CHAPTER 24

Section 24.2

1. First, suppose Σa_n and Σb_n both converge, say to A and B, respectively. Then, to each $\epsilon > 0$ there correspond an N_1 and N_2 such that

$$\left|\sum_1^N a_n - A\right| < \epsilon/2 \quad \text{for all } N > N_1,$$
$$\left|\sum_1^N b_n - B\right| < \epsilon/2 \quad \text{" } \quad \text{" } \quad N > N_2.$$

Let $N_0 = \max\{N_1, N_2\}$. Then

$$\begin{aligned}\left|\sum_1^N (a_n + ib_n) - (A+iB)\right| &= \left|\left(\sum_1^N a_n - A\right) + i\left(\sum_1^N b_n - B\right)\right| \\ &\le \left|\sum_1^N a_n - A\right| + \left|\sum_1^N b_n - B\right| \qquad \text{since if } z = a+ib,\ |z| \le |a|+|b| \\ &\le \ \epsilon/2 \ + \ \epsilon/2 \ = \epsilon\end{aligned}$$

for all $N > N_0$, so $\sum_1^\infty c_n$ converges to $A+iB$.
Second, suppose Σc_n converges, say to $A+iB$. Then, to each $\epsilon > 0$ there corresponds an M such that

$$\left|\sum_1^N (a_n + ib_n) - (A+iB)\right| < \epsilon \quad \text{for all } N > M$$

or, $\left|\left(\sum_1^N a_n - A\right) + i\left(\sum_1^N b_n - B\right)\right| < \epsilon$ for all $N > M$. Surely it follows from the latter that

$$\left|\sum_1^N a_n - A\right| < \epsilon \quad \text{for all } N > M$$
$$\text{and} \quad \left|\sum_1^N b_n - B\right| < \epsilon \quad \text{for all } N > M,$$

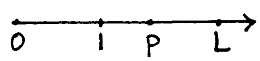
so Σa_n converges to A and Σb_n converges to B, which completes the proof.

2. The triangle inequality states that $|z_1 + z_2| \le |z_1| + |z_2|$. It follows that

$$|z_1 + z_2 + z_3| = |z_1 + (z_2 + z_3)| \le |z_1| + |z_2 + z_3| \le |z_1| + |z_2| + |z_3|.$$

Similarly, $$\begin{aligned}|z_1 + z_2 + z_3 + z_4| = |z_1 + (z_2 + z_3 + z_4)| &\le |z_1| + |z_2 + z_3 + z_4| \\ &\le |z_1| + |z_2| + |z_3| + |z_4|,\end{aligned}$$

and so on.

3. Choose any number p such that $1 < p < L$. 
Then by the definition of the convergence of $|c_{n+1}/c_n|$ to L, it follows that given p there must exist an N such that $|c_{n+1}/c_n| > p$ for all $n > N$. Hence, $|c_{n+1}| > |c_n|$ for all $n > N$. However, Theorem 24.2.2 says that $c_n \to 0$ as $n \to \infty$ is necessary for convergence, so Σc_n must be divergent.

4. Applying the ratio test (Thm 24.2.4) to $\sum_0^\infty a_n (z-a)^n$,

$$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = \left(\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|\right)|z-a| = L|z-a| \begin{array}{l} < 1 \text{ for convergence,} \\ > 1 \text{ for divergence.}\end{array}$$

so (for $L \neq 0, \infty$) $|z-a| < 1/L$ gives convergence and $|z-a| > 1/L$ gives divergence. If $L = 0$ then

$$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = \left(\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|\right)|z-a| = 0\,|z-a| \text{ is } < 1 \text{ for all } z, \text{ so}$$

we have convergence for all z; i.e., for $|z-a|<\infty$. If $L=\infty$ then the latter gives $\infty|z-a|$ which is >1 for all $z\neq a$; hence we have divergence for all $z\neq a$. At $z=a$ we have convergence, of course, because the series is $a_0+0+0+0+\cdots$ which converges to a_0.

5. (a) $\lim_{n\to\infty}\left|\dfrac{(n+1)/(2+i)^{n+1}}{n/(2+i)^n}\right| = \lim_{n\to\infty}\left|\dfrac{n+1}{n}\dfrac{1}{2+i}\right| = \dfrac{1}{\sqrt{5}}<1$, hence convergent by the ratio test.

(b) $\lim_{n\to\infty}\dfrac{(n+1)^{50}/3^{n+1}}{n^5/3^n} = \lim_{n\to\infty}\left(\dfrac{n+1}{n}\right)^{50}\dfrac{1}{3} = \dfrac{1}{3}<1$, hence conv. by ratio test.

(c) Let $M_n = 1/2^n$. Then $|c_n|\le M_n$ for each $n\ge 2$. Since $\sum^\infty M_n = \sum^\infty (1/2)^n$ is a convergent geometric series (conv. because $1/2<1$), $\sum^\infty c_n$ converges by the comparison test.

(d) $c_n\to 1$ as $n\to\infty$; hence divergent by Theorem 24.2.2.

(e) $\lim_{n\to\infty}\left|\dfrac{(1+3i)^{n+1}/(n+1)^{100}}{(1+3i)^n/n^{100}}\right| = \lim_{n\to\infty}\left(\dfrac{\sqrt{10}^{\,n+1}}{\sqrt{10}^{\,n}}\left(\dfrac{n}{n+1}\right)^{100}\right) = \sqrt{10}>1$, so div. by ratio test.

(f) $\lim_{n\to\infty}\left|\dfrac{(n+1)^4 e^{-(5-i)(n+1)}}{n^4 e^{-(5-i)n}}\right| = \lim_{n\to\infty}\left(\dfrac{n+1}{n}\right)^4|e^{-5+i}| = |e^{-5}e^{i}| = e^{-5}|e^{i}| = e^{-5}<1$, so conv. by ratio test.

(g) $|e^{-in}|=1$ for all n so $c_n=e^{-in}$ does not $\to 0$ as $n\to\infty$. Hence, div. by Theorem 24.2.2.

(h) $|c_n| = \left|\sin\left(\dfrac{1+i}{2-i}\right)^n\right| \le \left|\left(\dfrac{1+i}{2-i}\right)^n\right| = \sqrt{\tfrac{2}{5}}^{\,n}$. Since $\sum^\infty M_n = \sum^\infty\left(\sqrt{\tfrac{2}{5}}\right)^n$ is a conv. geometric series, it follows from the comparison test that $\sum c_n$ is convergent.

6. (a) $\sum^\infty z^{2n} = \sum^\infty (z^2)^n$ is a geometric series, which conv. if $|z^2|<1$ (i.e., if $|z|<1$) and div. if $|z^2|>1$ (i.e., if $|z|>1$).

(b) Use Thm. 24.2.5. $L=\lim|(n+1)^2/n^2|=1$, so conv. in $|z-3|<1$ and div. in $|z-3|>1$.

(c) Again use Thm. 24.2.5. $L=\lim (n+1)!/n! = \lim(n+1)=\infty$ so conv. only at $z=-5$.

(d) $L=\lim(e^{n+1}/e^n)=e$ so conv. in $|z+i|<1/e$, div. in $|z+i|>1/e$.

(e) $L=\lim(e^{-(n+1)}/e^{-n})=e^{-1}$ so conv. in $|z|<e$, div. in $|z|>e$.

(f) $L = \lim \left(\frac{n+1}{n}\right)^{100} \frac{n!}{(n+1)!} = \lim \frac{1}{n+1} = 0$ so conv. for all z

(g) $|e^{in}| = 1$, so $L = 1$, so conv. in $|z|<1$, div. in $|z|>1$

(h) $\lim \frac{a_{n+1}}{a_n} = \lim\left(\frac{\cos(n+1)}{\cos n} \frac{n^2+1}{(n+1)^2+1}\right) = \lim \frac{\cos(n+1)}{\cos n}$ does not exist, so the ratio test (Thm 24.2.5) does not apply. We can, at least, say that $|C_n| = \left|\frac{\cos n}{n^2+1} z^n\right| < |z|^n < r^n$ inside the disk $|z|<r$.

Now, if $r<1$ then $\sum^{\infty} r^n$ is a convergent geometric series, so we can at least say that the given series converges in $|z|<r$ for each $r<1$ i.e., the series converges in $|z|<1$. (No information for $|z|\geq 1$.)

(i) It's a geometric series: conv. in $|(2-i)z|<1$, i.e., in $|z|<1/\sqrt{5}$, and div. in $|z|>1/\sqrt{5}$.

(j) $L = \lim \frac{e^{(n+1)^2}}{(n+1)!} \frac{n!}{e^{n^2}} = \lim \frac{e^{2n+1}}{n+1} \overset{\text{l'Hôpital}}{=} \lim \frac{2e^{2n+1}}{1} = \infty$ so, by Theorem 24.2.5, the series converges only at $z=0$.

7. $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x=0. \end{cases}$, $f'(x) = 2e^{-1/x^2}/x^3$ for $x\neq 0$.

$$f'(0) = \lim_{x\to 0} \frac{e^{-1/x^2}-0}{x} = \lim_{x\to 0} \frac{1}{x(1+\frac{1}{x^2}+\frac{1}{2!}\frac{1}{x^4}+\cdots)} = 0$$

$$f''(0) = \lim_{x\to 0} \frac{2e^{-1/x^2}/x^3-0}{x} = 2\lim_{x\to 0}\frac{e^{-1/x^2}}{x^4} = 2\lim_{x\to 0}\frac{1}{x^4(1-\frac{1}{x^2}+\frac{1}{2!}\frac{1}{x^4}-\cdots)} = 0$$

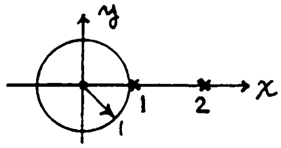
and similarly for $f'''(0), \ldots$.

8.(a) $L = \lim\left|\frac{(-1)^{n+2}(n+1)}{(-1)^{n+1}n}\right| = 1$, so the power series converges in $|z-1|<1$ and diverges in $|z-1|>1$. It is the Taylor series of its sum function in $|z-1|<1$.

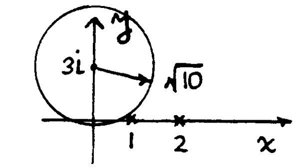
(b) $L = \lim \frac{4^{n+1}}{4^{n+2}} = \frac{1}{4}$, so the power series converges in $|z|<1/4$ and diverges in $|z|>1/4$. It is the Taylor series of its sum function in $|z|<1/4$.

(c) It is a geometric series (missing the first several terms) $\sum^{\infty}\left[\left(\frac{z+i}{1+i}\right)^2\right]^n$ so it converges in $\left|\frac{z+i}{1+i}\right|^2<1$, i.e., in $|z+i|<\sqrt{2}$, and diverges in $|z+i|>\sqrt{2}$. It is the Taylor series of its sum function in $|z+i|<\sqrt{2}$.

9. (a) $z^2-3z+2=0$ at $z=1,2$ so the TS about $z=0$ will converge in $|z|<1$.



(b)



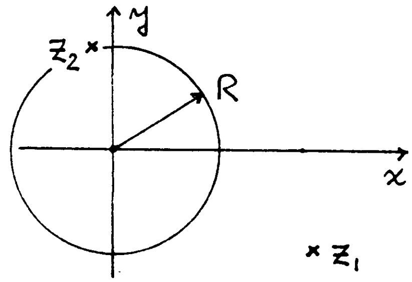
Conv. in $|z-3i|<\sqrt{10}$

(c) Conv. in $|z-(1-5i)|<5$ (d) Conv. in $|z-(5-i)|<\sqrt{10}$

10. (a) denominator $= z^2-2z+3i+1=0$ at

$z=(1+\sqrt{\tfrac{3}{2}})-\sqrt{\tfrac{3}{2}}\,i \equiv z_1$

and $(1-\sqrt{\tfrac{3}{2}})+\sqrt{\tfrac{3}{2}}\,i \equiv z_2$.



The numerator does not vanish at either of these points so z_1, z_2 are indeed singular points of the given function.

$$R=|z_2| = \sqrt{(1-\sqrt{3/2})(1-\sqrt{3/2})+3/2} = \sqrt{4-\sqrt{6}}.$$

(b) $R=|10i-z_2| = \sqrt{(\sqrt{\tfrac{3}{2}}-1)^2+(10-\sqrt{\tfrac{3}{2}})^2} = \sqrt{104-22\sqrt{\tfrac{3}{2}}}$ since it is evident that z_2 is closer to $10i$ than z_1

(c) $R=|2-5i-z_1|$ since it is evident that z_1 is closer to $2-5i$ than z_1

$= \sqrt{(1-\sqrt{3/2})^2+(5-\sqrt{3/2})^2} = \sqrt{29-6\sqrt{6}}$

(d) $R=|20-z_1| = \sqrt{(19-\sqrt{3/2})^2+(\sqrt{3/2})^2} = \sqrt{364-19\sqrt{6}}$

11. (a) $\sin z = z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots$, $R=\infty$.

(b) $\sin z = \sin a + (\cos a)(z-a) - \frac{\sin a}{2!}(z-a)^2 - \frac{\cos a}{3!}(z-a)^3 + \frac{\sin a}{4!}(z-a)^4+\cdots$, $R=\infty$,
where $\sin a = \sin(2-i) = \sin 2\cos i - \sin i \cos 2 = \sin 2\cosh 1 - i\sinh 1\cos 2$
and $\cos a = \cos(2-i) = \cos 2\cos i + \sin 2\sin i = \cos 2\cosh 1 + i\sin 2\sinh 1$.

(c) $\cos 2z = \cosh 6 - 2i(\sinh 6)(z-3i) - 2(\cosh 6)(z-3i)^2 + \frac{4}{3}i(\sinh 6)(z-3i)^3+\cdots$,
$R=\infty$.

(d) Can reduce the labor by letting $z^6\equiv w$, say. Then, expanding e^w in powers of w will give the desired series in powers of z:

$$e^{z^6} = e^w = 1+w+w^2/2!+\cdots = 1+z^6+\tfrac{1}{2!}z^{12}+\tfrac{1}{3!}z^{18}+\cdots, \quad R=\infty.$$

(e) To use the known geometric series re-express as

$$\frac{1}{i+z} = \frac{1}{i(1-iz)} = -i\frac{1}{1-iz} = -i[1+iz+(iz)^2+(iz)^3+\cdots]$$
$$= -i+z+iz^2-z^3-\cdots \quad \text{in } |iz|<1, \text{ i.e.,}$$

in $|z|<1$, so $R=1$.

(f) Get in geometric series form:

$$\frac{z^3}{2-iz} = \frac{z^3}{2}\frac{1}{1-\frac{iz}{2}} = \frac{z^3}{2}\left(1+\frac{iz}{2}+\left(\frac{iz}{2}\right)^2+\left(\frac{iz}{2}\right)^3+\cdots\right) = \frac{1}{2}z^3+\frac{i}{4}z^4-\frac{1}{8}z^5-\frac{i}{16}z^6\cdots$$

or, in summation form, $= \sum_0^\infty \frac{i^n}{2^{n+1}} z^{n+3}$; $R=2$ since we need $\left|\frac{iz}{2}\right|<1$

(g) Let $z^8 \equiv w$, say. Then $\sin z^8 = \sin w = w - \frac{1}{3!}w^3 + \frac{1}{5!}w^5 - \cdots$

$$= z^8 - \frac{1}{3!}z^{24} + \frac{1}{5!}z^{40} - \cdots$$

or, in summation form,

$$= \sum_1^\infty (-1)^{n+1}\frac{z^{16n-8}}{(2n-1)!}, \quad R=\infty.$$

The $z^8=w$ idea was important so we don't need to waste our time working out all the in-between terms, the coefficients of which are 0.

(h) $z^3 = (-2i)^3 + 3(-2i)^2(z+2i) + \frac{6(-2i)}{2!}(z+2i)^2 + \frac{6}{3!}(z+2i)^3$

$= 8i - 12(z+2i) - 6i(z+2i)^2 + (z+2i)^3$; $R=\infty$.

The series terminates. NOTE: If you want a Taylor series about $-2i$, do not expand the powers on the right-hand side and simplify, which would merely give z^3 !

(i) $1/(1+2z^{35}) = 1 - 2z^{35} + 4z^{70} - 8z^{105} + \cdots$, or, $= \sum_0^\infty (-2z^{35})^n = \sum_0^\infty (-2)^n z^{35n}$; need $|2z^{35}|<1$ or $|z|< 1/2^{1/35}$; $R = 1/2^{1/35}$.

(j) $z^2 - iz = (-4+2) + 3i(z-2i) + \frac{2}{2!}(z-2i)^2 = -2 + 3i(z-2i) + (z-2i)^2$; $R=\infty$.

12. (b) $\frac{1}{(3-z)^2} = \frac{1}{(3-i)^2\left[1-\left(\frac{z-i}{3-i}\right)\right]^2}$ so "z" is $\frac{z-i}{3-i}$ and "m" is 2

$$= \frac{1}{(3-i)^2}\sum_0^\infty \frac{(2+n-1)!}{(2-1)!\,n!}\left(\frac{z-i}{3-i}\right)^n = \sum_0^\infty \frac{(n+1)!}{n!}\frac{(z-i)^n}{(3-i)^{n+2}} = \sum_0^\infty \frac{n+1}{(3-i)^{n+2}}(z-i)^n$$

in $|\text{"}z\text{"}| = \left|\frac{z-i}{3-i}\right| = \frac{|z-i|}{\sqrt{10}} < 1$, i.e., in $|z-i| < \sqrt{10}$.

13. (a) $\frac{1}{(2z+1)^3} = \frac{1}{[1-(-2z)]^3} = \sum_0^\infty \frac{(3+n-1)!}{(3-1)!\,n!}(-2z)^n = \sum_0^\infty \frac{(n+2)!}{2n!}(-2z)^n$

$$= \frac{1}{2}\sum_0^\infty (-1)^n (n+2)(n+1)2^n z^n$$

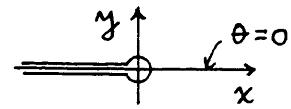
in $|-2z| = 2|z| < 1$, i.e., in $|z| < 1/2$.

(b) $\dfrac{1}{(2z+1)^3} = \dfrac{1}{[2(z-2)+5]^3} = \dfrac{1}{125}\,\dfrac{1}{\left[1-\left(-\frac{2(z-2)}{5}\right)\right]^3}$ so "z" is $\dfrac{-2(z-2)}{5}$, "m" is 3

$$= \frac{1}{125}\sum_0^\infty \frac{(3+n-1)!}{2!\,n!}\left[-\tfrac{2}{5}(z-2)\right]^n = \frac{1}{250}\sum_0^\infty (-1)^n(n+1)(n+2)\left(\tfrac{2}{5}\right)^n (z-2)^n$$

in $\left|-\frac{2}{5}(z-2)\right| < 1$ or, $|z-2| < 5/2$.

14. (a) $f(z) = \sqrt{z}$, $f' = \frac{1}{2}z^{-1/2}$, $f'' = -\frac{1}{4}z^{-3/2}, \ldots$



$a=1$: $f(z) = \sqrt{1} + \frac{1}{2}1^{-1/2}(z-1) - \frac{1}{4\cdot 2!}1^{-3/2}(z-1)^2 + \cdots$ and we need to evaluate these coefficients according to the branch cut chosen: $1^{1/2} = (1e^{i0})^{1/2} = 1$, $1^{-1/2} = (1e^{i0})^{-1/2} = 1$, $1^{-3/2} = (1e^{i0})^{-3/2} = 1$, so

$$\sqrt{z} = 1 + \tfrac{1}{2}(z-1) - \tfrac{1}{8}(z-1)^2 + \cdots$$ in $|z-1| < 1$, since if we make the circle any larger it will contain part of the branch cut so f will not be analytic throughout that disk.

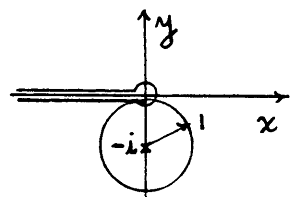
(b) $a=-i$: $f(z) = (-i)^{1/2} + \frac{1}{2}(-i)^{-1/2}(z+i) - \frac{1}{4\cdot 2!}(-i)^{-3/2}(z+i)^2 - \cdots$

where

$(-i)^{1/2} = (1e^{-\pi i/2})^{1/2} = e^{-\pi i/4} = (1-i)/\sqrt{2}$,

$(-i)^{-1/2} = (\quad " \quad)^{-1/2} = e^{\pi i/4} = (1+i)/\sqrt{2}$,

$(-i)^{-3/2} = (\quad " \quad)^{-3/2} = e^{3\pi i/4} = (-1+i)/\sqrt{2}$



and so on. Thus,

$$\sqrt{z} = \frac{1-i}{\sqrt{2}} + \frac{1}{2}\,\frac{1+i}{\sqrt{2}}(z+i) - \frac{1}{8}\,\frac{(-1+i)}{\sqrt{2}}(z+i)^2 + \cdots$$ in $|z+i| < 1$.

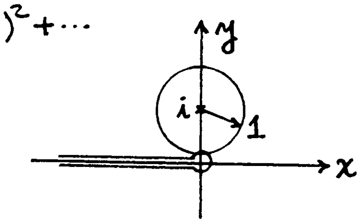
(c) $a=i$: $f(z) = i^{1/2} + \frac{1}{2}i^{-1/2}(z-i) + \left(-\frac{1}{4\cdot 2!}\right)i^{-3/2}(z-i)^2 + \cdots$

where

$(i)^{1/2} = (e^{\pi i/2})^{1/2} = e^{\pi i/4} = (1+i)/\sqrt{2}$

$(i)^{-1/2} = (\quad " \quad)^{-1/2} = e^{-\pi i/4} = (1-i)/\sqrt{2}$

$(i)^{-3/2} = (\quad " \quad)^{-3/2} = e^{-3\pi i/4} = (-1-i)/\sqrt{2}$



and so on. Thus,

$$\sqrt{z} = \frac{1+i}{\sqrt{2}} + \frac{1}{2}\,\frac{1-i}{\sqrt{2}}(z-i) - \frac{1}{4\cdot 2!}\,\frac{-1-i}{\sqrt{2}}(z-i)^2 + \cdots$$ in $|z-i| < 1$

15. $$1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \frac{z^5}{120} + \cdots = (a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + \cdots) \times \left(1 - \frac{z^2}{2} + \frac{z^4}{24} - \cdots\right)$$

z^4: $1/24 = a_0/24 - a_2/2 + a_4$ gives $a_4 = \frac{1}{24} - \frac{1}{24} + \frac{1}{2} = \frac{1}{2}$

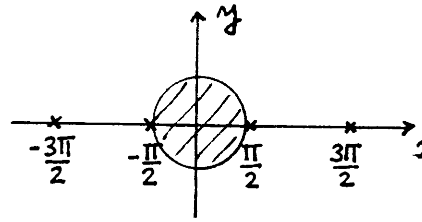
z^5: $1/120 = a_1/24 - a_3/2 + a_5$ gives $a_5 = \frac{1}{120} - \frac{1}{24} + \frac{1}{3} = \frac{3}{10}$

so $$\frac{e^z}{\cos z} = 1 + z + z^2 + \frac{2}{3}z^3 + \frac{1}{2}z^4 + \frac{3}{10}z^5 + \cdots$$

16.(a) $\tan z = \dfrac{\sin z}{\cos z} = \dfrac{z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \cdots}{1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \cdots} = a_0 + a_1 z + a_2 z^2 + \cdots$

$$z - \tfrac{1}{6}z^3 + \tfrac{1}{120}z^5 - \cdots = (1 - \tfrac{1}{2}z^2 + \tfrac{1}{24}z^4 - \cdots)(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots)$$

z^0: $0 = a_0$
z^1: $1 = a_1$
z^2: $0 = -\frac{1}{2}a_0 + a_2 \to a_2 = 0$
z^3: $-1/6 = a_3 - \frac{1}{2}a_1 \to a_3 = 1/3$
z^4: $0 = a_4 - \frac{1}{2}a_2 + \frac{1}{24}a_0 \to a_4 = 0$
z^5: $1/120 = a_5 - \frac{1}{2}a_3 + \frac{1}{24}a_1 \to \quad a_5 = 2/15$



and so on, so

$$\tan z = z + \tfrac{1}{3}z^3 + \tfrac{2}{15}z^5 + \cdots$$

in $|z| < \pi/2$ since $\tan z$ is singular at the zeros of $\cos z$, namely, at $\pm\pi/2, \pm 3\pi/2, \ldots$. The distance from $z=0$ to the closest of these is $\pi/2$.

(b) $\sec z = 1/\cos z$ so $1 = (1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \cdots)(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots)$

z^0: $1 = a_0$
z^1: $0 = a_1$
z^2: $0 = a_2 - \frac{1}{2}a_0 \to a_2 = 1/2$
z^3: $0 = a_3 - \frac{1}{2}a_1 \to a_3 = 0$
z^4: $0 = a_4 - \frac{1}{2}a_2 + \frac{1}{24}a_0 \to a_4 = 5/24$

and so on, so

$$\sec z = 1 + \tfrac{1}{2}z^2 + \tfrac{5}{24}z^4 + \cdots \quad \text{in } |z| < \pi/2, \text{ as in part (a).}$$

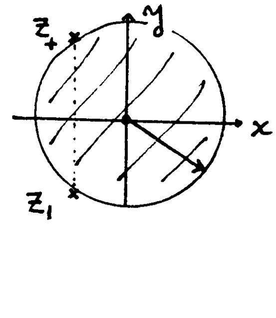
(c) $\csc z = 1/\sin z$ does not admit a Taylor (i.e., Maclaurin) series about $z=0$ because it is singular at $z=0$ (since $\sin 0 = 0$).

(d) $1 + z = (1 + 2z + 3z^2)(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots)$

z^0: $1 = a_0$
z^1: $1 = a_1 + 2a_0 \to a_1 = -1$
z^2: $0 = a_2 + 2a_1 + 3a_0 \to a_2 = -1$
z^3: $0 = a_3 + 2a_2 + 3a_1 \to a_3 = 5$
z^4: $0 = a_4 + 2a_3 + 3a_2 \to a_4 = -7$

and so on, so

$$\frac{1+z}{1+2z+3z^2} = 1 - z - z^2 + 5z^3 - 7z^4 + \cdots .$$



$3z^2 + 2z + 1 = 0$ gives $z = (-2 \pm \sqrt{-8})/6 = (-1 \pm i\sqrt{2})/3 \equiv z_\pm$

so convergence is in $|z| < 1/\sqrt{3}$

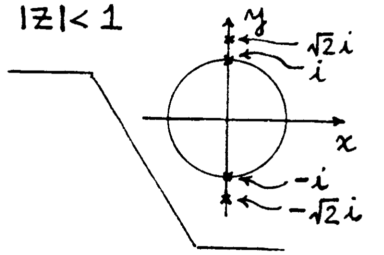
(e) Let us merely use Maple:

```
taylor((3-z)/(2+3*z^2+z^4), z=0, 8);
```

gives

$$\frac{3-z}{2+3z^2+z^4} = \frac{3}{2}-\frac{1}{2}z-\frac{9}{4}z^2+\frac{3}{4}z^3+\frac{21}{8}z^4-\frac{7}{8}z^5-\frac{45}{16}z^6+\frac{15}{16}z^7+\cdots$$

In what disk? $z^4+3z^2+2=0$ gives $z^2=(-3\pm\sqrt{1})/2=-1,-2$, so $z=\pm i, \pm\sqrt{2}\,i$. Thus, the series converges in $|z|<1$



(f) $e^z/\sin 2z$ does not admit a Taylor series about $z=0$ because it is not analytic there.

(h) $1=(4-\frac{z^2}{2}+\frac{z^4}{24}-\cdots)(a_0+a_1z+a_2z^2+a_3z^3+a_4z^4+\cdots)$

z^0: $1=4a_0$

z^1: $0=4a_1 \rightarrow a_1=0$

z^2: $0=4a_2-\frac{1}{2}a_0 \rightarrow a_2=1/32$

z^3: $0=4a_3-\frac{1}{2}a_1 \rightarrow a_3=0$

z^4: $0=4a_4-\frac{1}{2}a_2+\frac{1}{24}a_0 \rightarrow a_4=1/768$

and so on, so

$$\frac{1}{3+\cos z}=\frac{1}{4}+\frac{1}{32}z^2+\frac{1}{768}z^4+\cdots$$

NOTE: actually, we could have omitted the $a_1z, a_3z^3, \ldots$ terms since $1/(3+\cos z)$ is an even function of z.

In what disk? Set $3+\cos z=0$. $3+(e^{iz}+e^{-iz})/2=0$. Let e^{iz} be t.
$t^2+6t+1=0$, $t=(-6\pm\sqrt{32})/2=-3\pm2\sqrt{2}$ (both negative)

so $iz=\log(-3\pm2\sqrt{2})=\ln(3\mp2\sqrt{2})+i(\pi+2n\pi)$

$z=(\pi+2n\pi)-i\ln(3\mp2\sqrt{2})$, $\quad n=0,\pm1,\pm2,\ldots$

of which the smallest one (i.e., the one closest to the point of expansion, which is the origin) is $z=\pi-i\ln(3+2\sqrt{2})$ (actually, $\ln(3-2\sqrt{2})$ is $=-\ln(3+2\sqrt{2})$, so either one will do), so

$$R=\sqrt{\pi^2+[\ln(3+2\sqrt{2})]^2}.$$

Section 24.3

1. $t=(t-\frac{1}{6}t^3+\frac{1}{120}t^5-\cdots)(a_0+a_2t^2+a_4t^4+a_6t^6$

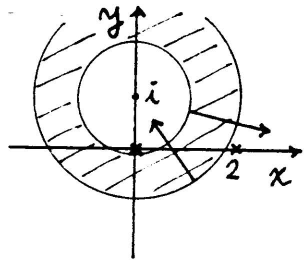
t: $1=a_0$

t^3: $0=a_2-\frac{1}{6}a_0 \rightarrow a_2=1/6$

t^5: $0=a_4-\frac{1}{6}a_2+\frac{1}{120}a_0 \rightarrow a_4=7/360$

t^7: $0=a_6-\frac{1}{6}a_4+\frac{1}{120}a_2-\frac{1}{5040}a_0 \rightarrow a_6=31/15120$

so $t/\sin t = 1+(1/6)/t^2+(7/360)t^4+(31/15120)t^6+\cdots$



2. $f(z)=\frac{1}{z(z-2)} = -\frac{1}{2}\frac{1}{z}+\frac{1}{2}\frac{1}{z-2}$

Expand the $1/(z-2)$ in $|z-i|<\sqrt{5}$ in a TS (Taylor series) and expand the $1/z$ in $|z-i|>1$ in a LS (Laurent series), as indicated by the arrows at the right.

$$f(z)=-\frac{1}{2}\frac{1}{i+(z-i)}+\frac{1}{2}\frac{1}{-2+i+(z-i)} = -\frac{1}{2}\frac{1}{i+t}+\frac{1}{2}\frac{1}{-2+i+t}$$

$$= -\frac{1}{2t}\frac{1}{1+\frac{i}{t}}+\frac{1}{2}\frac{1}{-2+i}\frac{1}{1+\frac{t}{-2+i}} = -\frac{1}{2t}\sum_0^\infty\left(-\frac{i}{t}\right)^n-\frac{2+i}{10}\frac{1}{1-\frac{2+i}{5}t}$$

$$= -\frac{1}{2}\sum_0^\infty (-i)^n t^{-n-1}-\frac{2+i}{10}\sum_0^\infty\left(\frac{2+i}{5}\right)^n t^n$$

$$= \underbrace{\underbrace{-\frac{1}{2}\sum_0^\infty (-i)^n (z-i)^{-n-1}}_{\text{Conv. in } |z-i|>1}\;\underbrace{-\frac{1}{2}\sum_0^\infty\left(\frac{2+i}{5}\right)^{n+1}(z-i)^n}_{\text{Conv. in } |z-i|<\sqrt{5}}}_{\text{Valid in the overlap } 1<|z-i|<\sqrt{5}.}$$

3. $f(z)=\frac{1}{z(z-2)}=\frac{1}{i+(z-i)}\,\frac{1}{i-2+(z-i)}=\frac{1}{(z-i)^2}\,\frac{1}{1+\frac{i}{z-i}}\,\frac{1}{1+\frac{i-2}{z-i}}$ (let $z-i$ be t)

$$=\frac{1}{t^2}\sum_{n=0}^\infty\left(-\frac{i}{t}\right)^n\sum_{m=0}^\infty\left(\frac{2-i}{t}\right)^m=\frac{1}{t^2}\sum_{n=0}^\infty\sum_{m=0}^\infty(-i)^n(2-i)^m t^{-(m+n)}$$

Now let $p=m+n$, $q=m$ (or n; either way is fine)

or, $n=p-q$
$m=q$

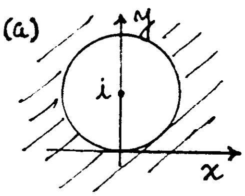
so the boundaries $m=0, n=0$ of the m,n quarter plane map into $0=p-q$ (ie, $p=q$) and $q=0$, hence the image is the wedge from $\pi/4$ to $\pi/2$, as shown in the figure in the text. It will be best, in writing the iterated sum on p and q, to sum on q first because the q limits will then be finite.

$$=\sum_{p=0}^\infty\left(\sum_{q=0}^p(-i)^{p-q}(2-i)^q\right)t^{-(p+2)}$$ or, writing out through $p=3$, say,

$$=\left((-i)^{0-0}(2-i)^0\right)t^{-(0+2)}+\left((-i)^{1-0}(2-i)^0+(-i)^{1-1}(2-i)^1\right)t^{-(1+2)}$$
$$+\left((-i)^{2-0}(2-i)^0+(-i)^{2-1}(2-i)^1+(-i)^{2-2}(2-i)^2\right)t^{-(2+2)}+\cdots$$

$$=t^{-2}+(2-2i)t^{-3}+(1-6i)t^{-4}+\cdots$$, which does agree with (33).

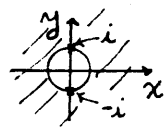
4. (a)



$$\frac{1}{z}=\frac{1}{i+(z-i)}=\frac{1}{t}\frac{1}{1+\frac{i}{t}}=\frac{1}{t}\left(1-\frac{i}{t}+\frac{i^2}{t^2}-\frac{i^3}{t^3}+\cdots\right)=\sum_0^\infty\frac{(-i)^n}{(z-i)^{n+1}}$$

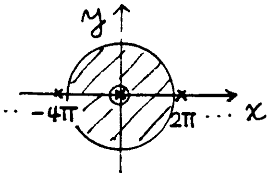
$$=\frac{1}{z-i}-i\frac{1}{(z-i)^2}-\frac{1}{(z-i)^3}+\frac{i}{(z-i)^4}-\cdots \quad \text{in } 1<|z-i|<\infty$$

(b) $\dfrac{1}{z^2+1} = \dfrac{1}{z^2(1+\frac{1}{z^2})} = \dfrac{1}{z^2}\sum_0^\infty \left(-\dfrac{1}{z^2}\right)^n = \sum_0^\infty (-1)^n \dfrac{1}{z^{2(n+1)}}$ in $1<|z|<\infty$



(c) $\dfrac{z^2+3}{z} = \dfrac{3}{z} + z$ in $0<|z|<\infty$

(d) $\dfrac{1}{e^z-1} = ?$ Singularities at the roots of $e^z=1$, namely, at $z=\log 1 = 2n\pi i$ $(n=0,\pm1,\pm2,\dots)$



$e^z-1 = 1+z+\dfrac{z^2}{2!}+\cdots-1 = z+\cdots$, hence there is a first order pole at $z=0$. Thus, write

$\dfrac{1}{e^z-1} = \dfrac{1}{z}\dfrac{z}{e^z-1}$ ← analytic at $z=0$ and in $|z|<2\pi$, so set

$\dfrac{z}{e^z-1} = a_0+a_1z+\cdots$ or, $z = (z+\frac{1}{2}z^2+\frac{1}{6}z^3+\frac{1}{24}z^4+\cdots)(a_0+a_1z+a_2z^2+\cdots)$

z: $1=a_0$,

z^2: $0=a_1+\frac{1}{2}a_0 \to a_1=-1/2$,

z^3: $0=a_2+\frac{1}{2}a_1+\frac{1}{6}a_0 \to a_2 = 1/12$,

and so on, so

$$\frac{1}{e^z-1} = \frac{1}{z}\left(1-\frac{1}{2}z+\frac{1}{12}z^2+\cdots\right) = \frac{1}{z}-\frac{1}{2}+\frac{1}{12}z+\cdots \quad \text{in } 0<|z|<2\pi$$

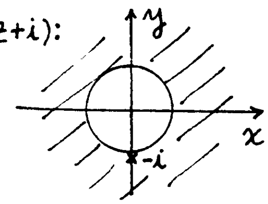
(e) $\dfrac{1}{z(z^3+2)} = \dfrac{1}{2z}\,\dfrac{1}{1+z^3/2} = \underbrace{\dfrac{1}{2z}}_{\text{one-term LS in } 0<|z|<\infty}\underbrace{\left(1-\dfrac{z^3}{2}+\dfrac{z^6}{4}-\cdots\right)}_{\text{TS in } 0\le|z|<\sqrt[3]{2}}$

$= \dfrac{1}{2z}-\dfrac{1}{4}z^2+\dfrac{1}{8}z^5-\dfrac{1}{16}z^8+\cdots$ in $0<|z|<\sqrt[3]{2}$

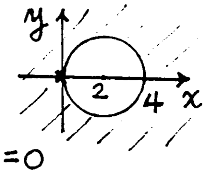
(f) $\dfrac{1}{z}+\dfrac{z}{z+i} = \dfrac{1}{z}+\dfrac{1}{1+\frac{i}{z}} = \underset{\substack{\uparrow \\ \text{LS in } 0<|z|<\infty}}{\dfrac{1}{z}} + \underset{\substack{\uparrow \\ \text{LS in } 1<|z|<\infty}}{\left(1-\dfrac{i}{z}+\dfrac{i^2}{z^2}-\dfrac{i^3}{z^3}+\cdots\right)}$

$= 1+(1-i)\dfrac{1}{z}-\dfrac{1}{z^2}+i\dfrac{1}{z^3}-\cdots$ in $1<|z|<\infty$

$z/(z+i)$:



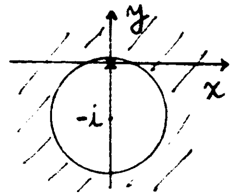
(g) $\dfrac{1}{z^3} = \dfrac{1}{[2+(z-2)]^3} = \dfrac{1}{t^3(1+\frac{2}{t})^3} = \dfrac{1}{t^3}\underbrace{\left(1+\dfrac{2}{t}\right)^{-3}}_{\text{Do a T.S. in } y=2/t \text{ about } y=0}$ $\quad (t=z-2)$



$= \dfrac{1}{t^3}\left(1-3\left(\dfrac{2}{t}\right)+6\left(\dfrac{2}{t}\right)^2-10\left(\dfrac{2}{t}\right)^3+\cdots\right) = \dfrac{1}{(z-2)^3}-6\dfrac{1}{(z-2)^4}+24\dfrac{1}{(z-2)^5}-80\dfrac{1}{(z-2)^6}+\cdots$

in $2<|z-2|<\infty$.

(h) $\dfrac{1}{z^2} = \dfrac{1}{[(z+i)-i]^2} = \dfrac{1}{t^2}\dfrac{1}{(1-\frac{i}{t})^2} \quad (t = z+i)$

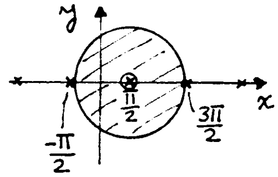


$= \dfrac{1}{t^2}\left(1 + \dfrac{2i}{t} + 3\left(\dfrac{i}{t}\right)^2 + 4\left(\dfrac{i}{t}\right)^3 + \cdots\right)$ ← TS in $\varphi = i/t$, in $|\varphi|<1$ (i.e., in $|t|>1$)

$= \dfrac{1}{(z+i)^2} + 2i\dfrac{1}{(z+i)^3} + 3i^2\dfrac{1}{(z+i)^4} + 4i^3\dfrac{1}{(z+i)^5} + \cdots$ in $1<|z+i|<\infty$

$= \displaystyle\sum_0^\infty \frac{(n+1)i^n}{(z+i)^{n+2}}$

(i) $\cos z = 0$ at $z = \pm\pi/2, \pm 3\pi/2, \ldots$.



$1/\cos z = 1/\cos[\pi/2 + (z-\pi/2)] = 1/[\overset{0}{\cos\frac{\pi}{2}}\cos(z-\frac{\pi}{2}) - \sin\frac{\pi}{2}\sin(z-\frac{\pi}{2})]$

$= -\dfrac{1}{\sin(z-\frac{\pi}{2})} = -\dfrac{1}{\sin t} = -\dfrac{1}{t}\dfrac{t}{\sin t}$ ← analytic in $0 \le |t| < \pi$ so Taylor expand it in that disk

$\dfrac{t}{\sin t} = a_0 + a_1 t + a_2 t^2 + \cdots$ (Can omit $a_1 t, a_3 t^3, \ldots$ since $t/\sin t$ is even in t.)

$t = (a_0 + a_2 t^2 + a_4 t^4 + \cdots)(t - \frac{1}{6}t^3 + \frac{1}{120}t^5 - \cdots)$

t: $1 = a_0$

t^3: $0 = a_2 - \frac{1}{6}a_0 \to a_2 = 1/6$

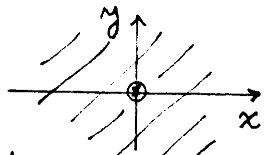
t^5: $0 = a_4 - \frac{1}{6}a_2 + \frac{1}{120}a_0 \to a_4 = 7/360$

and so on, so

$$\frac{1}{\cos z} = -\frac{1}{t}\left(1 + \frac{1}{6}t^2 + \frac{7}{360}t^4 + \cdots\right) = -\frac{1}{z-\frac{\pi}{2}} - \frac{1}{6}\frac{1}{(z-\frac{\pi}{2})^3} - \frac{7}{360}\frac{1}{(z-\frac{\pi}{2})^5} - \cdots$$

in $0 < |z - \frac{\pi}{2}| < \pi$

5. (a) $\sin 1/z$ is singular only at the point of expansion, $a=0$, so there is only one expansion possible, a LS in $0<|z|<\infty$. To obtain it, write

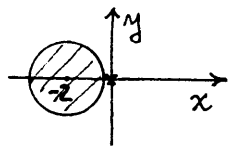


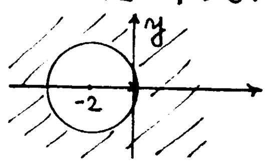
$\sin\frac{1}{z} = \sin t \quad (t = 1/z)$

and do a TS of $\sin t$ in $0 \le |t| < \infty$, which is equivalent to $0<|z|<\infty$:

$\sin\frac{1}{z} = \sin t = t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \cdots = \frac{1}{z} - \frac{1}{3!}\frac{1}{z^3} + \frac{1}{5!}\frac{1}{z^5} - \cdots$ in $0<|z|<\infty$

(b) Two possible expansions: In $0 \le |z+2| < 2$: In $2<|z+2|<\infty$:





In $0 \le |z+2| < 2$:

$\dfrac{1}{z} = \dfrac{1}{-2+(z+2)} = -\dfrac{1}{2}\dfrac{1}{1-\frac{z+2}{2}} = -\dfrac{1}{2}\left(1 + \dfrac{z+2}{2} + \left(\dfrac{z+2}{2}\right)^2 + \cdots\right) = -\dfrac{1}{2} - \dfrac{1}{4}(z+2) - \dfrac{1}{8}(z+2)^2 - \cdots$

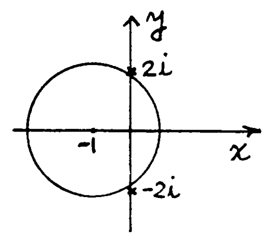
In $2<|z+2|<\infty$:

$\dfrac{1}{z} = \dfrac{1}{-2+(z+2)} = \dfrac{1}{z+2}\dfrac{1}{1-\frac{2}{z+2}} = \dfrac{1}{z+2}\left(1 + \dfrac{2}{z+2} + \left(\dfrac{2}{z+2}\right)^2 + \left(\dfrac{2}{z+2}\right)^3 + \cdots\right) = \dfrac{1}{z+2} + \dfrac{2}{(z+2)^2} + \cdots$

(c) Singular only at $z=0$ so the only expansion possible, about $z=0$, is in $0<|z|<\infty$:

$$e^{-1/z^3}=e^{-t}=1-t+\frac{1}{2!}t^2-\frac{1}{3!}t^3+\cdots \quad \text{in } 0\le |t|<\infty \qquad (t=1/z^3)$$
$$=1-\frac{1}{z^3}+\frac{1}{2!}\frac{1}{z^6}-\frac{1}{3!}\frac{1}{z^9}+\cdots \quad \text{in } 0<|z|<\infty .$$

(d) In $0\le |z+1|<\sqrt{5}$: TS gives



$$\frac{z^2+5}{z^2+4}=\frac{6}{5}+\frac{2}{25}(z+1)-\frac{1}{125}(z+1)^2-\frac{12}{625}(z+1)^3-\frac{19}{3125}(z+1)^4-\cdots$$

In $\sqrt{5}<|z+1|<\infty$: LS

$$\frac{z^2+5}{z^2+4}=\frac{z^2+5}{4i}\left(\frac{1}{z-2i}-\frac{1}{z+2i}\right). \text{ Then, with } t=z+1,$$

$$\frac{1}{z-2i}=\frac{1}{(z+1)-(2i+1)}=\frac{1}{z+1}\,\frac{1}{1-\frac{2i+1}{z+1}}=\frac{1}{t}\left[1+\frac{2i+1}{t}+\frac{4i-3}{t^2}+\cdots\right] \qquad ①$$

and, merely changing $i\to -i$,

$$\frac{1}{z+2i}=\frac{1}{t}\left[1+\frac{-2i+1}{t}+\frac{-4i-3}{t^2}+\cdots\right] \qquad ②$$

Also, $z^2+5=(t-1)^2+5=t^2-2t+6$, so

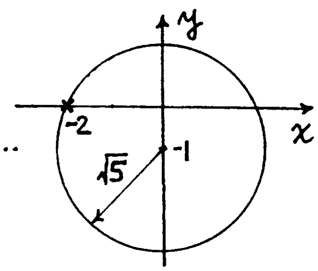
$$\frac{z^2+5}{z^2+4}=\frac{t^2-2t+6}{4it}\left[\cancel{1}+\frac{1+2i}{t}-\frac{3i-4}{t^2}+\cdots-\cancel{1}-\frac{1-2i}{t}-\frac{-3-4i}{t^2}-\cdots\right] \qquad ③$$

$$=\frac{1}{4i}\left(t-2+\frac{6}{t}\right)\left(\frac{4i}{t}+\frac{7+i}{t^2}+\cdots\right)=\frac{1}{4i}\left(4i+\frac{7+i}{t}+\cdots - \frac{8i}{t}-\cdots+\cdots\right)$$

$$=\frac{1}{4i}\left(4i+\frac{7-7i}{t}+\cdots\right)=1-\frac{7}{4}(1+i)\frac{1}{z+1}+\cdots . \qquad ④$$

I thought I would obtain the first 3 terms, in ④, by carrying ① and ② through 3 terms, but the cancelling 1's in ③ reduced ④ to only 2 terms. Thus, we need to include at least one more term in ① and in ②.

(e) In $0\le |z+i|<\sqrt{5}$: TS gives



$$\frac{1+z}{2+z}=\left(\frac{3-i}{5}\right)+\left(\frac{3+4i}{25}\right)(z+i)-\left(\frac{2+11i}{125}\right)(z+i)^2 -\left(\frac{7-24i}{625}\right)(z+i)^3+\left(\frac{38-41i}{3125}\right)(z+i)^4+\cdots$$

In $\sqrt{5}<|z+i|<\infty$: LS gives

$$\frac{1+z}{2+z}=\frac{1-i+(z+i)}{2-i+(z+i)}=\frac{1}{z+i}\left[(1-i)+(z+i)\right]\frac{1}{1+\frac{2-i}{z+i}}$$

$$=\left(1+\frac{1-i}{z+i}\right)\left[1-\left(\frac{2-i}{z+i}\right)+\left(\frac{2-i}{z+i}\right)^2-\cdots\right]$$

$$=1-\frac{1}{z+i}+(2-i)\frac{1}{(z+i)^2}-\cdots$$

(f) Singular only at $z=0$ so the only expansion is the LS in $0<|z|<\infty$.

$$\frac{\sin z}{z^4} = \frac{1}{z^4}\left(z-\frac{1}{3!}z^3+\frac{1}{5!}z^5-\cdots\right) = \frac{1}{z^3}-\frac{1}{6}\frac{1}{z}+\frac{1}{120}z-\cdots$$

(g) Singular only at $z=-i$ so the only expansion (about $z=-i$, that is) is the LS in $0<|z+i|<\infty$. The $1/(z+i)^2$ is already in powers of $(z+i)$ so leave it alone, and Taylor expand $\cos 2z$ about $-i$:

$$\frac{\cos 2z}{(z+i)^2} = \frac{1}{(z+i)^2}\left[\cosh 2+2i\sinh 2\,(z+i)-2\cosh 2\,(z+i)^2-\frac{4}{3}i\sinh 2\,(z+i)^3-\cdots\right]$$

$$= \cosh 2\,\frac{1}{(z+i)^2}+2i\sinh 2\,\frac{1}{z+i}-2\cosh 2-\frac{4}{3}i\sinh 2\,(z+i)-\cdots$$

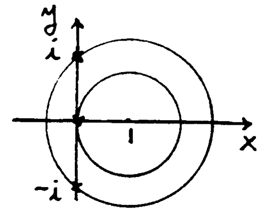
(h) Analytic everywhere, so we have only the TS

$$e^{-z^2} = 1-z^2+\frac{1}{2!}z^4-\frac{1}{3!}z^6+\cdots \quad \text{in } 0\le|z|<\infty$$

(i) We have only the LS

$$e^{-1/z} = 1-\frac{1}{z}+\frac{1}{2!}\frac{1}{z^2}-\frac{1}{3!}\frac{1}{z^3}+\cdots \quad \text{in } 0<|z|<\infty$$

(j) TS in $0\le|z-1|<1$:



$$\frac{1}{z(z^2+1)} = \frac{1}{2}-(z-1)+\frac{5}{4}(z-1)^2-\frac{5}{4}(z-1)^3+\frac{9}{8}(z-1)^4-\cdots$$

LS in $1<|z-1|<\sqrt{2}$:

$$\frac{1}{z(z^2+4)} = \frac{1}{1+(z-1)}\,\frac{1}{[(z-1)+1+2i][(z-1)+1-2i]} = \frac{1}{(1+t)(t+(1+2i))(t+(1-2i))}$$

$$= \frac{1}{4}\,\frac{1}{t+1}-\frac{1}{8}\,\frac{1}{t+(1+2i)}-\frac{1}{8}\,\frac{1}{t+(1-2i)}$$

Key step:
$$= \frac{1}{4t}\,\frac{1}{1+\frac{1}{t}}-\frac{1}{8(1+2i)}\,\frac{1}{1+\frac{t}{1+2i}}-\frac{1}{8(1-2i)}\,\frac{1}{1+\frac{t}{1-2i}}$$

$$= \frac{1}{4t}\left(1-\frac{1}{t}+\frac{1}{t^2}+\cdots\right)-\frac{1}{8(1+2i)}\left[1-\frac{t}{1+2i}+\frac{t^2}{(1+2i)^2}-\cdots\right]$$

$$-\frac{1}{8(1-2i)}\left[1-\frac{t}{1-2i}+\frac{t^2}{(1-2i)^2}-\cdots\right]$$

$$= \cdots-\frac{1}{4}\frac{1}{t^2}+\frac{1}{4}\frac{1}{t}-\frac{1}{20}-\frac{3}{100}t+\frac{11}{500}t^2-\cdots,\ \text{where } t=z-1.$$

6. $\frac{1}{z^2}+\frac{1}{z^3}+\cdots = \frac{1}{z^2}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right)$ in $1<|z|<\infty$

so $f(z) = \frac{1}{z^2}\,\frac{1}{1-\frac{1}{z}} = \frac{1}{z(z-1)}$ everywhere (except at $z=0,1$)

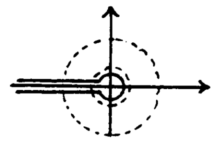
so $f(2)=(2i)=(-2+i)/10$, $f(i/3)=(-9+27i)/10$

7. $f(z) = \frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \cdots$ in $1 < |z| < \infty$

$$= -\left(1 - \frac{1}{z} + \frac{1}{z^2} - \cdots\right) + 1 = 1 - \frac{1}{1 + 1/z} = \frac{1}{z+1} \text{ everywhere (except at } z=-1)$$

so $f(2) = 1/3$, $f(1/3) = 3/4$.

8. No, there is no annulus of analyticity about $z=0$ due to the intrusion of the cut; i.e., $z=0$ is not an isolated singular point of $\log z$.



9. (a) $e^{\frac{x}{2}(z - \frac{1}{z})} = e^{\frac{x}{2}z} e^{-\frac{x}{2}\frac{1}{z}}$. The first factor is analytic everywhere and the second is analytic everywhere except at $z=0$ where it has an essential singularity. Thus, the LS on the RHS must be valid in $0 < |z| < \infty$.

10. "c_n" is $J_n(x)$, so $J_n(x) = \frac{1}{2\pi i}\oint_C \frac{e^{\frac{x}{2}(\zeta - 1/\zeta)}}{(\zeta - 0)^{n+1}}\,d\zeta$ but $\zeta = e^{i\theta}$ on C, so

$$J_n(x) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{e^{\frac{x}{2}(e^{i\theta} - e^{-i\theta})}}{(e^{i\theta})^{n+1}}\, i e^{i\theta}\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\sin\theta} e^{-in\theta}\, d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big[\underbrace{\cos(x\sin\theta - n\theta)}_{\text{even}} + i\,\cancelto{0}{\underbrace{\sin(x\sin\theta - n\theta)}_{\text{odd}}}\Big]\,d\theta = \frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta - n\theta)\,d\theta$$

Section 24.4

2.(a) $z^2 - z = 0$ at $z=0$ and $z=1$.

TS about 0 is $= -z + z^2 \sim -z$, so first-order zero at 0

TS about 1 is $= (z-1) + (z-1)^2 \sim (z-1)$ so " " " " 1

(b) $e^z - 1 = 0$ at $z = \log 1 = \ln 1 + 2n\pi i$ $(n = 0, \pm 1, \pm 2, \ldots)$

$= 2n\pi i$

TS about $2n\pi i$ is $= (z - 2n\pi i) + \frac{1}{2!}(z - 2n\pi i)^2 + \cdots \sim (z - 2n\pi i)$ so $e^z - 1$ has first-order zeros at $2n\pi i$ for $n = 0, \pm 1, \ldots$.

(c) $z\sin z = 0$ at $z = n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$.

TS about $z=0$ is $= z^2 - \frac{1}{3!}z^4 + \frac{1}{5!}z^6 - \cdots$, so 2nd-order zero at $z=0$

TS about $z = n\pi$ $(n \neq 0)$ is $= [n\pi + (z - n\pi)][\cos n\pi\,(z - n\pi) - \frac{1}{2!}(z - n\pi)^2 + \cdots]$

$\sim n\pi\cos n\pi\,(z - n\pi)$ so 1st order zeros at $n\pi$ $(n \neq 0)$.

(d) It's easiest to factor f as $(z)(\cos z)(\cos z)$.

TS about $z=0$ is $= (z)(1 + \cdots)(1 + \cdots) = z + \cdots \sim z$ so 1st order zero at 0.

TS about $z = n\pi/2$ (n an odd integer) is $= [\frac{n\pi}{2} + (z - \frac{n\pi}{2})][-(\sin\frac{n\pi}{2})(z - \frac{n\pi}{2}) + \cdots]^2$

$\sim \frac{n\pi}{2}(\sin\frac{n\pi}{2})^2(z-\frac{n\pi}{2})^2$ so 2nd order zeros at $n\pi/2$ (n odd)

(e) $(z^2+1)^3 = (z+i)^3(z-i)^3$. First and second factors have 0th and 3rd order zeros at i so f has a $0+3 =$ 3rd order zero at i. First and second factors have 3rd and 0th order zeros at $-i$ so f has a $3+0 =$ 3rd order zero at $-i$.

(f) zeros at $z = \log(-2) = \ln 2 + (2n+1)\pi i$ for $n = 0, \pm 1, \pm 2, \ldots$. TS about that point is $= -2[z-(\ln 2+(2n+1)\pi i)] + \cdots$, so f has 1st order zeros at those points

(g) 1st order zeros at $z = (-1+\sqrt{3}i)/2$ and at $z = (-1-\sqrt{3}i)/2$.

(h) $1-z^4 = 0$ at $z = 1^{1/4} = 1, i, -1, -i$, at each of which f has a 1st order zero

3. (a) Singular only at $z=0$; 2nd order pole

(b) Singular at $z = 2n\pi i$ ($n = \pm 1, \pm 2, \ldots$ but not $n=0$ because at $z=0$ the 2nd order zero in the numerator overpowers the 1st order zero in the denominator); 1st order poles

(c) 1st order poles at each of the 3 one-third roots of 1, namely, at $1, -\frac{1}{2}+\frac{\sqrt{3}}{2}i, -\frac{1}{2}-\frac{\sqrt{3}}{2}i$

(d) 1st order pole at -2

(e) $\sinh z$ has 1st order zeros at $z = n\pi i$ ($n = 0, \pm 1, \pm 2, \ldots$) so $1/\sinh z$ has first order poles there

(f) $\cosh z = \cos iz$ has 1st order zeros at $iz = n\pi/2$ ($n = \pm 1, \pm 2, \ldots$), that is, at $z = m\pi i/2$ ($m = \pm 1, \pm 2, \ldots$) so $1/\cosh z$ has 1st order poles at those points

(g) $\sin z$ has 1st order zeros at $n\pi$ ($n = 0, \pm 1, \pm 2, \ldots$) so $\sin^3 z$ has 3rd order zeros at those points. Thus, $z/\sin^3 z$ has 3rd order poles at $z = n\pi$ ($n = \pm 1, \pm 2, \ldots$) but a 2nd order pole at $z=0$ (since the numerator has a first order pole there).

(h) Singular only at $z=0$, where it has an essential singularity

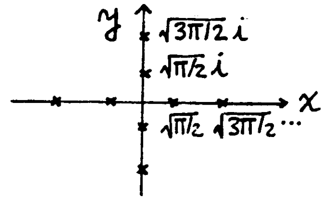
(i)-(m) Same as for (h)

(n) 2nd order pole at $z=1$

(o) e^{-z} is analytic for all z. (e^z has no zeros)

(p) $\tan z^2 = \frac{\sin z^2}{\cos z^2}$. Since $\sin^2 z^2 + \cos^2 z^2 = 1$ for all z it follows that $\cos z^2$ and $\sin z^2$ cannot vanish at the same point so we need merely attend to the zeros of the denominator, $\cos z^2$, namely, $z^2 = n\pi/2$ ($n = \pm 1, \pm 3, \ldots$) or $z = \begin{cases} \pm\sqrt{n\pi/2} & \text{for } n = 1, 3, \ldots \\ \pm i\sqrt{|n|\pi/2} & \text{for } n = -1, -3, \ldots \end{cases}$

At any of those points, say z_n, $\cos z^2 = 0 - \underbrace{(2z\sin z^2)}_{\neq 0}\big|_{z_n}(z-z_n) + \cdots$



so $\cos z^2$ has a 1st order zero and $\tan z^2$ has a 1st order pole there.

(q) Same idea as in (p): 1st order poles where $1/z^2 = n\pi/2$ ($n=\pm1, \pm3, \dots$), namely, at the points $z = \pm 1/\sqrt{n\pi/2}$ for $n=1,3,\dots$ and $\pm i/\sqrt{|n|\pi/2}$ for $n=-1,-3,\dots$

(r) No singular points since e^z (and hence e^{z^2}) $\neq 0$ for all z.

(s) 1st order poles where $z-2 = n\pi$, namely, at $z = 2+n\pi$ for $n=0,\pm1,\pm2,\dots$

(t) 1st order poles where $1/z = n\pi$, namely, at $z = 1/n\pi$ for $n=\pm1,\pm2,\dots$ (I'll omit $n=0$ since that would give $z=0$, whereas we are considering here only the finite z plane.)

4. (a) With $t=1/z$ (so $z=\infty \to t=0$), $\frac{e^z-1}{z^3} = (e^{1/t}-1)t^3$ which has an essential singularity at $t=0$ and hence at $z=\infty$.

(b) $\frac{z^2}{e^z-1} = \frac{1}{t^2(e^{1/t}-1)}$ has an essential singularity at $t=0$, hence at $z=\infty$. Why? CRUDELY put, for small t the -1 is inconsequential so the $1/(e^{1/t}-1)$ is "like" $1/e^{1/t} = e^{-1/t}$, which has an essential singularity at $t=0$. More convincingly, let us seek the Laurent expansion of $1/(e^{1/t}-1)$ about $t=0$. (We can ignore the $1/t^2$ because if $1/(e^t-1)$ has an essential sing. at $t=0$ then so does $1/t^2$ times it.) The form

$$\frac{1}{e^{1/t}-1} = t + a_0 + \frac{a_1}{t} + \frac{a_2}{t^2} + \cdots$$

will work since $1 = \left(\frac{1}{t} + \frac{1}{2!}\frac{1}{t^2} + \frac{1}{3!}\frac{1}{t^3} + \cdots\right)\left(t + a_0 + \frac{a_1}{t} + \frac{a_2}{t^2} + \cdots\right)$ gives:

1: $\quad 1 = 1$ ✓

t^{-1}: $\quad 0 = a_0 + \frac{1}{2!} \to a_0 = -1/2!$

t^{-2}: $\quad 0 = a_1 + \frac{a_0}{2!} + \frac{1}{3!} \to a_1 =$ etc

t^{-3}: $\quad 0 = a_2 + \frac{a_1}{2!} + \frac{a_0}{3!} + \frac{1}{4!} \to a_2 =$ etc.,

and so on.

(c) $\frac{1}{z^3-1} = \frac{t^3}{1-t^3}$ is analytic at $t=0$ and hence at $z=\infty$.

(d) $\frac{1}{1+\frac{1}{1+z}} = \frac{z+1}{z+2} = \frac{1+t}{1+2t}$ ($t=1/z$) is analytic at $t=0$ and hence at $z=\infty$.

5. (a) $f(z) = \frac{1}{z^2}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right) = \frac{1}{z^2}\frac{1}{1-\frac{1}{z}} = \frac{1}{z(z-1)}$, which has a 1st-order pole at $z=0$.

(b) $f(z) = -\left[1-\frac{1}{2z}+\left(\frac{1}{2z}\right)^2-\left(\frac{1}{2z}\right)^3+\cdots\right]+1 = 1-\frac{1}{1+\frac{1}{2z}} = 1-\frac{2z}{2z-1} = \frac{1}{1-2z}$,

which is analytic at $z=0$.

(c) f has a 4th-order pole at $z=0$.

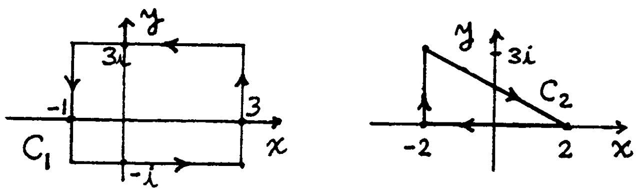
(d) $f(z) = \left[1+\frac{1}{1!}\left(\frac{1}{z}\right)+\frac{1}{2!}\left(\frac{1}{z}\right)^2+\cdots\right]-1-\frac{1}{z} = e^{1/z}-1-\frac{1}{z}$ has an essential singularity at $z=0$

(e) $f(z) = \frac{1}{z^5}\frac{1}{1+\frac{2}{z^3}} = \frac{1}{z^2(z^3+2)}$ has a 2nd order pole at $z=0$.

6. "has an infinite number of negative powers of z" is incorrect; it has an infinite number of positive powers of z in the denominator, which is not the same thing. Indeed, $1/e^z = e^{-z}$ is analytic for all z and has the TS $e^{-z} = 1-\frac{1}{1!}z+\frac{1}{2!}z^2-\cdots$ in $0 \le |z| < \infty$.

Section 24.5

1.



(a) $\sin 2z = 0$ at $z = n\pi/2 = 0, \pm\pi/2, \pm\pi, \ldots$, of which 0 and $\pi/2$ are within C_1. Thus,

$$\int_{C_1} \frac{dz}{\sin 2z} = 2\pi i\left(\text{Res}@0 + \text{Res}@\tfrac{\pi}{2}\right)$$
$$= 2\pi i\left(\lim_{z\to 0}\frac{z}{\sin 2z} + \lim_{z\to\pi/2}\frac{z-\pi/2}{\sin 2z}\right)$$
$$= 2\pi i\left(\frac{1}{2}-\frac{1}{2}\right) \text{ by l'Hôpital's rule}$$
$$= 0$$

(b) Second order pole @ 0 where Res $=-1$ since $\frac{1}{z^2e^z} = \frac{1}{z^2}e^{-z} = \frac{1}{z^2}(1-z+\cdots)$

Thus, $\int_{C_1}\frac{dz}{z^2e^z} = 2\pi i(-1) = -2\pi i$

(c) $\sinh 2z = -i\sin(i2z) = 0$ at $z = 0, \pm\pi i/2, \pm\pi i, \pm 3\pi i/2, \pm 4\pi i/2, \ldots$, of which $0, \pi i/2$ are within C_1. $z^2/\sinh 2z$ is analytic at 0, however, and has 1st-order pole @ $\pi i/2$, with Res $= \lim_{z\to\pi i/2}\left(\frac{z-\pi i/2}{\sinh 2z}z^2\right)$

$= \left(\lim_{z\to\pi i/2}\frac{z-\frac{\pi i}{2}}{\sinh 2z}\right)\left(\frac{\pi i}{2}\right)^2 = \frac{1}{2\cosh\pi i}\left(-\frac{\pi^2}{4}\right) = \frac{\pi^2}{8}$, so $I = 2\pi i(\pi^2/8) = \pi^3 i/4$

(d) Integrand has 3rd-order pole at 1 and (with $z-1=t$)

$$\left(\frac{z+1}{z-1}\right)^3 = \frac{(2+t)^3}{t^3} = \frac{8}{t^3} + \frac{12}{t^2} + \frac{6}{t} + 1 \quad \text{so Res@1} = 6 \quad \text{and } I = 2\pi i(6) = 12\pi i$$

(e) $z^2 - 2iz - 2 = [z-(1+i)][z-(-1+i)]$ so the integrand has 1st order poles at $1+i$ and $-1+i$, of which $1+i$ is outside of C_2 and $-1+i$ is inside, so

$$I = -2\pi i \,\text{Res@}(-1+i) = -2\pi i \lim_{z\to -1+i} [z-\cancel{(-1+i)}]\frac{1}{[z-(1+i)][z-\cancel{(-1+i)}]} = \pi i$$

(f) $\cosh(\pi z/2) = \cos(i\pi z/2) = 0$ at $i\pi z/2 = \pm\pi/2, \pm 3\pi/2, \dots$
or, $z = \pm i, \pm 3i, \dots$,
of which only $+i$ is within C_2. at i, $1/\cosh^2(\pi z/2)$ has a 2nd-order pole, so

$$\text{Res@}i = \lim_{z\to i}\frac{d}{dz}[(z-i)^2/\cosh^2(\pi z/2)]$$

$$= \lim_{z\to i}\left\{\frac{2(z-i)}{\cosh^2\frac{\pi z}{2}} + \frac{(z-i)^2(-2)(\pi/2)\sinh\frac{\pi z}{2}}{\cosh^3\frac{\pi z}{2}}\right\}$$

$$= \lim_{z\to i}\frac{2(z-i)\cosh\frac{\pi z}{2} - \pi(z-i)^2\sinh\frac{\pi z}{2}}{\cosh^3\frac{\pi z}{2}}$$

$$= \lim_{z\to i}\frac{2\cosh\frac{\pi z}{2} + \pi(z-i)\sinh\frac{\pi z}{2} - 2\pi(z-i)\sinh\frac{\pi z}{2} - \frac{\pi^2}{2}(z-i)^2\cosh\frac{\pi z}{2}}{\frac{3\pi}{2}\cosh^2\frac{\pi z}{2}\sinh\frac{\pi z}{2}}$$

by l'Hôpital, but we still need to apply l'Hôpital again – twice in fact, and it is looking tedious, so let's try evaluating the Res more directly, by developing the LS of the integrand about $z=i$: With $z-i=t$,

$\cos\pi/2 = 0$

$$\cosh\frac{\pi z}{2} = \cosh\frac{\pi}{2}(t+i) = \cosh\frac{\pi t}{2}\cosh\frac{\pi i}{2} + \sinh\frac{\pi t}{2}\sinh\frac{\pi i}{2}$$
$$= i\sinh\frac{\pi t}{2} = i\left(\frac{\pi t}{2} + \frac{\pi^3 t^3}{6} + \cdots\right)$$

*

so
$$\frac{1}{\cosh^2\frac{\pi z}{2}} = \frac{1}{(i\frac{\pi t}{2})^2}\frac{1}{\left(1+\frac{\pi^2 t^2}{3}+\cdots\right)^2}$$
$$= -\frac{4}{\pi^2 t^2}\Big[1+\underbrace{\left(\frac{\pi^2 t^2}{3}+\cdots\right)}\Big]^{-2}$$

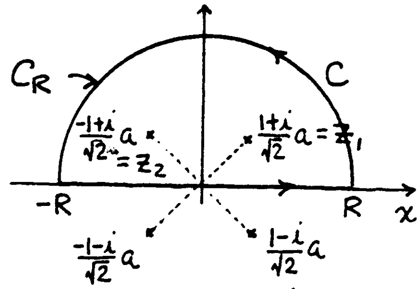
Calling this μ, say,

$$(1+\mu)^{-2} = 1 - 2\mu + 3\mu^2 - \cdots$$
$$= 1 - 2\left(\frac{\pi^2 t^2}{3}+\cdots\right) + 3\left(\frac{\pi^2 t^2}{3}+\cdots\right)^2 - \cdots$$
$$= 1 - \frac{2\pi^2}{3}t^2 + (\text{etc})t^4 + (\text{etc})t^6 + \cdots$$

so the residue (i.e., the coeff. of $1/t$) is seen to be 0. Hence, $I = 0$.

NOTE: The * method is useful and might be worth discussing in class.

2.(a) $I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{x^4+a^4} = ?$ Consider



$J = \oint_C \frac{dz}{z^4+a^4} = 2\pi i\,(\text{Res}\,@\,z_1 + \text{Res}\,@\,z_2)$

$= 2\pi i\left(\lim_{z\to z_1} \frac{z-z_1}{z^4+a^4} + \lim_{z\to z_2} \frac{z-z_2}{z^4+a^4}\right)$

$= 2\pi i\left(\frac{1}{4z_1^3} + \frac{1}{4z_2^3}\right)$ by l'Hôpital. Best to express z_1, z_2 in polar form now: $z_1 = a e^{\pi i/4}$, $z_2 = a e^{3\pi i/4}$ or, better yet, $z_2 = -a e^{-\pi i/4}$

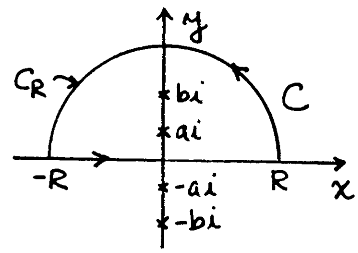
$= \frac{2\pi i}{4a^3}\left(\frac{e^{-3\pi i/4} - e^{+3\pi i/4}}{2i}\right)2i = \frac{2\pi i}{4a^3}\left(\sin\frac{3\pi}{4}\right)(2i) = -\frac{\pi}{a^4}\left(-\frac{1}{\sqrt{2}}\right) = \frac{\pi}{\sqrt{2}a^4}$

$= \lim_{R\to\infty}\int_{-R}^{R} \frac{dx}{x^4+a^4} + \lim_{R\to\infty}\int_{C_R}$

$\left|\int_{C_R}\right| \le \max\left|\frac{1}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}\right| \pi R \le \frac{\pi R}{(R-a)^4} \to 0$ as $R\to\infty$

$= \int_{-\infty}^{\infty} \frac{dx}{x^4+a^4} + 0 = 2I$, so $I = \frac{\pi}{2\sqrt{2}a^4}$

(b) $I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = ?$ Consider



$J = \oint_C \frac{dz}{(z^2+a^2)(z^2+b^2)} = 2\pi i\,(\text{Res}\,@\,ai + \text{Res}\,@\,bi)$

$= 2\pi i\left(\lim_{z\to ai} \frac{z-ai}{(z^2+a^2)(z^2+b^2)} + \lim_{z\to bi} \frac{z-bi}{(z^2+a^2)(z^2+b^2)}\right)$

$= 2\pi i\left(\frac{1}{2ai(b^2-a^2)} + \frac{1}{(a^2-b^2)\,2bi}\right) = \frac{\pi}{ab(a+b)}$

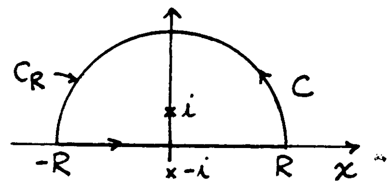
$= \lim_{R\to\infty}\int_{-R}^{R} \frac{dx}{(x^2+a^2)(x^2+b^2)} + \lim_{R\to\infty}\int_{C_R}$

Let $\max\{a,b\} \equiv \alpha$. Then $\left|\int_{C_R}\right| \le \frac{1}{(R-\alpha)^4}\pi R \to 0$ as $R\to\infty$

$= \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} + 0 = 2I$, so $I = \frac{\pi}{2ab(a+b)}$

(c) $\int_0^{\infty} \frac{x^2}{x^4+1}\,dx = \pi\sqrt{2}/4$ (by maple)

(d) $I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} = ?$ Consider



$$J = \oint_C \frac{dz}{(z^2+1)^2} = 2\pi i \,\text{Res}@i = 2\pi i \lim_{z\to i} \frac{d}{dz} \frac{(z-i)^2}{(z^2+1)^2}$$

$$= 2\pi i \lim_{z\to i} \frac{d}{dz}(z+i)^{-2} = \pi/2$$

$$= \lim_{R\to\infty}\int_{-R}^{R} \frac{dx}{(x^2+1)^2} + \lim_{R\to\infty}\int_{C_R}$$

$$\left|\int_{C_R}\right| \le \frac{1}{(R-1)^2}\pi R \sim \frac{\pi}{R} \to 0 \text{ as } R\to\infty$$

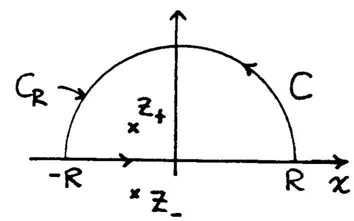
$$= \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} + 0 = 2I, \text{ so } I = \pi/4$$

(e) $I = \int_{-\infty}^{\infty} \frac{dx}{4x^2+2x+1} = ?$ Consider

$$J = \oint_C \frac{dz}{4z^2+2z+1} = \frac{1}{4}\oint_C \frac{dz}{(z-z_+)(z-z_-)}$$

where $z_+ = (-1+i\sqrt{3})/4$ is in C and $z_- = (-1-i\sqrt{3})/4$ is not

$$= 2\pi i \frac{1}{4}\,\text{Res}@z_+ = \frac{2\pi i}{4}\frac{1}{z_+ - z_-} = \frac{\pi}{\sqrt{3}}$$



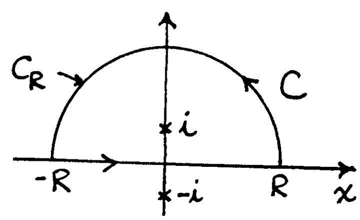
$$= \lim_{R\to\infty}\int_{-R}^{R} \frac{dx}{4x^2+2x+1} + \lim_{R\to\infty}\int_{C_R}$$

$$\left|\int_{C_R}\right| \le \frac{1}{4(R-|z_+|)^2}\pi R \sim \frac{\pi}{4R} \to 0$$

$$= \int_{-\infty}^{\infty} \frac{dx}{4x^2+2x+1} + 0 = I, \text{ so } I = \pi/\sqrt{3}.$$

(f) Maple gives $I = \pi/6$, namely, $2\pi i$ times the sum of the residues at the first order poles in the upper half plane, at $\pm\frac{\sqrt{3}}{2} + \frac{i}{2}$.

(g) $I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos 2x}{(x^2+1)^2}\,dx = ?$ Consider



$$J = \oint_C \frac{e^{i2z}}{(z^2+1)^2}\,dz = 2\pi i \,\text{Res}@i$$

$$= 2\pi i \lim_{z\to i} \frac{d}{dz} \frac{\cancel{(z-i)^2}\, e^{i2z}}{(z+i)^2\cancel{(z-i)^2}} = \frac{3\pi}{2e^2}$$

$$= \lim_{R\to\infty}\int_{-R}^{R} \frac{\cos 2x + i\sin 2x}{(x^2+1)^2}\,dx + \lim_{R\to\infty}\int_{C_R} \frac{e^{i2z}}{(z^2+1)^2}\,dz$$

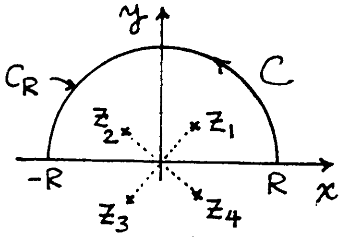
On C_R, $|e^{i2z}| = |e^{i2(x+iy)}| = e^{-2y} \le 1$ so

$$\left|\int_{C_R}\right| \le \frac{1}{(R-1)^4}\pi R \sim \frac{\pi}{R^3} \to 0 \text{ as } R\to\infty$$

$$= \int_{-\infty}^{\infty} \frac{\cos 2x + i\overset{0 \text{ by odd integrand}}{\sin 2x}}{(x^2+1)^2}\,dx + 0 = 2I, \text{ so } I = 3\pi/(4e^2)$$

(h) $x\sin x$ is even, as is x^4+16, so

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x\sin x}{x^4+16}\,dx = ? \quad \text{Consider}$$



$z_1 = 2e^{\pi i/4}$, $z_2 = 2e^{3\pi i/4} = -2e^{-\pi i/4}$

$$J = \oint_C \frac{ze^{iz}}{z^4+16}\,dz = 2\pi i\,(\text{Res}@z_1 + \text{Res}@z_2)$$

$$= 2\pi i\left(\lim_{z\to z_1} \frac{(z-z_1)ze^{iz}}{z^4+16} + \lim_{z\to z_2}\frac{(z-z_2)ze^{iz}}{z^4+16}\right)$$

$$= 2\pi i\left(z_1e^{iz_1}\lim_{z\to z_1}\frac{z-z_1}{z^4+16} + z_2e^{iz_2}\lim_{z\to z_2}\frac{z-z_2}{z^4+16}\right)$$

$$= 2\pi i\left(z_1e^{iz_1}\frac{1}{4z_1^3} + z_2e^{iz_2}\frac{1}{4z_2^3}\right) = \frac{2\pi i}{4}\left(\frac{e^{iz_1}}{z_1^2} + \frac{e^{iz_2}}{z_2^2}\right)$$

$$= \frac{\pi i}{2}\left[\frac{e^{i(2)\left(\frac{1+i}{\sqrt{2}}\right)}}{4i} + \frac{e^{i(2)\left(\frac{-1+i}{\sqrt{2}}\right)}}{-4i}\right] = \frac{\pi}{8}e^{-\sqrt{2}}\left(\frac{e^{i\sqrt{2}}-e^{-i\sqrt{2}}}{2i}\right)2i = \frac{\pi i}{4}e^{-\sqrt{2}}\sin\sqrt{2}$$

$$= \lim_{R\to\infty}\int_{-R}^{R}\frac{xe^{ix}}{x^4+16}\,dx + \lim_{R\to\infty}\int_{C_R}\frac{ze^{iz}}{z^4+16}\,dz$$

On C_R, $|e^{iz}| = |e^{i(x+iy)}| = e^{-y} \le 1$

so $\left|\int_{C_R}\right| \le \frac{R}{(R-2)^4}\pi R \sim \frac{\pi}{R^2} \to 0$ as $R\to\infty$

$$= \int_{-\infty}^{\infty}\frac{x(\cos x + i\sin x)}{x^4+16}\,dx + 0 = \underset{\text{since } x\cos x/(x^4+16) \text{ is odd}}{0} + i2I$$

so $2iI = \frac{\pi i}{4}e^{-\sqrt{2}}\sin\sqrt{2}$, $I = \frac{\pi e^{-\sqrt{2}}}{8}\sin\sqrt{2}$

NOTE: I tried checking this result by the maple command

```
int( x* sin(x)/(x^4+16), x=0..infinity );
```

but it didn't give an answer. Using numerical integration, however,

```
evalf( int (x*sin(x)/(x^4+16), x=0..infinity));
```

gives $I = 0.094303712$, which does agree with our analytical result.

(i) $I = \int_{-\infty}^{\infty}\frac{\cos x\,dx}{8x^2+12x+5} = ?$ Consider $J = \oint_C \frac{e^{iz}\,dz}{8z^2+12z+5}$ where C is the "usual" contour, as above. Then $8z^2+12z+5 = 8(z-z_1)(z-z_2)$ where $z_1 = (-3+i)/4$ is in C and $z_2 = (-3-i)/4$ is not, so

$$J = 2\pi i\,\text{Res}@z_1 = 2\pi i\,\frac{e^{iz_1}}{8(z_1-z_2)} = \frac{\pi}{2}e^{-1/4}\left(\cos\frac{3}{4} - i\sin\frac{3}{4}\right)$$

and (omitting the $\lim_{R\to\infty}$ steps for brevity) this $= \int_{-\infty}^{\infty} \frac{\cos x + i\sin x}{8x^2+12x+5}\,dx$

so equating real parts gives $I = \int_{-\infty}^{\infty} \frac{\cos x\,dx}{8x^2+12x+5} = \frac{\pi}{2} e^{-1/4}\cos\frac{3}{4}$

which is verified using the Maple evalf(int( )) command.

3. (a) $I = \frac{1}{2}\int_{-\pi}^{\pi} \sin^2 x\,dx = \frac{1}{2}\oint_C \left(\frac{z^2-1}{2iz}\right)^2 \frac{dz}{iz} = \frac{1}{-8i}\oint_C \frac{z^4-2z^2+1}{z^3}\,dz$

$= -\frac{1}{8i}\,2\pi i(0-2+0) = \pi/2$.

(b) $I = \frac{1}{2}\int_{-\pi}^{\pi} \cos^2 x\,dx = \frac{1}{2}\oint_C \left(\frac{z^2+1}{2z}\right)^2 \frac{dz}{iz} = \frac{1}{8i}\oint_C \frac{z^4+2z^2+1}{z^3}\,dz = \frac{2}{8i}2\pi i = \frac{\pi}{2}$

(c) Sketching the graph of $\sin^2 x$ it is evident that $\int_0^{\pi/2} \sin^2 x\,dx = \frac{1}{4}\int_{-\pi}^{\pi}\sin^2 x\,dx$. Then proceed as in (a). We obtain $I = \pi/4$.

(d) Sketching the graph of $\cos^2 x$ it is evident that $\int_{\pi/2}^{\pi}\cos^2 x\,dx = \frac{1}{4}\int_{-\pi}^{\pi}\cos^2 x\,dx$ Then, proceeding as in (b), we obtain $I = \pi/4$.

(e) $I = \frac{1}{2}\int_{-\pi}^{\pi} \sin^4 x\,dx = \frac{1}{2}\oint_C \left(\frac{z^2-1}{2iz}\right)^4 \frac{dz}{iz} = \frac{1}{32i}\oint_C \frac{(z^4-2z^2+1)(z^4-2z^2+1)}{z^5}\,dz$

To evaluate the residue merely pick out the coefficient of the z^4 term in the numerator, namely, 6, so $I = \frac{1}{32i}2\pi i(6) = 3\pi/8$

(g) Proceeding as in (e),

$I = \frac{1}{2}\oint_C \left(\frac{z^2-1}{2iz}\right)^6 \frac{dz}{iz} = -\frac{1}{128i}\oint_C \frac{(z^4-2z^2+1)(z^4-2z^2+1)(z^4-2z^2+1)}{z^7}\,dz$

$= -\frac{1}{128i}(2\pi i)(-20) = 5\pi/16$

(i) $I = \oint_C \frac{1}{7+\frac{z^2+1}{2z}}\frac{dz}{iz} = \frac{2}{i}\oint_C \frac{dz}{z^2+14z+1} = \frac{2}{i}\oint_C \frac{dz}{(z-z_1)(z-z_2)}$ $\left\{\begin{matrix} z_1 = -7+4\sqrt{3} \\ z_2 = -7-4\sqrt{3}\end{matrix}\right\}$

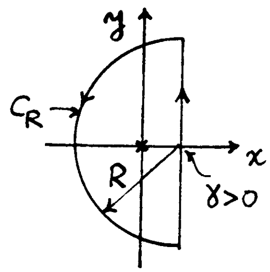
z_1 in inside C and z_2 is outside, so

$I = \frac{2}{i}2\pi i \operatorname{Res}@z_1 = \frac{2}{i}2\pi i\frac{1}{z_1-z_2} = \pi/2\sqrt{3}$ or $\pi\sqrt{3}/6$

4. $I = \frac{1}{2}\int_0^{2\pi} \frac{\cos t\,dt}{1-2a\cos t+a^2} = \frac{1}{2}\oint_C \frac{\frac{z+1/z}{2}\,\frac{dz}{iz}}{1+a^2-a(z+\frac{1}{z})} = -\frac{1}{4ai}\oint_C \frac{(z^2+1)dz}{z\underbrace{[z^2-(\frac{1+a^2}{a})z+1]}_{(z-\frac{1}{a})(z-a),}}$ where $|a|<1$

$= -\frac{1}{4ai}[\operatorname{Res}@0+\operatorname{Res}@a]2\pi i = -\frac{\pi}{2a}\left[1+\frac{a^2+1}{a(a-\frac{1}{a})}\right] = \frac{\pi a}{1-a^2}$

5. (a) Consider $J = \oint_C \frac{e^{st}}{s^2}\,ds$. $J = 2\pi i \,\text{Res}@s=0 = 2\pi i t$.



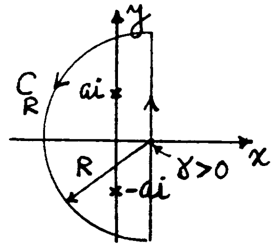
also, $J = \int_{\gamma-iR}^{\gamma+iR} \frac{e^{st}}{s^2}\,ds + \int_{C_R}$

But $\left|\int_{C_R}\right| \le \frac{\max|e^{(x+iy)t}|}{\min|s|^2}\pi R = \frac{e^{\gamma t}}{R^2}\pi R \to 0$ as $R\to\infty$

so $2\pi i t = \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{st}}{s^2}\,ds$, $L^{-1}\left\{\frac{1}{s^2}\right\} = \frac{1}{2\pi i}2\pi i t = t$ (for $t>0$)

(b) Like (a), except Res@$s=0$ is $= t^5/5!$. We obtain $L^{-1}\left\{\frac{1}{s^6}\right\} = \frac{t^5}{5!}$ (for $t>0$)

(c) Consider $J = \oint_C \frac{e^{st}}{s^2+a^2}\,ds = 2\pi i(\text{Res}@ai + \text{Res}@{-ai})$
$= 2\pi i\left(\frac{e^{iat}}{2ai} + \frac{e^{-iat}}{-2ai}\right)$
$= \frac{2\pi i}{a}\sin at$

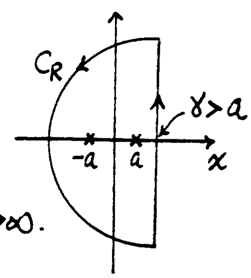


also, $J = \int_{\gamma-iR}^{\gamma+iR} \frac{e^{st}}{s^2+a^2}\,ds + \int_{C_R}$

But $\left|\int_{C_R}\right| \le \frac{\max|e^{(x+iy)t}|}{\min|s-ai|\,\min|s+ai|}\pi R = \frac{e^{\gamma t}}{(R-\sqrt{a^2+\gamma^2})^2}\pi R \sim \frac{\pi e^{at}}{R} \to 0$ as $R\to\infty$

so, letting $R\to\infty$ in
$\frac{2\pi i}{a}\sin at = \int_{\gamma-iR}^{\gamma+iR} \frac{e^{st}}{s^2+a^2}\,ds + \int_{C_R}$
gives
$\frac{2\pi i}{a}\sin at = \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{st}}{s^2+a^2}\,ds + 0$
$= 2\pi i\, L^{-1}\left\{\frac{1}{s^2+a^2}\right\}$ so $L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a}$ (for $t>0$)

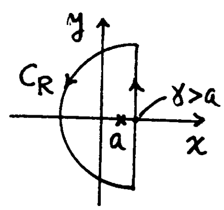
(d) Consider $J = \oint_C \frac{e^{st}}{s^2-a^2}\,ds = 2\pi i(\text{Res}@a + \text{Res}@{-a})$
$= 2\pi i\left(\frac{e^{at}}{2a} + \frac{e^{-at}}{-2a}\right) = 2\pi i\frac{\sinh at}{a}$



But $\left|\int_{C_R}\right| \le \frac{\max|e^{(x+iy)t}|}{\min|s-a|\,\min|s+a|}\pi R = \frac{e^{\gamma t}}{(R+a)(R-a)}\pi R \sim \frac{\pi e^{\gamma t}}{R} \to 0$ as $R\to\infty$.

So, letting $R\to\infty$ in
$2\pi i\frac{\sinh at}{a} = \int_{\gamma-iR}^{\gamma+iR} \frac{e^{st}}{s^2-a^2}\,ds + \int_{C_R}$
gives $2\pi i\frac{\sinh at}{a} = 2\pi i L^{-1}\left\{\frac{1}{s^2-a^2}\right\} + 0$ so $L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{\sinh at}{a}$ ($t>0$)

(e) Consider $J = \oint_C \frac{e^{st}}{(s-a)^4}\,ds = 2\pi i\,\text{Res}@a$
$= 2\pi i \frac{1}{3!}\frac{d^3}{ds^3}\left(\frac{\cancel{(s-a)^4}e^{st}}{\cancel{(s-a)^4}}\right)\Big|_{s\to a}$
$= 2\pi i\, t^3 e^{at}/3!$



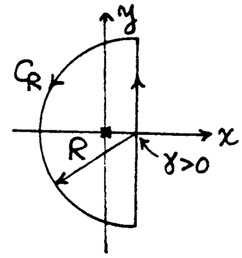
But $\left|\int_{C_R}\right| \le \dfrac{\max|e^{(x+iy)t}|}{\min|s-a|^4}\pi R = \dfrac{e^{\gamma t}}{(R-a)^4}\pi R \sim \dfrac{\pi e^{\gamma t}}{R^3} \to 0$ as $R\to\infty$

so, letting $R\to\infty$ in

$$2\pi i\, t^3 e^{at}/3! = \int_{\gamma-iR}^{\gamma+iR} \frac{e^{st}}{(s-a)^4}\,ds + \int_{C_R}$$

gives $2\pi i\, t^3 e^{at}/3! = 2\pi i\, L^{-1}\left\{\frac{1}{(s-a)^4}\right\} + 0$ so $L^{-1}\left\{\frac{1}{(s-a)^4}\right\} = t^3 e^{at}/6 \quad (t>0)$

(k) Consider $J = \oint_C e^{st}\dfrac{e^{-as}}{s^3}\,ds = 2\pi i\,\mathrm{Res}@s=0$
$= 2\pi i \dfrac{(t-a)^2}{2!}$



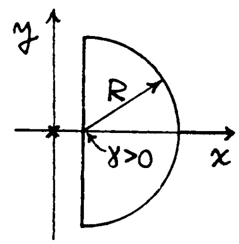
$\left|\int_{C_R}\right| \le \dfrac{\max|e^{(x+iy)(t-a)}|}{\min|s|^3}\pi R = \dfrac{e^{\gamma(t-a)}}{(R-\gamma)^3}$ if $t \geq a$

so, letting $R\to\infty$ in

$$2\pi i\frac{(t-a)^2}{2} = \int_{\gamma-iR}^{\gamma+iR}\frac{e^{s(t-a)}}{s^3}\,ds + \int_{C_R}$$

gives $\quad 2\pi i\dfrac{(t-a)^2}{2} = 2\pi i\, L^{-1}\left\{\dfrac{e^{-as}}{s^3}\right\} + 0$ so $L^{-1}\left\{\dfrac{e^{-as}}{s^3}\right\} = \dfrac{(t-a)^2}{2}$ for $t \geq a$.

If $t<a$, close C on the right, as shown:
In that case the residue theorem (or Cauchy's theorem) gives $J=0$. Also,



$\left|\int_{C_R}\right| \le \dfrac{\max|e^{-(x+iy)(a-t)}|}{\min|s|^3}\pi R = \dfrac{e^{-\gamma(a-t)}}{(R^2+\gamma^2)^{3/2}}\pi R \sim \dfrac{\pi e^{-\gamma(a-t)}}{R^2}$

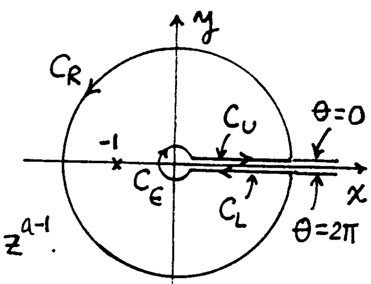
$\to 0$ as $R\to\infty$

so $\quad 0 = \int_{\gamma-i\infty}^{\gamma+i\infty}\dfrac{e^{s(t-a)}}{s^3}\,ds$. Thus,

$$L^{-1}\left\{\frac{e^{-as}}{s^3}\right\} = \begin{cases}(t-a)^2/2, & t>a\\ 0, & t<a\end{cases} = H(t-a)\frac{(t-a)^2}{2}$$

6. (a) $I = \int_0^\infty \dfrac{x^{a-1}}{x+1}\,dx \quad (0<a<1)$

Consider $J = \oint_C \dfrac{z^{a-1}}{z+1}\,dz$ where C is as shown:



$J = 2\pi i\,\mathrm{Res}@-1 = 2\pi i(-1)^{a-1} = 2\pi i(1e^{\pi i})^{a-1} = 2\pi i\, e^{(a-1)\pi i}$
per the branch cut for z^{a-1}.

Also,

$$J = \int_{C_R} + \int_R^\epsilon \frac{(xe^{i2\pi})^{a-1}}{x+1}\,dx + \int_{C_\epsilon} + \int_\epsilon^R \frac{(xe^{i0})^{a-1}}{x+1}\,dx$$

so $\quad 2\pi i\, e^{(a-1)\pi i} = \int_{C_R} + \int_{C_\epsilon} + \int_\epsilon^R \dfrac{x^{a-1}}{x+1}\,dx + e^{2\pi(a-1)i}\int_R^\epsilon \dfrac{x^{a-1}}{x+1}\,dx.$

Now,

$\left|\int_{C_R}\right| \le \frac{R^{a-1}}{R-1}\pi R \sim \frac{\pi}{R^{1-a}} \to 0$ as $R \to \infty$, and

$\left|\int_{C_\epsilon}\right| \le \frac{\epsilon^{a-1}}{1-\epsilon}2\pi\epsilon \sim 2\pi\epsilon^{a} \to 0$ as $\epsilon \to 0$.

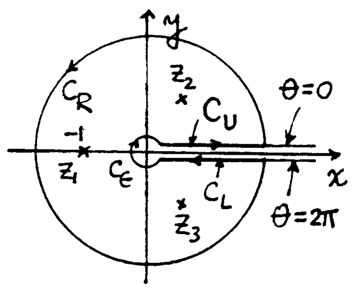
Thus, letting $R \to \infty$ and $\epsilon \to 0$ in ♀ gives

$$2\pi i e^{(a-1)\pi i} = (1-e^{2\pi(a-1)i})\int_0^\infty \frac{x^{a-1}}{x+1}\,dx$$

so

$$\pi = \frac{e^{-(a-1)\pi i} - e^{(a-1)\pi i}}{2i}\int_0^\infty \frac{x^{a-1}}{x+1}\,dx, \text{ so } I = \frac{\pi}{\sin(1-a)\pi} = \frac{\pi}{\sin a\pi}.$$

(b) $I = \int_0^\infty \frac{\sqrt{x}}{x^3+1}\,dx$. Consider $J = \oint_C \frac{\sqrt{z}}{z^3+1}\,dz$ where C is:



The roots of $z^3+1=0$ are $z=-1\equiv z_1$, $z_2 = e^{i\pi/3}$, and $z = e^{i5\pi/3}$ (not $e^{-i\pi/3}$, per the cut), so

$$J = 2\pi i(\text{Res}@z_1 + \text{Res}@z_2 + \text{Res}@z_3)$$

$$= 2\pi i\left(\frac{(z-z_1)}{z^3+1}\sqrt{z}\Big|_{z\to z_1} + \frac{(z-z_2)}{z^3+1}\sqrt{z}\Big|_{z\to z_2} + \frac{(z-z_3)}{z^3+1}\sqrt{z}\Big|_{z\to z_3}\right)$$

or, by l'Hôpital,

$$= 2\pi i\left(\frac{\sqrt{z_1}}{3z_1^2} + \frac{\sqrt{z_2}}{3z_2^2} + \frac{\sqrt{z_3}}{3z_3^2}\right)$$

$$= \frac{2\pi i}{3}\left(z_1^{-3/2} + z_2^{-3/2} + z_3^{-3/2}\right)$$

where, by the branch cut,

$$z_1^{-3/2} = (1e^{\pi i})^{-3/2} = e^{-3\pi i/2} = i$$

$$z_2^{-3/2} = (1e^{\pi i/3})^{-3/2} = e^{-\pi i/2} = -i$$

$$z_3^{-3/2} = (1e^{5\pi i/3})^{-3/2} = e^{-5\pi i/2} = -i$$

$$= \frac{2\pi i}{3}(i - i - i) = 2\pi/3.$$

Also, $J = \int_{C_R} + \int_R^\epsilon \frac{(xe^{2\pi i})^{1/2}}{x^3+1}\,dx + \int_{C_\epsilon} + \int_\epsilon^R \frac{(xe^{i0})^{1/2}}{x^3+1}\,dx \quad *$

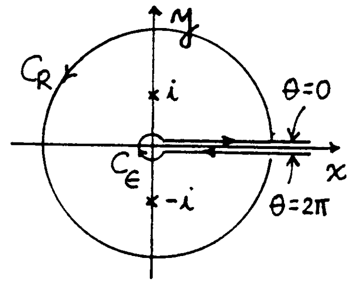
Now, $\left|\int_{C_R}\right| \le \frac{\sqrt{R}}{(R-1)^3}2\pi R \sim 2\pi R^{-3/2} \to 0$ as $R \to \infty$

$\left|\int_{C_\epsilon}\right| \le \frac{\sqrt{\epsilon}}{(1-\epsilon)^3}2\pi\epsilon \sim 2\pi\epsilon^{3/2} \to 0$ as $\epsilon \to 0$

so, letting $R \to \infty$ and $\epsilon \to 0$ in $*$,

$$\frac{2\pi}{3} = 0 + \int_\infty^0 \frac{-\sqrt{x}}{x^3+1}\,dx + 0 + \int_0^\infty \frac{\sqrt{x}}{x^3+1}\,dx,$$

so $\int_0^\infty \frac{\sqrt{x}}{x^3+1}\,dx = \frac{\pi}{3}$.



(c) $I = \int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)^2}$. Consider $J = \oint_C \frac{dz}{\sqrt{z}(z^2+1)^2}$

$= 2\pi i\,(\text{Res@}i + \text{Res@}-i)$

Let's work out the residues.

@i: $\frac{1}{\sqrt{z}(z^2+1)^2} = \frac{1}{\sqrt{i+(z-i)}\,[2i+(z-i)]^2[z-i]^2}$

$= \frac{1}{\sqrt{i}(2i)^2}(1+t)^{-1/2}\left(1+\frac{t}{2i}\right)^{-2}\frac{1}{t^2} \qquad (t = z-i)$

$= \frac{i^{-1/2}}{-4}\left(1-\tfrac{1}{2}t+\cdots\right)\left(1-2\frac{t}{2i}+\cdots\right)\frac{1}{t^2}$

so $\text{Res@}i = -\frac{i^{-1/2}}{4}\left(-\frac{1}{2}-\frac{1}{i}\right) = \frac{(e^{\pi i/2})^{-1/2}}{4}\left(\frac{1}{2}+\frac{1}{i}\right) = \frac{-1-3i}{8\sqrt{2}}$

@$-i$: $\frac{1}{\sqrt{z}(z^2+1)^2} = \frac{1}{\sqrt{-i+(z+i)}\,[z+i]^2[-2i+(z+i)]^2}$

is same as above, with $i \to -i$, so

$\text{Res@}-i = -\frac{(-i)^{-1/2}}{4}\left(-\frac{1}{2}+\frac{1}{i}\right) = \frac{(e^{3\pi i/2})^{-1/2}}{4}\left(\frac{1}{2}-\frac{1}{i}\right) = \frac{1-3i}{8\sqrt{2}}$

so $J = 2\pi i\left(\frac{-1-3i}{8\sqrt{2}} + \frac{1-3i}{8\sqrt{2}}\right) = \frac{3\pi}{2\sqrt{2}}$

Also, $J = \int_{C_R} + \int_R^\epsilon \frac{dx}{\sqrt{xe^{2\pi i}}(x^2+1)^2} + \int_{C_\epsilon} + \int_\epsilon^R \frac{dx}{\sqrt{xe^{i0}}(x^2+1)^2} \qquad \ddagger$

Now,

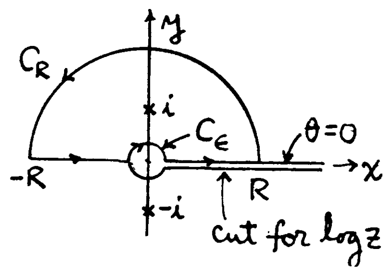
$\left|\int_{C_R}\right| \le \frac{1}{\sqrt{R}(R-1)^4}2\pi R \sim 2\pi R^{-7/2} \to 0$ as $R\to\infty$

$\left|\int_{C_\epsilon}\right| \le \frac{1}{\sqrt{\epsilon}(1-\epsilon)^4}2\pi\epsilon \sim 2\pi\sqrt{\epsilon} \to 0$ as $\epsilon \to 0$

so, letting $R\to\infty$ and $\epsilon\to 0$ in $\ddagger$ gives

$\frac{3\pi}{2\sqrt{2}} = 0 + \int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)^2} + 0 + \int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)^2}$

so $\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)^2} = \frac{3\pi}{4\sqrt{2}}$



(d) $I = \int_0^\infty \frac{\ln x}{x^2+1}dx$. Consider $J = \oint_C \frac{\log z}{z^2+1}dz$ where C:

$J = 2\pi i\,\text{Res@}i = 2\pi i\frac{\log i}{2i} = \pi(\underbrace{\ln 1}_{0} + i\tfrac{\pi}{2}) = i\frac{\pi^2}{2}$

Also,

$J = \int_{C_R} + \int_{-R}^{-\epsilon} \frac{\log(-xe^{i\pi})}{x^2+1}dx + \int_{C_\epsilon} + \int_\epsilon^R \frac{\log(xe^{i0})}{x^2+1}dx$

so $\quad i\frac{\pi^2}{2} = \int_{C_R} + \int_{-R}^{-\epsilon} \frac{\ln(-x)+i\pi}{x^2+1}\,dx + \int_{C_\epsilon} + \int_\epsilon^R \frac{\ln x + i0}{x^2+1}\,dx \quad \bigstar$

Now, $\left|\int_{C_R}\right| \leq \frac{\max|\log(Re^{i\theta})|}{(R-1)^2}\pi R = \frac{\sqrt{(\ln R)^2+\pi^2}}{(R-1)^2}\pi R \sim \pi\frac{\ln R}{R} \to 0$ as $R\to\infty$

$\left|\int_{C_\epsilon}\right| \leq \frac{\max|\log(\epsilon e^{i\theta})|}{(1-\epsilon)^2}\pi\epsilon = \frac{\sqrt{(\ln\epsilon)^2+\pi^2}}{(1-\epsilon)^2}\pi\epsilon \sim \pi\epsilon|\ln\epsilon| \to 0$ as $\epsilon\to 0$

so letting $R\to\infty$ and $\epsilon\to 0$ in $\bigstar$ gives

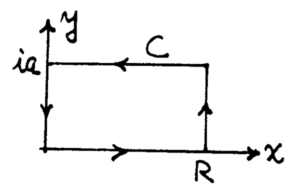
$$i\frac{\pi^2}{2} = 0 + \int_{-\infty}^0 \frac{\ln(-x)+i\pi}{x^2+1}\,dx + 0 + \int_0^\infty \frac{\ln x}{x^2+1}\,dx$$

$$= \int_\infty^0 \frac{\ln t + i\pi}{t^2+1}(-dt) + \int_0^\infty \frac{\ln x}{x^2+1}\,dx = 2\int_0^\infty \frac{\ln x}{x^2+1}\,dx + i\pi\int_0^\infty \frac{dx}{x^2+1}$$

and equating real and imaginary parts gives

$$\int_0^\infty \frac{\ln x}{x^2+1}\,dx = 0 \quad \text{and (as a "bonus")} \int_0^\infty \frac{dx}{x^2+1} = \frac{\pi}{2}.$$

7. Consider $J = \oint_C e^{-z^2}dz$. $J = 0$ since e^{-z^2} is analytic everywhere. Thus,



$$0 = \int_0^R e^{-x^2}dx + \int_0^a e^{y^2-i2Ry-R^2}\,i\,dy + \int_R^0 e^{-x^2-i2ax+a^2}dx + \int_a^0 e^{y^2}i\,dy$$

$$\equiv K+L+M+N,$$

say. Now, $K \to \sqrt{\pi}/2$ as $R\to\infty$

$$|L| \leq \max_{0\leq y\leq a}|e^{y^2-i2Ry-R^2}|\cdot a = \max_{0\leq y\leq a} a e^{y^2-R^2} = ae^{a^2-R^2} \to 0 \text{ as } R\to\infty$$

$$M = -\int_0^R e^{a^2}e^{-x^2}(\cos 2ax - i\sin 2ax)\,dx \to -e^{a^2}\int_0^\infty e^{-x^2}\cos 2ax\,dx$$

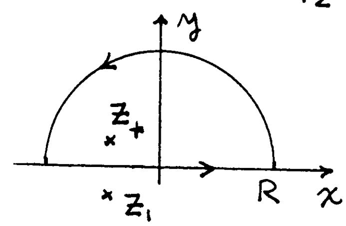
plus imaginary term

$N =$ imaginary

so, equating real parts gives $0 = \frac{\sqrt{\pi}}{2} - e^{a^2}\int_0^\infty e^{-x^2}\cos 2ax\,dx$

or, $\int_0^\infty e^{-x^2}\cos 2ax\,dx = \frac{\sqrt{\pi}}{2}e^{-a^2}$

8. (a) $J = \oint_C \frac{dz}{z^2+z+1}$. $z^2+z+1 = 0 \to z = \frac{-1\pm\sqrt{3}i}{2} \equiv z_\pm$



$2\pi i\ \mathrm{Res}@z_+ = \int_{-\infty}^\infty \frac{dx}{x^2+x+1}$ (by letting $R\to\infty$)

$= \int_{-\infty}^0 \frac{dx}{x^2+x+1} + \int_0^\infty \frac{dx}{x^2+x+1}$ ($x = -t$ in first)

$= \int_\infty^0 \frac{-dt}{t^2-t+1} + \text{"} = +\int_0^\infty \frac{dx}{x^2-x+1} + \int_0^\infty \frac{dx}{x^2+x+1},$

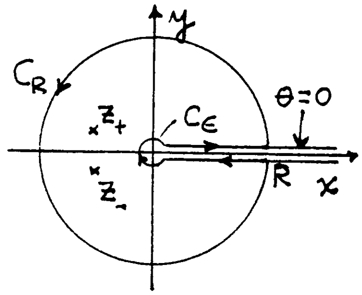
which is one equation in the two unknown integrals.

(b) $J = \oint_C \frac{\log z}{z^2+z+1}\,dz = 2\pi i\,(\text{Res@}z_+ + \text{Res@}z_-)$

$= 2\pi i\left(\frac{\log z_+}{z_+-z_-} + \frac{\log z_-}{z_--z_+}\right)$

$= 2\pi i\left(\frac{\ln 1 + i2\pi/3}{\sqrt{3}i} + \frac{\ln 1 + i4\pi/3}{-\sqrt{3}i}\right)$

$= -\frac{4\pi^2}{3\sqrt{3}}i$



$\left|\int_{C_R}\right| \le \frac{\max|\ln R + i\theta|}{\min|z-z_+|\,\min|z-z_-|}2\pi R = \frac{\sqrt{(\ln R)^2+4\pi^2}}{(R-1)^2}2\pi R \sim 2\pi\frac{\ln R}{R} \to 0 \text{ as } R\to\infty$

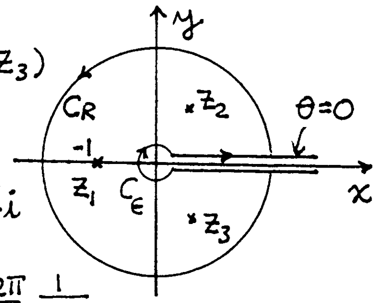
$\left|\int_{C_\epsilon}\right| \le \frac{\max|\ln\epsilon + i\theta|}{\min|z-z_+|\,\min|z-z_-|}2\pi R = \frac{\sqrt{(\ln\epsilon)^2+4\pi^2}}{(1-\epsilon)^2}2\pi\epsilon \sim 2\pi\epsilon|\ln\epsilon| \to 0 \text{ as } \epsilon\to 0$

so letting $R\to\infty$ and $\epsilon\to 0$ in

$$-\frac{4\pi^2}{3\sqrt{3}}i = \int_{C_R} + \int_R^\epsilon \frac{\cancel{\ln x} + i2\pi}{x^2+x+1}dx + \int_{C_\epsilon} + \int_\epsilon^R \frac{\cancel{\ln x}+i0}{x^2+x+1}dx$$

gives $-\frac{4\pi^2}{3\sqrt{3}}i = -i2\pi\int_0^\infty \frac{dx}{x^2+x+1}$ or, $\int_0^\infty \frac{dx}{x^2+x+1} = \frac{2\pi}{3\sqrt{3}}$

9. (a) Consider $J = \oint_C \frac{\log z}{z^3+1}dz = 2\pi i(\text{Res@}z_1 + \text{Res@}z_2 + \text{Res@}z_3)$



$z_1 = -1 = e^{\pi i},\ z_2 = e^{\pi i/3},\ z_3 = e^{5\pi i/3}$

$\text{Res@}z_1 = \frac{\log z_1}{(z_1-z_2)(z_1-z_3)} = \frac{\ln 1 + \pi i}{\left[-1-\left(\frac{1+\sqrt{3}i}{2}\right)\right]\left[-1-\left(\frac{1-\sqrt{3}i}{2}\right)\right]} = \frac{\pi}{3}i$

$\text{Res@}z_2 = \frac{\log z_2}{(z_2-z_1)(z_2-z_3)} = \frac{\ln 1 + \pi i/3}{\left[\frac{1+\sqrt{3}i}{2}+1\right]\left[\frac{1+\sqrt{3}i}{2} - \frac{1-\sqrt{3}i}{2}\right]} = \frac{2\pi}{9}\frac{1}{\sqrt{3}+i}$

$\text{Res@}z_3 = \frac{\log z_3}{(z_3-z_1)(z_3-z_2)} = \text{etc} = \frac{10\pi}{9}\frac{1}{i-\sqrt{3}}$

so $2\pi i\left(\frac{\pi}{3}i + \frac{2\pi}{9}\frac{1}{\sqrt{3}+i} + \frac{10\pi}{9}\frac{1}{i-\sqrt{3}}\right) = \int_\epsilon^R \frac{\cancel{\ln x}}{x^3+1}dx + \int_{C_R} + \int_R^\epsilon \frac{\cancel{\ln x}+i2\pi}{x^3+1}dx + \int_{C_\epsilon}$ $\quad *$

Bounds: $\left|\int_{C_R}\right| \le \frac{\max|\ln R+i\theta|}{(R-1)^3}2\pi R = \sqrt{(\ln R)^2+4\pi^2}\,\frac{2\pi R}{(R-1)^3} \sim 2\pi\frac{\ln R}{R^2} \to 0 \text{ as } R\to\infty$

$\left|\int_{C_\epsilon}\right| \le \frac{\max|\ln\epsilon + i\theta|}{(1-\epsilon)^3}2\pi\epsilon = \sqrt{(\ln\epsilon)^2+4\pi^2}\,\frac{2\pi\epsilon}{(1-\epsilon)^3} \sim 2\pi\epsilon|\ln\epsilon| \to 0 \text{ as } \epsilon\to 0$

so, letting $R\to\infty$ and $\epsilon\to 0$ in $*$ gives

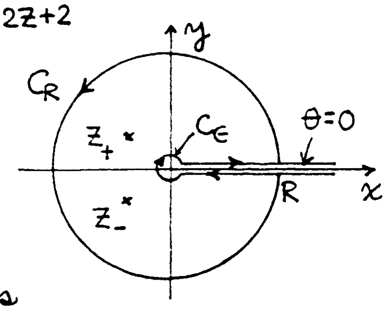
$-\frac{4\pi^2}{3\sqrt{3}}i = -2\pi i\int_0^\infty \frac{dx}{x^3+1}$ so $\int_0^\infty \frac{dx}{x^3+1} = \frac{2\pi}{3\sqrt{3}}$

(b) Using Maple, $I = \frac{\pi}{108} - \frac{1}{180} - \frac{1}{54}\tan^{-1}\left(\frac{1}{3}\right)$

(c) Using Maple, $I = 2\pi/(3\sqrt{3})$

(d) Set $t = (1-x)/x$ to send $\int_0^1$ to $\int_0^\infty$.

$I = \int_0^1 \frac{dx}{x^2+1} = \int_0^\infty \frac{dt}{t^2+2t+2}$, so consider $J = \oint_C \frac{\log z}{z^2+2z+2}\,dz$



$$2\pi i\,(\underbrace{\text{Res @} -1+i}_{z_+}) + 2\pi i\,(\underbrace{\text{Res @} -1-i}_{z_-})$$

$$= \int_\epsilon^R \frac{\ln x + i0}{x^2+2x+2}\,dx + \int_{C_R} + \int_R^\epsilon \frac{\ln x + i2\pi}{x^2+2x+2}\,dx + \int_{C_\epsilon}$$

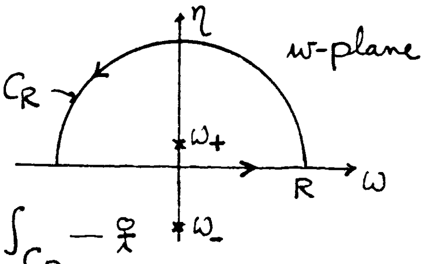
We can (but won't) show that $\int_{C_R} \to 0$ as $R\to\infty$ and that $\int_{C_\epsilon} \to 0$ as $\epsilon \to 0$ so the latter becomes

$$\cancel{2\pi} i\left(\frac{\log z_+}{z_+ - z_-} + \frac{\log z_-}{z_- - z_+}\right) = \cancel{2\pi} i\int_\infty^0 \frac{dx}{x^2+2x+2}$$

so $I = -\left(\frac{\cancel{\ln\sqrt{2}} + i3\pi/4}{2i} + \frac{\cancel{\ln\sqrt{2}} + i5\pi/4}{-2i}\right) = \frac{\pi}{4}$

10.(a) $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega x}}{\omega^2 + i\omega + 2}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega x}}{(\omega-\omega_+)(\omega-\omega_-)}\,d\omega$ where $\omega_\pm = (-1\pm\sqrt{3})i/2$

Consider $J = \frac{1}{2\pi}\oint_C \frac{e^{ixw}}{(w-\omega_+)(w-\omega_-)}\,dw$



$$= 2\pi i\,(\text{Res @ } \omega_+) = 2\pi i \frac{1}{2\pi}\frac{e^{ix\omega_+}}{\omega_+ - \omega_-}$$

$$= \frac{e^{-(\sqrt{3}-1)x/2}}{\sqrt{3}} \quad \text{also} = \frac{1}{2\pi}\int_{-R}^{R} \frac{e^{i\omega x}}{\omega^2+i\omega+2}\,d\omega + \int_{C_R} \quad \text{—} (*)$$

Now, $\left|\int_{C_R}\right| \le \frac{\max\left|e^{ix(\omega+i\eta)}\right|}{\min|w-\omega_+|\,\min|w-\omega_-|\;2\pi}\,\pi R = \frac{\pi R}{(R-|\omega_+|)\sqrt{R^2+|\omega_-|^2}}\frac{1}{2\pi} \sim \frac{1}{2R}$

$\to 0$ as $R\to\infty$

so letting $R\to\infty$ in $(*)$ gives

$$\frac{e^{-(\sqrt{3}-1)x/2}}{\sqrt{3}} = \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\omega x}}{\omega^2+i\omega+2}\,d\omega}_{f(x)} + 0$$

so $f(x) = e^{-(\sqrt{3}-1)x/2}/\sqrt{3}$.

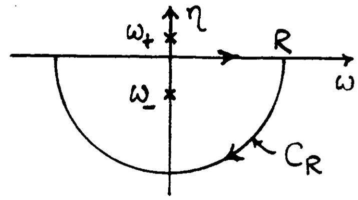
However, understand that this result has only for $x>0$, for if $x>0$ then $\max|\exp[ix(\omega+i\eta)]| = \max|\exp(ix\omega)\exp(-x\eta)|$
$= \max|\exp(ix\omega)|\,\max|\exp(-x\eta)| = 1\,\max e^{-x\eta} = 1$, provided that $x>0$.

If $x<0$ we need to close C on the bottom instead:

This time $I = \frac{1}{2\pi}\oint_C \frac{e^{ix\omega}}{(\omega-\omega_+)(\omega-\omega_-)}\,d\omega$

$$= -2\pi i\,(\mathrm{Res} \,@\, \omega_-) = -2\pi i \frac{1}{2\pi}\frac{e^{ix\omega_-}}{\omega_- - \omega_+}$$

$$= \frac{e^{(\sqrt{3}+1)x/2}}{\sqrt{3}}, \text{ also } = \frac{1}{2\pi}\int_{-R}^{R}\frac{e^{i\omega x}}{\omega^2+i\omega+2}\,d\omega + \int_{C_R} \text{———} \quad \ddagger$$



Now, $\left|\int_{C_R}\right| \le \frac{\max|e^{ix(\omega+i\eta)}|}{\min|\omega-\omega_+|\,\min|\omega-\omega_-|\,2\pi}\pi R$. This time $x<0$ but so is η,
so $\max|e^{ix(\omega+i\eta)}| = \max|e^{ix\omega}|\,\max|e^{-x\eta}| = \max e^{-x\eta} = 1$, and

$$\left|\int_{C_R}\right| \le \frac{\pi R}{\sqrt{R^2+|\omega_+|^2}\,(R-|\omega_-|)\,2\pi} \sim \frac{1}{2R} \to 0 \text{ as } R\to\infty$$

so letting $R\to\infty$ in $\ddagger$ gives

$$\frac{e^{(\sqrt{3}+1)x/2}}{\sqrt{3}} = \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\omega x}}{\omega^2+i\omega+2}\,d\omega}_{f(x)} + 0$$

so $f(x) = e^{(\sqrt{3}+1)x/2}/\sqrt{3}$ for $x<0$. Summarizing,

$$f(x) = \begin{cases} e^{(\sqrt{3}+1)x/2}/\sqrt{3}, & x<0 \\ e^{-(\sqrt{3}-1)x/2}/\sqrt{3}, & x>0 \end{cases}$$

Though not asked to do this let us check this with the Fourier transform table in Appendix D. Using partial fractions,

$$\frac{1}{\omega^2+i\omega+2} = \frac{1}{(\omega-\omega_+)(\omega-\omega_-)} = \frac{1}{\omega_+-\omega_-}\left(\frac{1}{\omega-\omega_+} - \frac{1}{\omega-\omega_-}\right) = \frac{1}{\sqrt{3}i}\left(\frac{1}{\omega-\omega_+} - \frac{1}{\omega-\omega_-}\right)$$

Using entries 2 and 3 we need to obtain the forms $\frac{1}{a+i\omega}$ or $\frac{1}{a-i\omega}$, where $\mathrm{Re}\, a>0$. Well,

$$\frac{1}{\omega^2+i\omega+2} = \frac{1}{\sqrt{3}}\left(\frac{1}{\left(\frac{\sqrt{3}-1}{2}\right)+i\omega} - \frac{1}{\left(\frac{-1-\sqrt{3}}{2}\right)+i\omega}\right)$$

$$= \frac{1}{\sqrt{3}}\left(\frac{1}{\left(\frac{\sqrt{3}-1}{2}\right)+i\omega} + \frac{1}{\left(\frac{\sqrt{3}+1}{2}\right)-i\omega}\right)$$

so entries 2 and 3, respectively, give the inverse as

$$f(x) = \frac{1}{\sqrt{3}}\left(H(x)e^{-(\sqrt{3}-1)x/2} + H(-x)e^{(\sqrt{3}+1)x/2}\right),$$

$$= \begin{cases} e^{-(\sqrt{3}-1)x/2}/\sqrt{3}, & x>0 \\ e^{(\sqrt{3}+1)x/2}/\sqrt{3}, & x<0, \end{cases}$$

as given above. ✓

(b) Same idea as in (a), but this time both roots $w = i, 2i$ are in the upper half plane so for $x > 0$ (closing the contour above) we get

$$f(x) = 2\pi i(\text{Res}@i + \text{Res}@2i) = 2\pi i \frac{1}{2\pi}\left(\frac{e^{-x}}{-i} + \frac{e^{-2x}}{i}\right) = -e^{-x} + e^{-2x}$$

and for $x < 0$ we get (closing the contour below)

$$f(x) = 0.$$

(c) Same idea as in (a). This time both roots $w = -i, -2i$ are in the lower half plane so for $x > 0$ (closing the contour above) we get

$$f(x) = 0$$

and for $x < 0$ (closing the contour below) we get

$$f(x) = -2\pi i(\text{Res}@-i + \text{Res}@-2i) = -2\pi i \frac{1}{2\pi}\left(\frac{e^{x}}{i} + \frac{e^{2x}}{-i}\right) = -e^{x} + e^{2x}$$

(d) Same idea as in (a). This time the only root $w = 2/i = -2i$ is in the lower half plane so for $x > 0$ (closing above) we get

$$f(x) = 0$$

and for $x < 0$ (closing below) we get what? The integrand is

$$\frac{1}{2\pi}\frac{e^{iwx}}{(2-iw)^2} = -\frac{1}{2\pi}\frac{e^{iwx}}{(w+2i)^2}$$

so for $x < 0$ we have

$$f(x) = -2\pi i(\text{Res}@-2i) = -2\pi i \frac{d}{dw}\left(-\frac{1}{2\pi}\cancel{(w+2i)^2}\frac{e^{iwx}}{\cancel{(w+2i)^2}}\right)\Big|_{w\to 2i}$$
$$= -xe^{-2x}$$

(g) Same idea as in (a). This time the integrand has

$$\frac{1}{2\pi}\frac{e^{iwx}}{(w^2+1)^2} = \frac{1}{2\pi}\frac{e^{iwx}}{(w-i)^2(w+i)^2}$$

has 2nd order poles at $w = i, -i$. For $x > 0$ we close the contour above and get

$$f(x) = 2\pi i(\text{Res}@i) = 2\pi i \frac{d}{dw}\left(\frac{\cancel{(w-i)^2}}{2\pi}\frac{e^{iwx}}{\cancel{(w-i)^2}(w+i)^2}\right)\Big|_{w=i} = \frac{1+x}{4}e^{-x}$$

and for $x < 0$ we close the contour below and get

$$f(x) = -2\pi i(\text{Res}@-i) = -2\pi i \frac{d}{dw}\left(\frac{\cancel{(w+i)^2}}{2\pi}\frac{e^{iwx}}{(w-i)^2\cancel{(w+i)^2}}\right)\Big|_{w=-i} = \frac{1-x}{4}e^{x},$$

so

$$f(x) = \frac{1+|x|}{4}e^{-|x|}$$

NOTE: If we use entry 4 of Appendix D we have $f(x) = \frac{1}{2}e^{-|x|} * \frac{1}{2}e^{-|x|}$ $= \frac{1}{4}\int_{-\infty}^{\infty} e^{-|x-\xi|}e^{-|\xi|}\,d\xi$, which does give the same result.

11. (a) $$\int_{-1}^{3}\frac{dx}{x} = \lim_{\substack{\epsilon_1\to0\\ \epsilon_2\to0}}\left\{\int_{-1}^{0-\epsilon_1}\frac{dx}{x} + \int_{0+\epsilon_2}^{3}\frac{dx}{x}\right\}$$
$$= \lim_{\epsilon_1\to0}\ln|x|\Big|_{-1}^{-\epsilon_1} + \lim_{\epsilon_2\to0}\ln|x|\Big|_{\epsilon_2}^{3}$$
$$= \lim_{\epsilon_1\to0}\ln\epsilon_1 + \ln 3 - \lim_{\epsilon_2\to0}\epsilon_2$$
does not exist because $\lim_{\epsilon_1\to}\ln\epsilon_1 = $ "$-\infty$" does not exist, and similarly for $\lim_{\epsilon_2\to0}\epsilon_2$. However,
$$\oint_{-1}^{3}\frac{dx}{x} = \lim_{\epsilon\to0}\left\{\int_{-1}^{0-\epsilon}\frac{dx}{x} + \int_{0+\epsilon}^{3}\frac{dx}{x}\right\} = \lim_{\epsilon\to0}\left(\cancel{\ln\epsilon} + \ln 3 - \cancel{\ln\epsilon}\right)$$
$$= \lim_{\epsilon\to0}\ln 3 \text{ \underline{does} exist, and it} = \ln 3.$$

(b) $$\int_{1}^{4}\frac{dx}{x(x-2)} = \lim_{\substack{\epsilon_1\to0\\ \epsilon_2\to0}}\left\{\int_{1}^{2-\epsilon_1}\frac{dx}{x(x-2)} + \int_{2+\epsilon_2}^{4}\frac{dx}{x(x-2)}\right\}$$

Note: By partial fractions $\frac{1}{x(x-2)} = -\frac{1}{2x} + \frac{1}{2}\frac{1}{x-2}$

$$= \lim_{\epsilon_1\to0}\left(-\tfrac{1}{2}\ln|x| + \tfrac{1}{2}\ln|x-2|\right)\Big|_{1}^{2-\epsilon_1}$$
$$+ \lim_{\epsilon_2\to0}\left(-\tfrac{1}{2}\ln|x| + \tfrac{1}{2}\ln|x-2|\right)\Big|_{2+\epsilon_2}^{4}$$
$$= \lim_{\epsilon_1\to0}\left(-\tfrac{1}{2}\ln|2-\epsilon_1| + \tfrac{1}{2}\ln\epsilon_1\right) - \left(-\tfrac{1}{2}\cancel{\ln 1} + \tfrac{1}{2}\cancel{\ln 1}\right)$$
$$+ \left(-\tfrac{1}{2}\ln 4 + \tfrac{1}{2}\ln 2\right) - \lim_{\epsilon_2\to0}\left(-\tfrac{1}{2}\ln|2+\epsilon_2| + \tfrac{1}{2}\ln\epsilon_2\right)$$
$$= \cancel{-\tfrac{1}{2}\ln 2} + \tfrac{1}{2}\lim_{\epsilon_1\to0}\ln\epsilon_1 - \tfrac{1}{2}\ln 4 + \tfrac{1}{2}\ln 2 + \cancel{\tfrac{1}{2}\ln 2} - \tfrac{1}{2}\lim_{\epsilon_2\to0}\ln\epsilon_2 \qquad *$$
does not exist because each of the limits in * fails to exist. However, if $\epsilon_1 = \epsilon_2$ then the two limit terms in * cancel and
$$\oint_{1}^{4}\frac{dx}{x(x-2)} \text{ does exist and} = -\tfrac{1}{2}\ln 4 + \tfrac{1}{2}\ln 2$$
$$= -\ln 2 + \tfrac{1}{2}\ln 2 = -\tfrac{1}{2}\ln 2$$

12. So that the $\int_{C_R}$ integral $\to 0$ as $R\to\infty$, consider
$$J = \oint_C \frac{e^{iz}}{z}\,dz$$
rather than $\oint_C \frac{\sin z}{z}\,dz$. However, whereas $\sin z/z$ is analytic everywhere, the $e^{iz}/z \sim 1/z$ as $z\to0$; that is, it has a singularity (1st order pole) right on the path of integration. Thus, modify the path C by "indenting"

C at the origin with a semicircle C_ϵ, as shown in the exercise. Inside the indented contour e^{iz}/z is analytic so (by Cauchy's theorem—or the residue theorem)

$$\oint = 0 = \left(\int_{-R}^{-\epsilon} + \int_{\epsilon}^{R}\right)\frac{e^{iz}}{z}dz + \int_{C_\epsilon} + \int_{C_R},$$

which holds for each R (no matter how large) and for each ϵ (no matter how small). Thus it holds in the limit as $R\to\infty$ and as $\epsilon\to 0$, so

$$0 = \text{⨍}_{-\infty}^{\infty} \frac{\cos x + i\sin x}{x}dx + \lim_{R\to\infty}\int_{C_R} + \lim_{\epsilon\to 0}\int_{C_\epsilon}. \qquad ☆$$

Now, on C_R we have $|e^{iz}| = |e^{i(x+iy)}| = |e^{ix}||e^{-y}| = e^{-y} \le 1$ and $|z|=R$, so

$$\left|\int_{C_R}\frac{e^{iz}}{z}dz\right| \le \frac{1}{R}\pi R = \pi,$$

which is not sharp enough to determine whether $\int_{C_\epsilon}\to 0$ or not. Thus, use the sharper bound

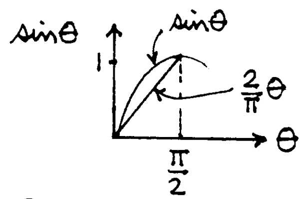
$$\left|\int_C f(z)\,dz\right| \le \int_C |f(z)||dz|$$

given in Exercise 7 of Sec. 23.2:

$$\left|\int_{C_R}\frac{e^{iz}}{z}dz\right| \le \int_0^{\pi}\frac{e^{-y}}{R}R\,d\theta = \int_0^{\pi} e^{-R\sin\theta}d\theta = 2\int_0^{\pi/2} e^{-R\sin\theta}d\theta \quad \left(\text{since } \sin\theta \text{ is symmetric about } \theta=\tfrac{\pi}{2}\right)$$

The latter is a hard integral, but since all we need is a bound, use the fact that

$$\sin\theta \ge \frac{2}{\pi}\theta$$



so

$$\left|\int_{C_R}\right| \le 2\int_0^{\pi/2} e^{-\frac{2R}{\pi}\theta}d\theta = \frac{\pi}{R}(1-e^{-R}) \to 0 \text{ as } R\to\infty.$$

Good. Now consider the C_ϵ integral. Following the hint, write the Laurent series

$$\frac{e^{iz}}{z} = \frac{1}{z} + i - \frac{z}{2} + \cdots \quad \text{in } 0<|z|<\infty.$$

Since the latter converges in $0<|z|<\infty$, then the part

$$g(z) \equiv i - \frac{z}{2} + \cdots \qquad ♀$$

does too. Since ♀ converges at $z=1$, say, it must converge (by theorem 24.2.1) inside $|z|<1$ and hence (by Theorem 24.2.8) be analytic there. Since $g(z)$ is analytic there it is bounded on C_ϵ by m, say. Thus,

$$\left|\int_{C_\epsilon} g(z)\,dz\right| \le (m)(\pi\epsilon) \to 0 \text{ as } \epsilon\to 0.$$

Further,

$$\int_{C_\epsilon}\frac{1}{z}dz = \int_{C_\epsilon}\frac{1}{\epsilon e^{i\theta}}d(\epsilon e^{i\theta}) = \int_{\pi}^{0}\frac{\epsilon i e^{i\theta}}{\epsilon e^{i\theta}}d\theta = -\pi i,$$

so $\lim\limits_{\epsilon\to 0} \int_{C_\epsilon} \frac{e^{iz}}{z} dz$ is $= -\pi i$

Hence, ★ becomes

$$0 = \fint_{-\infty}^{\infty} \frac{\cos x + i \sin x}{x} dx + 0 - \pi i$$

or $$\fint_{-\infty}^{\infty} \frac{\cos x}{x} dx + i \fint_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi i .$$

Equating real and imaginary parts give

$$\fint_{-\infty}^{\infty} \frac{\cos x}{x} dx = 0 \qquad ✡$$

and $$\fint_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi ,$$

of which the latter is actually

$$\fint_{-\infty}^{\infty} \frac{\sin x}{x} dx = \int_{-\infty}^{\infty} \frac{\sin x}{x} dx \quad \text{since } \frac{\sin x}{x} \text{ is nonsingular at } x=0$$

$$= 2\int_0^{\infty} \frac{\sin x}{x} dx$$

so $$\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2} .$$

The former, ✡, is a "bonus" result, but not very interesting since it follows from the antisymmetry of the integrand $\cos x / x$