\begin{pmatrix}c-1 & s\\ -s & c-1\end{pmatrix}\begin{pmatrix}G'\\ H'\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ where $c=\cos 2\pi k$, $s=\sin 2\pi k$. To avoid $G'=H'=0$, set

$$\begin{vmatrix}c-1 & s\\ -s & c-1\end{vmatrix}=(c-1)^2+s^2=0,\ \text{so}\ \left.\begin{aligned}&\cos 2\pi k=1\\ &\text{and}\ \sin 2\pi k=0\end{aligned}\right\}\Rightarrow k=n\ \ (n=1,2,\dots)$$

with G' and H' arbitrary. Thus far, then,

$$\Phi(r,\theta)=-\frac{\Gamma}{2\pi}\theta+\sum_1^\infty\left(r^n+\frac{a^{2n}}{r^n}\right)(G_n'\cos n\theta+H_n'\sin n\theta)+\cancel{C'}\ \ \text{can set}=0$$

Finally, $\Phi(r,\theta)\sim Ur\cos\theta$ as $r\to\infty\Rightarrow G_1'=U$ and all other G_n''s and H_n''s are zero, so

$$\Phi(r,\theta)=U\left(r+\frac{a^2}{r}\right)\cos\theta-\frac{\Gamma}{2\pi}\theta.$$

(b) Set $\underset{\sim}{v} = \underset{\sim}{\nabla}\Phi = \Phi_r \hat{e}_r + \frac{1}{r}\Phi_\theta \hat{e}_\theta = \underbrace{U(1-\frac{a^2}{r^2})\cos\theta}_{v_r} \hat{e}_r + \underbrace{[-U(r+\frac{a^2}{r})\sin\theta - \frac{\Gamma}{2\pi}]\frac{1}{r}}_{v_\theta}\hat{e}_\theta = \underset{\sim}{0}$

$v_r = 0$ gives $r=a$ or $\theta=\pi/2$ or $\theta = 3\pi/2$. Consider these one at a time:

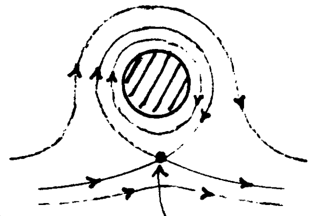
$r=a$: Then $v_\theta = 0$ gives $\theta = \sin^{-1}(-\Gamma/4\pi U a)$ provided that $\Gamma \leq 4\pi U a$ (let us consider Γ to be ≥ 0; if it is <0 the story is essentially the same but with the swirl counterclockwise and the "lift" force downward instead of upward)

$\theta = \pi/2$: Then $v_\theta = 0$ gives $r^2 + (\Gamma/2\pi U)r + a^2 = 0$, but this has no positive roots.

$\theta = 3\pi/2$: Then $v_\theta = 0$ gives $r^2 - (\Gamma/2\pi U)r + a^2 = 0$ so $r = \frac{\Gamma}{4\pi U} + \sqrt{(\frac{\Gamma}{4\pi U})^2 - a^2}$, which $\to a$ as $\Gamma \to 4\pi U a$.

Thus, we see that when $\Gamma = 0$ there are stagnation points on the cylinder at $\theta = 0, \pi$. As Γ increases the stagnation points move downward on the cylinder (as in the last figure in Exercise 7) and are located at the two roots of $\theta = \sin^{-1}(-\Gamma/4\pi U a)$. When $\Gamma = 4\pi U a$ the two stagnation points merge at $\theta = 3\pi/2$. As Γ increases beyond $4\pi U a$ the stagnation point leaves the surface of the cylinder and moves "south" along $\theta = 3\pi/2$ to $r = (\Gamma/4\pi U) + \sqrt{(\Gamma/4\pi U)^2 - a^2}$.

NOTE: An interesting and challenging project consists of seeing what the flow pattern looks like for the "supercritical" case where $\Gamma > 4\pi U a$. Qualitatively, the student should find that the pattern is somewhat (topologically, at least) as we have sketched at the right.



(c) $\underset{\sim}{v} = \underset{\sim}{\nabla}\Phi = U(1-\frac{a^2}{r^2})\cos\theta\, \hat{e}_r + \frac{1}{r}[-U(r+\frac{a^2}{r})\sin\theta - \frac{\Gamma}{2\pi}]\hat{e}_\theta$

$= 0\,\hat{e}_r - (2U\sin\theta + \Gamma/2\pi)\hat{e}_\theta$ on $r=a$.

By Bernoulli, $p|_{r=a} = \text{const.} - (\sigma/2)(2U\sin\theta + \Gamma/2\pi)^2 = \text{const.} - 2U^2\sin^2\theta + \frac{U\Gamma}{\pi a}\sin\theta$,

$L = \sigma \int_0^{2\pi} (-2U^2 \sin^2\theta + \frac{U\Gamma}{\pi a}\sin\theta)(a \sin\theta\, d\theta) = \sigma U \Gamma$

8. After applying boundedness we have

$$u(r,\theta) = E' + F'\theta + r^k(G'\cos k\theta + H'\sin k\theta) \qquad ①$$

$$u(r,0) - u(r,2\pi) = 0 = E' + G'r^k - [E' + 2\pi F' + r^k(G'\cos 2\pi k + H'\sin 2\pi k)]$$

$$u_\theta(r,0) - u_\theta(r,2\pi) = 0 = F' + kH'r^k - [F' + kr^k(-G'\sin 2\pi k + H'\cos 2\pi k)]$$

or,

$$-2\pi F' + r^k[(1-c)G' - sH'] = 0 \Rightarrow F' = 0 \text{ and } (1-c)G' - sH' = 0$$

and $\qquad r^k[sG' + (1-c)H'] = 0 \qquad\qquad sG' + (1-c)H' = 0,$

where $c = \cos 2\pi k$, $s = \sin 2\pi k$. To avoid $G' = H' = 0$ set $\begin{vmatrix} 1-c & -s \\ s & 1-c \end{vmatrix} = 0$ and obtain, as in Exercise 7(a), $k = n$ $(n = 1, 2, \ldots)$, where G' and H' are then arbitrary. Thus ① becomes

$$u(r,\theta) = E' + \sum_1^\infty r^n(G_n' \cos n\theta + H_n' \sin n\theta),$$

which is the same as (31). Then proceed as in Example 2.

9. $\frac{1}{2}+\sum_1^\infty \left(\frac{r}{b}\right)^n \cos n(\vartheta-\theta) = \frac{1}{2} + \text{Re}\sum_1^\infty \left(\frac{r}{b}\right)^n e^{in(\vartheta-\theta)} = \frac{1}{2}+\text{Re}\sum_1^\infty \left(\frac{r}{b}e^{i(\vartheta-\theta)}\right)^n$

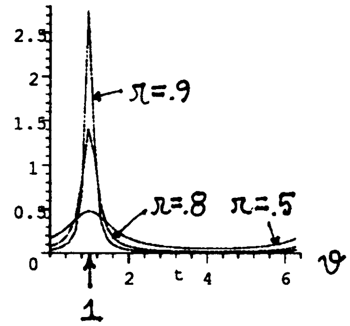
$= \frac{1}{2}-1+\text{Re}\sum_0^\infty \left(\frac{r}{b}e^{i(\vartheta-\theta)}\right)^n = -\frac{1}{2}+\text{Re}\,\frac{1}{1-\frac{r}{b}e^{i(\vartheta-\theta)}}$

$= -\frac{1}{2}+\text{Re}\,\frac{1}{1-\frac{r}{b}e^{i(\vartheta-\theta)}}\;\frac{1-\frac{r}{b}e^{-i(\vartheta-\theta)}}{1-\frac{r}{b}e^{-i(\vartheta-\theta)}}$ (since we multiply $\frac{1}{a+ib}$ by $\frac{a-ib}{a-ib}$ to get it into the form $c+id$.)

$= -\frac{1}{2}+\frac{1-\frac{r}{b}\cos(\vartheta-\theta)}{1-2\frac{r}{b}\cos(\vartheta-\theta)+\frac{r^2}{b^2}} = \frac{1}{2}\,\frac{b^2-r^2}{b^2-2br\cos(\vartheta-\theta)+r^2}$. ✓

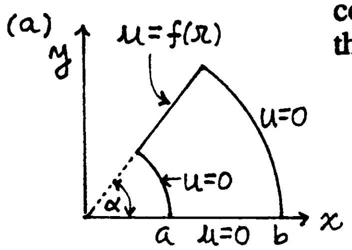
10. $d^2u/dx^2=0$ on $a<x<b$ with $u(a)=u_1$ and $u(b)=u_2$ given.
Solving gives $u(x)=u_1+\frac{u_2-u_1}{b-a}(x-a)$. That is, the solution is linear. Thus, surely u at the midpoint $(a+b)/2$ is the average value $(u_1+u_2)/2$, as is easily shown.

12.



(I forgot to include $r=0$, which gives the constant value $P(0,\vartheta-1)=1/2\pi$.)

13. NOTE: This is a nice problem pedagogically since it falls between the case where the Sturm-Liouville expansion is equivalent to (and could therefore been handled as) a half- or quarter-range cosine or sine series and the more sophisticated cases where the expansion is a series of special functions such as Bessel functions or Legendre polynomials. In this case the eigenfunctions are still elementary functions, but not merely cosines and sines. Also, this problem forms a natural companion for the problem where u is prescribed on one of the circular-arc edges, discussed as Example 1 in this section.



$\frac{r^2R''+rR'}{R} = -\frac{\Theta''}{\Theta} = -k^2$

$\rightarrow u(r,\theta) = (A+B\ln r)(E+F\theta) + [C\cos(k\ln r)+D\sin(k\ln r)][G\cosh k\theta+H\sinh k\theta]$

Since the St.-Liou. expansion will be in r, we need to be sure to do the $r=a$ and $r=b$ boundary conditions before any expansions in r can be attempted. Actually, it looks like we can also do the $u(r,0)=0$ condition early.

$u(r,0)=0=(A+B\ln r)E+[C\cos(k\ln r)+D\sin(k\ln r)]G \rightarrow E=G=0$, so

$u(r,\theta) = (A'+B'\ln r)\theta + [C'\cos(k\ln r)+D'\sin(k\ln r)]\sinh k\theta$

$$u(a,\theta)=0=(A'+B'\ln a)\theta+[C'\cos(\kappa\ln a)+D'\sin(\kappa\ln a)]\sinh\kappa\theta$$
$$u(b,\theta)=0=(A'+B'\ln b)\theta+[C'\cos(\kappa\ln b)+D'\sin(\kappa\ln b)]\sinh\kappa\theta$$

so $\left.\begin{array}{l}A'+B'\ln a=0\\ A'+B'\ln b=0\end{array}\right\}\rightarrow A'=B'=0.$

and
$$C'\cos(\kappa\ln a)+D'\sin(\kappa\ln a)=0 \quad ①$$
$$C'\cos(\kappa\ln b)+D'\sin(\kappa\ln b)=0 \quad ②$$

To avoid $C'=D'=0$, set
$$\begin{vmatrix}\cos(\kappa\ln a) & \sin(\kappa\ln a)\\ \cos(\kappa\ln b) & \sin(\kappa\ln b)\end{vmatrix}=\cos(\kappa\ln a)\sin(\kappa\ln b)-\cos(\kappa\ln b)\sin(\kappa\ln b)$$
$$=\sin(\kappa\ln a-\kappa\ln b)$$
$$=\sin(\kappa\ln\tfrac{a}{b})=0$$

so $\kappa\ln(a/b)=n\pi$ $(n=1,2,\ldots)$ or, $\kappa_n=n\pi/\ln(\frac{a}{b})$.

With that choice of κ we now need to solve ① and ② for the resulting nontrivial values of C', D'. With $\kappa_n=n\pi/\ln(\frac{a}{b})$, ② and ① will be redundant, so let us discard ②, say, and solve ① for D' in terms of C':
$$D'=-\cot(\kappa_n\ln a)\,C'.$$

Thus far,
$$u(r,\theta)=\sum_1^\infty C_n'\,[\cos(\kappa_n\ln r)-\cot(\kappa_n\ln a)\sin(\kappa_n\ln r)]\sinh\kappa_n\theta$$
$$=\sum_1^\infty C_n'\,\frac{\cos(\kappa_n\ln r)\sin(\kappa_n\ln a)-\cos(\kappa_n\ln a)\sin(\kappa_n\ln r)}{\sin(\kappa_n\ln a)}\sinh\kappa_n\theta$$
$$=\sum_1^\infty I_n\sin(\kappa_n\ln r-\kappa_n\ln a)\sinh\kappa_n\theta$$
$$=\sum_1^\infty I_n\sin(\kappa_n\ln\tfrac{r}{a})\sinh\kappa_n\theta \quad ③$$

where we've combined $-C_n'/\sin(\kappa_n\ln a)$ as "I_n" for simplicity.

(c) $$u(r,\alpha)=f(r)=\sum_1^\infty(I_n\sinh\kappa_n\alpha)\,\phi_n(r)\quad(a<r<b) \quad ④$$

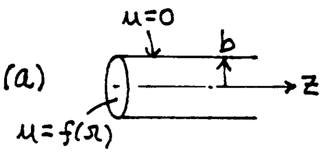
where $\phi_n(r)=\sin(\kappa_n\ln\frac{r}{a})$ are the eigenfunctions of the Sturm-Liouville problem
$$r^2R''+rR'+\kappa^2R=0\quad(a<r<b) \quad ⑤$$
$$R(a)=0,\ R(b)=0.$$

To identify the weight function for the inner product multiply ⑤ by $1/r$ so $(rR')'+\kappa^2\frac{1}{r}R=0$. Thus, the weight function is $1/r$, so ③ gives
$$I_n\sinh\kappa_n\alpha=\frac{\langle f,\phi_n\rangle}{\langle\phi_n,\phi_n\rangle}=\frac{\int_a^b f(r)\sin(\kappa_n\ln\frac{r}{a})\frac{1}{r}\,dr}{\int_a^b\sin^2(\kappa_n\ln\frac{r}{a})\frac{1}{r}\,dr}. \quad ⑥$$

Then the solution is given by ③ and ⑥.

14. $R'' + \frac{1}{r}R' - k^2R = 0$. To identify a, b, c in (50) (page 239), write out (46) (pg 238): $x^a y'' + a x^{a-1} y' + b x^c y = 0$
or, $y'' + a x^{-1} y' + b x^{c-a} y = 0$
so $a = 1$, $b = -k^2$, $c - a = 0$ so $c = a = 1$. Then $\alpha = 2/2 = 1$, $\nu = 0/2 = 0$, so (50) gives $R(r) = r^0 Z_0(\sqrt{1 \cdot k^2}\, r) = Z_0(kr)$.
Since $b < 0$, $Z_0 \to I_0, K_0$ so
$$R(r) = C I_0(kr) + D K_0(kr).$$

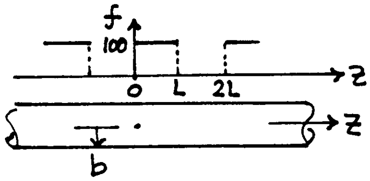
15. Same as in Exercise 14 but with $-k^2 \to +k^2$. Thus,
$$R(r) = r^0 Z_0(\sqrt{k^2}\, r) = Z_0(kr)$$
$$= C J_0(kr) + D Y_0(kr)$$

16. (a) 

Surely the eventual St.-Liou expansion will be on r rather than on z, so write $u(r,z) = R(r)Z(z)$ in $u_{rr} + \frac{1}{r}u_r + u_{zz} = 0$ and obtain
$\frac{R'' + \frac{1}{r}R'}{R} = -\frac{Z''}{Z} = -k^2$. Then
$$R = \begin{cases} A + B\ln r, & k = 0 \\ C J_0(kr) + D Y_0(kr), & k \neq 0 \end{cases}$$
$$Z = \begin{cases} E + Fz, & k = 0 \\ G e^{kz} + H e^{-kz}, & k \neq 0 \end{cases}$$
so $u(r,z) = (A + B\ln r)(E + Fz) + [C J_0(kr) + D Y_0(kr)](G e^{kz} + H e^{-kz})$
Boundedness as $r \to 0 \Rightarrow B = D = 0$, and bddness as $z \to \infty \Rightarrow G = 0$, so
$$u(r,z) = E' + F'z + C' J_0(kr) e^{-kz}.$$
Then
$u(b,z) = 0 = E' + F'z + C' J_0(kb) e^{-kz} \to E' = F' = 0$, $J_0(kb) = 0$ with $kb = z_n$ where the z_n's are the known positive roots of $J_0(z) = 0$. Thus,
$$u(r,z) = \sum_1^\infty C_n' J_0\left(z_n \frac{r}{b}\right) e^{-z_n z/b} \qquad ①$$
Finally,
$$u(r,0) = f(r) = \sum_1^\infty C_n' J_0\left(z_n \frac{r}{b}\right) \qquad (0 \le r < b)$$

where the $J_0(z_n \frac{r}{b})$'s are the eigenfunctions of the S-L problem
$$rR'' + R' + k^2 r R = 0, \quad (0 < r < b)$$
$$R(0) \text{ bdd}, \; R(b) = 0$$
with weight function r. Thus
$$C_n' = \frac{\langle f, J_0 \rangle}{\langle J_0, J_0 \rangle} = \frac{\int_0^b f(r) J_0(z_n \frac{r}{b})\, r\,dr}{\int_0^b J_0^2(z_n \frac{r}{b})\, r\,dr} = \frac{2}{b^2 J_1^2(z_n)} \int_0^b f(r) J_0(k_n r)\, r\,dr \quad ②$$
The solution is given by ① and ②.

(b)



This time we anticipate the expansion to be on the z variable so write

$$\frac{R''+\frac{1}{r}R'}{R} = -\frac{Z''}{Z} = +k^2$$

so $u(r,z) = (A+B\ln r)(E+Fz) + [CI_0(kr)+DK_0(kr)](G\cos kz + H\sin kz)$

Boundedness as $r\to 0 \Rightarrow B=0$ and $D=0$ and boundedness in z implies that $F=0$, so

$$u(r,z) = A' + I_0(kr)(G'\cos kz + H'\sin kz).$$

Understand that this means

$$u(r,z) = A' + I_0(k_1 r)(G_1'\cos k_1 z + H_1'\sin k_1 z) + \cdots + I_0(k_N r)(G_N'\cos k_N z + H_N'\sin k_N z)$$

for any set of k_j's. Looking ahead to the expansion of $f(z)$ in a classical Fourier series

$$f(z) = a_0 + \sum_1^\infty \left(a_n \cos\frac{n\pi z}{L} + b_n \sin\frac{n\pi z}{L}\right)$$

$$a_0 = \text{ave. value} = 50,\quad a_n = \frac{1}{L}\int_{-L}^{L} f(z)\cos\frac{n\pi z}{L}dz = \frac{100}{L}\int_0^L \cos\frac{n\pi z}{L}dz = 0,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(z)\sin\frac{n\pi z}{L}dz = \frac{100}{L}\int_0^L \sin\frac{n\pi z}{L}dz = \begin{cases} 200/n\pi, & n \text{ odd} \\ 0, & n \text{ even}\end{cases}$$

we can see that we should (and can) choose the k's as $n\pi/L$. Thus, write

$$u(r,z) = A' + \sum_1^\infty I_0(n\pi r/L)\left(G_n'\cos\frac{n\pi z}{L} + H_n'\sin\frac{n\pi z}{L}\right)$$

Finally,

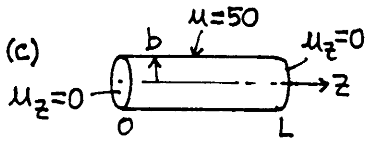
$$u(b,z) = f(z) = 50 + \sum_{1,3,..}^\infty \frac{200}{n\pi}\sin\frac{n\pi z}{L} = A' + \sum_1^\infty I_0\left(\frac{n\pi b}{L}\right)\left(G_n'\cos\frac{n\pi z}{L} + H_n'\sin\frac{n\pi z}{L}\right)$$

so $A'=50$, G_n''s all $=0$, $I_0\left(\frac{n\pi b}{L}\right)H_n' = \frac{200}{n\pi}$ for n odd and 0 for n even. Thus

$$u(r,z) = 50 + \frac{200}{\pi}\sum_{1,3,..}^\infty \frac{1}{n}\frac{I_0(n\pi r/L)}{I_0(n\pi b/L)}\sin\frac{n\pi z}{L}$$

NOTE: alternatively, we could have started with the form $u(r,z) = A + \sum_1^\infty \left[B_n(r)\cos\frac{n\pi z}{L} + C_n(r)\sin\frac{n\pi z}{L}\right]$.

(c)

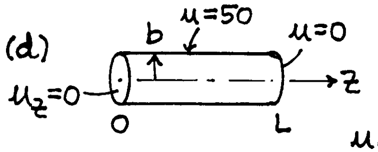


$$\frac{R''+\frac{1}{r}R'}{R} = -\frac{Z''}{Z} = +k^2$$

$\Downarrow$

$u(r,z) = 50$, which can be seen, by inspection, at the outset.

(d)



$$\frac{R''+\frac{1}{r}R'}{R} = -\frac{Z''}{Z} = +k^2$$

$$u(r,z) = (A+B\ln r)(C+Dz) + [EI_0(kr)+FK_0(kr)](G\cos kz + H\sin kz)$$

u bdd as $r\to 0 \Rightarrow B=F=0$, so

$$u(r,z) = C' + D'z + I_0(kr)(G'\cos kz + H'\sin kz)$$

$$u_z(r,0) = 0 = D' + I_0(kr)\,kH' \rightarrow D'=H'=0, \text{ so}$$

$$u(r,z) = C' + G'I_0(kr)\cos kz$$

$u(r,L) = 0 = C' + G' I_0(kr)\cos kL \rightarrow C' = 0$ and $kL = n\pi/2$ (n odd)

$$u(r,z) = \sum_{1,3,\dots} G_n' I_0\left(\frac{n\pi r}{2L}\right)\cos\frac{n\pi z}{2L}$$

$$u(b,z) = 50 = \sum_{1,3,\dots} G_n' I_0\left(\frac{n\pi b}{2L}\right)\cos\frac{n\pi z}{2L} \qquad (0<z<L)$$

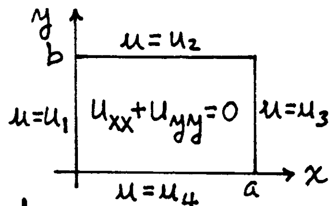
and we can use either a quarter-range cosine formula or the St.-Liou. theory. By QRC:

$$G_n' I_0\left(\frac{n\pi b}{2L}\right) = \frac{2}{L}\int_0^L 50\cos\frac{n\pi z}{2L}\,dz = \frac{200}{n\pi}\sin\frac{n\pi}{2}$$

so

$$u(r,z) = \frac{200}{\pi}\sum_{1,3,\dots}^{\infty}\frac{1}{n}\sin\frac{n\pi}{2}\,\frac{I_0(n\pi r/2L)}{I_0(n\pi b/2L)}\cos\frac{n\pi z}{2L} \qquad (1)$$

NOTE: Recall from Exercise 6 of Sec. 20.2 that for the problem shown at the right we can either write $\frac{X''}{X} = -\frac{Y''}{Y} = +k^2$ (giving cosines and sines on y), apply the u_4 and u_2 bc's first and then do the Fourier series expansions that will be needed on the western and eastern edges OR we can write $\frac{X''}{X} = -\frac{Y''}{Y} = -k^2$ (giving cosines and sines on x), apply the u_1 and u_3 bc's first and then do the Fourier series expansions that will be needed on the southern and northern edges. The resulting series solutions will look different but will sum to the same result. Likewise in this problem we could instead proceed as follows: $\frac{R'' + \frac{1}{r}R'}{R} = -\frac{Z''}{Z} = -k^2$



$\rightsquigarrow \quad u(r,z) = (A + B\ln r)(C + Dz) + [EJ_0(kr) + FY_0(kr)](G\cosh kz + H\sinh kz)$

u bdd $\rightarrow \quad u(r,z) = C' + D'z + J_0(kr)(G'\cosh kz + H'\sinh kz)$

Next we must do

$$u(b,z) = 50 = C' + D'z + J_0(kb)(\quad \cdots \quad \cdots \quad)$$

so $C' = 50$, $D' = 0$ and $k_n = z_n/b$ where z_n's are the positive roots of $J_0(z) = 0$.

Thus, $\quad u(r,z) = 50 + \sum_1^{\infty} J_0(k_n r)(G_n'\cosh k_n z + H_n'\sinh k_n z)$

Then,

$$u_z(r,0) = 0 = \sum_1^{\infty} k_n H_n' J_0(k_n r) \rightarrow H_n'\text{'s} = 0$$

so

$$u(r,z) = 50 + \sum_1^{\infty} G_n' J_0(k_n r)\cosh k_n z \qquad (2)$$

Finally,

$$u(r,L) = 0 = 50 + \sum_1^{\infty} G_n' J_0(k_n r)\cosh k_n L$$

or,

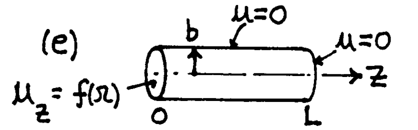
$$-50 = \sum_1^{\infty}(G_n'\cosh k_n L)J_0(k_n r)$$

gives

$$G_n'\cosh k_n L = \frac{\langle -50, J_0(k_n r)\rangle}{\langle J_0(k_n r), J_0(k_n r)\rangle} = \text{etc.} \qquad (3)$$

The solution given by ② and ③ is equivalent to that given by ① but in a different form. My choice would be ① since it is simpler to work with, if only because we need the k_n's in ② and ③ — i.e., the z_n roots of $J_0(z)=0$.

(e)



$$\frac{R''+\frac{1}{r}R'}{R} = -\frac{Z''}{Z} = -k^2.$$ (This time $f(r)$ is not a constant so we have no choice but to do our expansion in r. Hence, choose $-k^2$.)

$$u(r,z) = (A+B\ln r)(C+Dz) + \left[EJ_0(kr)+FY_0(kr)\right](G\cosh kz + H\sinh kz)$$

u bdd as $r\to 0 \Rightarrow$ $u(r,z) = C'+D'z + J_0(kr)(G'\cosh kz + H'\sinh kz)$

$$u(b,z)=0 = C'+D'z + J_0(kb)(G'\cosh kz + H'\sinh kz)$$

so $C'=D'=0$ and $k = z_n/b \equiv k_n$ where $J_0(z_n)=0$.

$$u(r,z) = \sum_1^\infty J_0(k_n r)(G_n'\cosh k_n z + H_n'\sinh k_n z) \qquad ①$$

Then

$$u_z(r,0) = f(r) = \sum_1^\infty k_n H_n' J_0(k_n r) \qquad (0<r<b)$$

so $$k_n H_n' = \frac{\langle f, J_0\rangle}{\langle J_0, J_0\rangle}, \quad H_n' = \frac{1}{k_n}\frac{\int_0^b f(r)J_0(k_n r)\,r\,dr}{\int_0^b J_0^2(k_n r)\,r\,dr} \qquad ②$$ ← denom. can be evaluated

Finally,

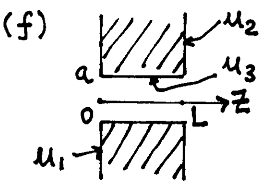
$$u(r,L) = 0 = \sum_1^\infty (G_n'\cosh z_n L + H_n'\sinh z_n L)J_0(k_n r)$$

gives

$$G_n'\cosh z_n L + H_n'\sinh z_n L = 0 \quad \text{or} \quad G_n' = -(\tanh z_n L)H_n' \qquad ③$$

and the solution is given by ①-③.

(f)



$$\frac{R''+\frac{1}{r}R'}{R} = -\frac{Z''}{Z} = k^2$$

so $u(r,z) = (A+B\ln r)(C+Dz) + [EI_0(kr)+FK_0(kr)](G\cos kz + H\sin kz)$

u bdd as $r\to\infty \Rightarrow B=E=0$ so

$$u(r,z) = C'+D'z + K_0(kr)(G'\cos kz + H'\sin kz)$$

$u(r,0) = u_1 = C' + K_0(kr)G' \rightarrow C'=u_1$ and $G'=0$ so

$$u(r,z) = u_1 + D'z + H'K_0(kr)\sin kz$$

$u(r,L) = u_2 = u_1 + D'L + H'K_0(kr)\sin kL$ so $D' = \frac{u_2-u_1}{L}$, $k = n\pi/L$

$$u(r,z) = u_1 + (u_2-u_1)\frac{z}{L} + \sum_1^\infty H_n' K_0(n\pi r/L)\sin(n\pi z/L) \qquad ①$$

Finally,

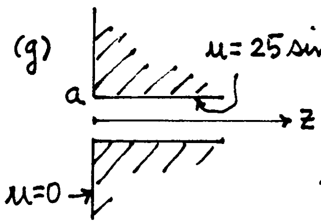
$$u(a,z) = u_3 = u_1 + (u_2-u_1)\frac{z}{L} + \sum_1^\infty H_n' K_0(n\pi a/L)\sin(n\pi z/L)$$

or

$$u_3 - u_1 - (u_2-u_1)\frac{z}{L} = \sum_1^\infty H_n' K_0(n\pi a/L)\sin(n\pi z/L) \qquad (0<z<L)$$

HRS: $$H_n' K_0(n\pi a/L) = \frac{2}{L}\int_0^L \left[u_3 - u_1 - (u_2-u_1)\frac{z}{L}\right]\sin\frac{n\pi z}{L}\,dz = \text{etc.} \qquad ②$$

and the solution is given by ① and ②.

(g) 

$$\frac{R''+\frac{1}{r}R'}{R} = -\frac{Z''}{Z} = +k^2$$

gives $u(r,z) = (A+B\ln r)(C+Dz) + [EI_0(kr)+FK_0(kr)](G\cos kz + H\sin kz)$

u bdd $\Rightarrow B = E = 0$ and $D = 0$, so

$$u(r,z) = A' + K_0(kr)(G'\cos kz + H'\sin kz)$$

and $u(a,z) = 25\sin(3z/2) = A' + K_0(ka)(G'\cos kz + H'\sin kz)$

gives $A'=0$, $G'=0$, $K_0(ka)H' = 25$ and $k = 3/2$, so

$$u(r,z) = 25\,\frac{K_0(3r/2)}{K_0(3a/2)}\sin\frac{3z}{2}$$

17. Students often have trouble with this one.

$$\Phi'' + \cot\phi\,\Phi' + k^2\Phi = 0$$

With $\mu = \cos\phi$, $\frac{d}{d\phi} = \frac{d}{d\mu}\frac{d\mu}{d\phi} = -\sin\phi\frac{d}{d\mu} = -\sqrt{1-\mu^2}\frac{d}{d\mu}$, so we have

$$\left(-\sqrt{1-\mu^2}\frac{d}{d\mu}\right)\left(-\sqrt{1-\mu^2}\frac{d\Phi}{d\mu}\right) + \frac{\mu}{\sqrt{1-\mu^2}}\left(-\sqrt{1-\mu^2}\frac{d\Phi}{d\mu}\right) + k^2\Phi = 0$$

$$(1-\mu^2)\frac{d^2\Phi}{d\mu^2} + \sqrt{1-\mu^2}\left(\tfrac{1}{2}\right)\frac{(-2\mu)}{\sqrt{1-\mu^2}}\frac{d\Phi}{d\mu} - \mu\frac{d\Phi}{d\mu} + k^2\Phi = 0$$

$$(1-\mu^2)\frac{d^2\Phi}{d\mu^2} - 2\mu\frac{d\Phi}{d\mu} + k^2\Phi = 0$$

18. $A_n = \frac{2n+1}{2c^n}\int_{-1}^{1} f P_n\,d\mu = \frac{2n+1}{2c^n}\int_0^1 100 P_n\,d\mu = 50\,\frac{2n+1}{c^n}\int_0^1 P_n(\mu)\,d\mu.$

The maple commands

```
with(orthopoly):
int(P(j,x), x=0..1);
```

gives, for $j=0,1,2,\dots,8$, the values $1, 1/2, 0, -1/8, 0, 1/16, 0, -5/128, 0, \dots$

Thus,

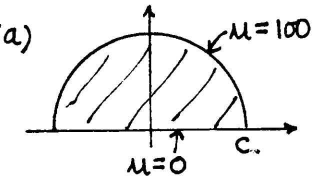
$A_0 = 50$, $A_1 = \frac{150}{c}\left(\frac{1}{2}\right) = \frac{75}{c}$, $A_2 = 0$, $A_3 = \frac{350}{c^3}\left(-\frac{1}{8}\right) = -\frac{175}{4c^3}$, $A_4 = 0$, $A_5 = \frac{550}{c^5}\left(\frac{1}{16}\right) = \frac{275}{8c^5}$,

$A_6 = 0$, $A_7 = \frac{750}{c^7}\left(-\frac{5}{128}\right) = -\frac{1875}{64c^7}$, $A_8 = 0, \dots$

so (81) gives

$$\begin{aligned} u(\rho,\phi) &= \sum_0^\infty A_n \rho^n P_n(\cos\phi) \\ &= 50P_0 + \frac{75}{c}\rho P_1 - \frac{175}{4c^3}\rho^3 P_3 + \frac{275}{8c^5}\rho^5 P_5 - \frac{1875}{64c^7}\rho^7 P_7 + \cdots \\ &= 50\left[P_0(\cos\phi) + \frac{3}{2}\left(\frac{\rho}{c}\right)P_1(\cos\phi) - \frac{7}{8}\left(\frac{\rho}{c}\right)^3 P_3(\cos\phi)\right. \\ &\quad \left. + \frac{11}{16}\left(\frac{\rho}{c}\right)^5 P_5(\cos\phi) - \frac{75}{128}\left(\frac{\rho}{c}\right)^7 P_7(\cos\phi) + \cdots\right] \end{aligned}$$

19. (a)



By (81), $u(\rho,\phi)=\sum_0^\infty A_n \rho^n P_n(\cos\phi)$

$u(\rho,\pi/2)=0=\sum_0^\infty A_n P_n(0)\rho^n$.

Now, the $P_n(0)$'s are 0 if $n=$ odd, so $A_0=A_2=A_4=\cdots=0$ and $u(\rho,\phi)=\sum_{1,3,\ldots} A_n \rho^n P_n(\cos\phi)$

Then,

$$u(c,\phi)=100=\sum_{1,3,\ldots} A_n c^n P_n(\cos\phi) \qquad (0<\phi<\pi/2 \text{ or } 0<\mu<1)$$

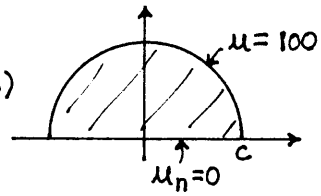
$$\text{so } A_n c^n=\frac{\langle 100,P_n\rangle}{\langle P_n,P_n\rangle}=\frac{\int_0^1 100P_n(\mu)\,d\mu}{\int_0^1 P_n^2(\mu)\,d\mu}=\overset{n=1}{150},\ \overset{3}{-\frac{175}{2}},\ \overset{5}{\frac{275}{4}},\ \overset{7}{-\frac{1875}{32}},\ \overset{9}{\frac{3325}{64}},\ \ldots$$

so

$$u(\rho,\phi)=150\left(\frac{\rho}{c}\right)P_1(\cos\phi)-\frac{175}{2}\left(\frac{\rho}{c}\right)^3P_3(\cos\phi)+\frac{275}{4}\left(\frac{\rho}{c}\right)^5P_5(\cos\phi)$$

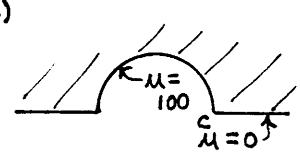
$$-\frac{1875}{32}\left(\frac{\rho}{c}\right)^7P_7(\cos\phi)+\frac{3325}{64}\left(\frac{\rho}{c}\right)^9P_9(\cos\phi)-\cdots$$

(b)



We can see by inspection that $u(\rho,\phi)=$ constant $=100$. Of course, a detailed solution will lead to this result; see that solution outlined in the Answers to Selected Exercises.

(c)



In Example 5 we set $B=0$ in (80) to give boundedness at $\rho=0$, but in this case $\rho=0$ is not relevant since $c<\rho<\infty$. Rather, boundedness as $\rho\to\infty \Rightarrow A_n=0$ for $n\geq 1$ so $u(\rho,\phi)=A_0+\sum_1^\infty \frac{B_n}{\rho^{n+1}}P_n(\cos\phi)$

Then,

$$u(\rho,\pi/2)=0=A_0+\sum_1^\infty \frac{B_n}{\rho^{n+1}}P_n(0)$$

Now, $P_n(0)=0$ if n is odd, so $A_0=B_2=B_4=\cdots=0$ and

$$u(\rho,\phi)=\sum_{1,3,\ldots}\frac{B_n}{\rho^{n+1}}P_n(\cos\phi).$$

Then,

$$u(c,\phi)=100=\sum_{1,3,\ldots}^\infty \frac{B_n}{c^{n+1}}P_n(\cos\phi) \qquad (0<\phi<\pi/2 \text{ or } 0<\mu<1)$$

so $\quad B_n/c^{n+1}=\dfrac{\langle 100,P_n\rangle}{\langle P_n,P_n\rangle}=$ same values as in part (a) above.

Thus,

$$u(\rho,\phi)=\frac{B_1}{\rho^2}P_1(\cos\phi)+\frac{B_3}{\rho^4}P_3(\cos\phi)+\frac{B_5}{\rho^6}P_5(\cos\phi)+\cdots$$

$$=150\left(\frac{c}{\rho}\right)^2P_1(\cos\phi)-\frac{175}{2}\left(\frac{c}{\rho}\right)^4P_3(\cos\phi)+\frac{275}{4}\left(\frac{c}{\rho}\right)^6P_5(\cos\phi)$$

$$-\frac{1875}{32}\left(\frac{c}{\rho}\right)^8P_7(\cos\phi)+\frac{3325}{64}\left(\frac{c}{\rho}\right)^{10}P_9(\cos\phi)-\cdots$$

For ex., at $\rho=2c$ and $\phi=0$, this gives $u(2c,0)=37.5-5.47+1.07-0.23+0.05-\cdots=32.92$,