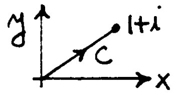
CHAPTER 23

Section 23.2

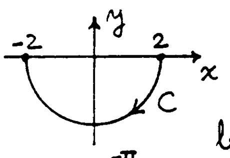
1. (a)



$$I = \int_C |z|^2\,dz = \int_C (x^2+y^2)(dx+i\,dy) \quad \text{but } y=x \text{ on } C, \text{ so this}$$
$$= \int_0^1 2x^2(1+i)\,dx = (2+2i)/3$$

(b) $I = \int_C \bar{z}\,dz = \int_C (x-iy)(dx+i\,dy) = \int_0^1 (1-i)x(1+i)\,dx = 2x^2/2\big|_0^1 = 1$

(c)

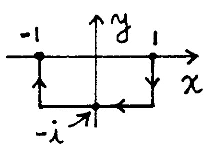


$$I = \int_C \bar{z}\,dz = \int_C (x-iy)(dx+i\,dy) \qquad (x=2\cos\theta,\ y=2\sin\theta)$$
$$= \int_0^{-\pi} 2(\cos\theta - i\sin\theta)(-2\sin\theta + i2\cos\theta)\,d\theta,$$

but easier to use $z = 2e^{i\theta}$, $\bar{z} = 2e^{-i\theta}$ on C:

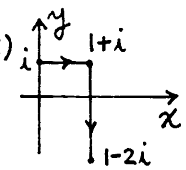
$$I = \int_0^{-\pi} 2e^{-i\theta}\,d(2e^{i\theta}) = 4\int_0^{-\pi} e^{-i\theta} i e^{i\theta}\,d\theta = 4i\int_0^{-\pi} d\theta = -4\pi i$$

(d)



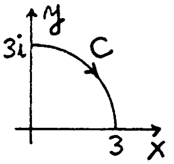
$$I = \int_C \frac{1}{z}\,dz = \int_0^{-1} \frac{i\,dy}{1+iy}\,\frac{1-iy}{1-iy} + \int_1^{-1} \frac{dx}{x-i}\,\frac{x+i}{x+i} + \int_{-1}^{0} \frac{i\,dy}{-1+iy}\,\frac{-1-iy}{-1-iy}$$
$$= \int_0^{-1} \frac{i+y}{1+y^2}\,dy + \int_1^{-1} \frac{i+x}{1+x^2}\,dx - \int_{-1}^{0} \frac{i-y}{1+y^2}\,dy$$
$$= (i\tan^{-1}y + \ln\sqrt{1+y^2})\Big|_0^{-1} + (i\tan^{-1}x + \ln\sqrt{1+x^2})\Big|_1^{-1} - (i\tan^{-1}y - \ln\sqrt{1+y^2})\Big|_{-1}^{0}$$
$$= -i\frac{\pi}{4} + \ln\sqrt{2} - 0 - 0 - i\frac{\pi}{4} + \ln\sqrt{2} - i\frac{\pi}{4} - \ln\sqrt{2} - 0 - 0 - i\frac{\pi}{4} - \ln\sqrt{2} = -\pi i$$

(e)



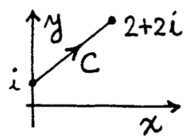
$$I = \int_C e^z\,dz = \int_C e^x(\cos y + i\sin y)(dx+i\,dy)$$
$$= \int_C e^x(\cos y\,dx - \sin y\,dy) + i\int_C e^x(\sin y\,dx + \cos y\,dy)$$
$$= \int_0^1 e^x\cos 1\,dx + i\int_0^1 e^x \sin 1\,dx + \int_1^{-2} -e^1\sin y\,dy + i\int_1^{-2} e^1\cos y\,dy$$
$$= (\cancel{e}-1)\cos 1 + i(\cancel{e}-1)\sin 1 + e(\cos 2 - \cancel{\cos 1}) + ie(-\sin 2 - \cancel{\sin 1})$$
$$= (e\cos 2 - \cos 1) - i(\sin 1 + e\sin 2)$$

(f)



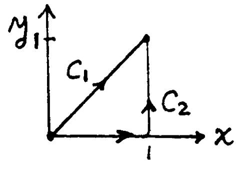
$$I = \int_C (\operatorname{Re} z)\,dz = \int_C x(dx+i\,dy) = \int_0^3 x\,dx + i\int_{\pi/2}^{0} (3\cos\theta)(3\cos\theta\,d\theta) = \frac{9}{2} - \frac{9\pi}{4}i$$

(g)



$$I = \int_C (\operatorname{Im} z)\,dz = \int_C y(dx+i\,dy) = \int_0^2 (\tfrac{1}{2}x+1)\,dx + i\int_1^2 y\,dy = 1+2+i\tfrac{3}{2} = 3+\tfrac{3}{2}i$$

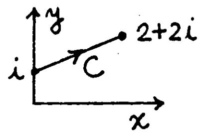
2.



$$\int_{C_1} (x-iy)(dx+i\,dy) = (1-i)(1+i)\int_0^1 x\,dx = 1$$
$$\int_{C_2} (x-iy)(dx+i\,dy) = \int_0^1 x\,dx + \int_0^1 (1-iy)\,i\,dy + \tfrac{1}{2} + i + \tfrac{1}{2} = 1+i.$$

$\int_{C_1} \neq \int_{C_2}$ so the integral is path dependent.

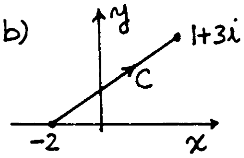
3.(a)



$$\left|\int_C z^5 dz\right| \le \underbrace{(\sqrt{8})^5}_{M} \underbrace{\sqrt{4+1}}_{L} = 128\sqrt{10} \quad \text{(or larger, of course)}$$

↑ since the maximum $|z|$ on C is $|2+2i| = \sqrt{8}$

(b)



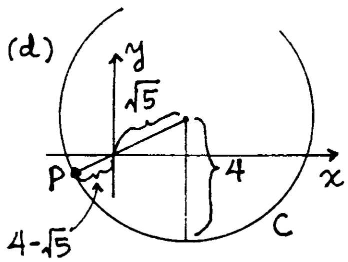
$|e^z| = |e^{x+iy}| = |e^x||e^{iy}| = e^x \le e^1$ on C and $L = \sqrt{9+9} = 3\sqrt{2}$, so

$$\left|\int_C e^z dz\right| \le 3\sqrt{2}\, e.$$

(c) $|e^{-z}| = |e^{-x-iy}| = |e^{-x}||e^{-iy}| = e^{-x} \le e^{-(-2)} = e^2$ on C, and $L = 3\sqrt{2}$ again, so

$$\left|\int_C e^{-z} dz\right| \le 3\sqrt{2}\, e^2.$$

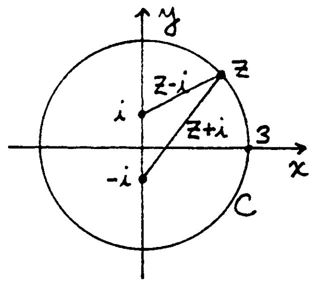
(d)



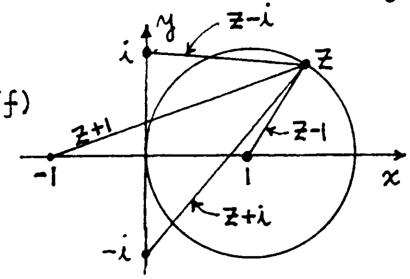
$\max\left|\frac{1}{z}\right| = \max\frac{1}{|z|} = \frac{1}{\min|z|} = \frac{1}{4-\sqrt{5}}$ since the point on C that is closest to the origin (to minimize $|z|$) is P. Also, $L = (2\pi)(4)$, so $\left|\int_C \frac{dz}{z}\right| \le \frac{8\pi}{4-\sqrt{5}}$

(e) $\max\left|\frac{1}{z^2+1}\right| = \frac{1}{\min|z^2+1|} \le \frac{1}{(\min|z-i|)(\min|z+i|)} = \frac{1}{2}\frac{1}{2}$

and $L = (2\pi)(3)$, so $\left|\int_C \frac{dz}{z^2+1}\right| \le \frac{6\pi}{4} = \frac{3\pi}{2}$



(f)



$$\max\left|\frac{z^2+1}{z^2-1}\right| \le \frac{\max|(z-i)(z+i)|}{\min|(z-1)(z+1)|}$$

$$\le \frac{\max|z-i|\max|z+i|}{\min|z+1|} \quad \text{since } |z-1| = 1 \text{ everywhere on } C$$

$$= \frac{(\sqrt{2}+1)(\sqrt{2}+1)}{1} = 3+2\sqrt{2}$$

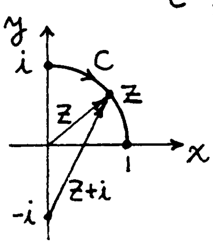
Alternatively, we could say $\max\left|\frac{z^2+1}{z^2-1}\right| \le \frac{\max|z^2+1|}{\min|z^2-1|} \le \frac{\max(|z^2|+|1|)}{\min|z+1|}$ (by triangle inequality)

$$= \frac{\max|z|^2+1}{1} = \frac{5}{1} = 5.$$

Let's use the latter in place of the former since 5 is smaller than $3+2\sqrt{2}$. Also, $L = (2\pi)(1)$, so we have

$$\left|\int_C \frac{z^2+1}{z^2-1} dz\right| \le (5)(2\pi) = 10\pi.$$

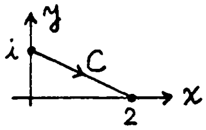
(g)



$\max\left|\frac{1}{z(z+i)}\right| = \max\left(\frac{1}{|z|}\frac{1}{|z+i|}\right) = \max\frac{1}{|z+i|}$ since $|z| = 1$ on C

$= 1/\min|z+i| = 1/\sqrt{2}$. Also, $L = (2\pi)(1)/4$, so

$$\left|\int_C \frac{dz}{z(z+i)}\right| \le \frac{\pi}{2\sqrt{2}} = \frac{\pi\sqrt{2}}{4}.$$

(h) 

$$\max\left|\frac{e^z}{z}\right| \le \frac{\max|e^z|}{\min|z|} = \frac{\max|e^{x+iy}|}{1} = \max|e^x| = e^2$$

and $L=\sqrt{5}$, so $\left|\int_C e^z dz/z\right| \le (e^2)(\sqrt{5}) = \sqrt{5}\,e^2$

(i) $$\max\left|\frac{\cos z}{z}\right| \le \frac{\max|\cos x\cosh y - i\sin x\sinh y|}{\min|z|} = \max\sqrt{\cos^2 x\cosh^2 y + \sin^2 x\sinh^2 y} \le \sqrt{\cosh^2 1 + \sinh^2 1}$$

and $L=\sqrt{5}$, so $\left|\int_C \frac{\cos z}{z}\,dz\right| \le \sqrt{\cosh^2 1+\sinh^2 1}\,\sqrt{5} \approx 4.34$

4. $$\left|\frac{\sin z}{z(z^2+9)}\right| = \frac{\sqrt{\sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y}}{5\,|(x+iy)-3i|\,|(x+iy)+3i|}$$

$$= \frac{1}{5}\sqrt{\frac{\sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y}{[x^2+(y-3)^2][x^2+(y+3)^2]}} \equiv f(\theta),$$ since $x=5\cos\theta$, $y=5\sin\theta$ on C.

Let us use maple to find $\max\limits_{0\le\theta\le 2\pi} f(\theta)$:

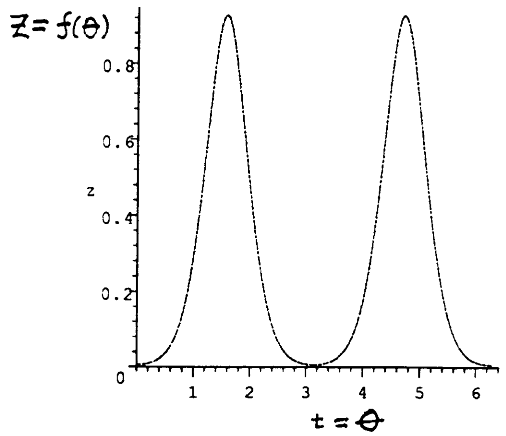
```
> x:=5*cos(t):
> y:=5*sin(t):
> f:=.2*sqrt((sin(x)^2*cosh(y)^2+cos(x)^2*sinh(y)^2)/((x^2+(y-3)^2)*
  (x^2+(y+3)^2)));
```

$$f := .2\sqrt{\frac{\sin(5\cos(t))^2\cosh(5\sin(t))^2+\cos(5\cos(t))^2\sinh(5\sin(t))^2}{(25\cos(t)^2+(5\sin(t)-3)^2)\,(25\cos(t)^2+(5\sin(t)+3)^2)}}$$

```
> g:=diff(f(t),t):
> fsolve(g=0,t,0..2*Pi);
```

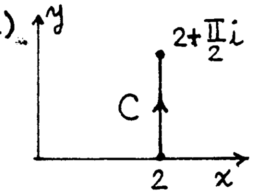
Unfortunately, maple does not give a response, so let us plot f:

```
> with(plots):
> implicitplot(z=f,t=0..2*Pi,z=0..10,grid=[200,200]);
```



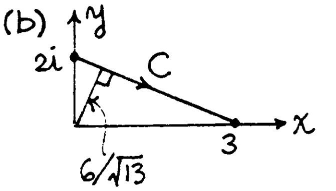
Perhaps if we go back to the fsolve command and narrow the search by changing the 0..2*Pi search interval to 1.4..1.6 (as seen from the plot), but, unfortunately, it still doesn't work. So let it suffice to observe, directly from the plot, that $\max f(\theta) \approx 0.72$

5. (a)



$$|e^z+1| = |e^2(\cos y + i\sin y)+1| = \sqrt{(e^2\cos y+1)^2+(e^2\sin y)^2}$$
$$= \sqrt{e^4+2e^2\cos y+1} \geq \sqrt{e^4+1} \quad \text{on } 0 \leq y \leq \pi/2$$
so $\left|\frac{1}{e^z+1}\right| \leq \frac{1}{\sqrt{e^4+1}}$. Also, $L=\pi/2$, so $\left|\int_C \frac{dz}{e^z+1}\right| \leq \frac{\pi}{2\sqrt{e^4+1}}$.

(b)



$$|\cos z| = |\cos(x+iy)| = |\cos x\cosh y - i\sin x\sinh y|$$
$$= \sqrt{(\cos^2 x\cosh^2 y + \sin^2 x\sinh^2 y)}$$
$$= \sqrt{[\cos^2 x\cosh^2 y + (1-\cos^2 x)\sinh^2 y]}$$
$$= \sqrt{[\cos^2 x(\cosh^2 y - \sinh^2 y) + \sinh^2 y]}$$
$$= \sqrt{(\cos^2 x + \sinh^2 y)} \leq \sqrt{1+\sinh^2 2} = \sqrt{\cosh^2 2} = \cosh 2$$

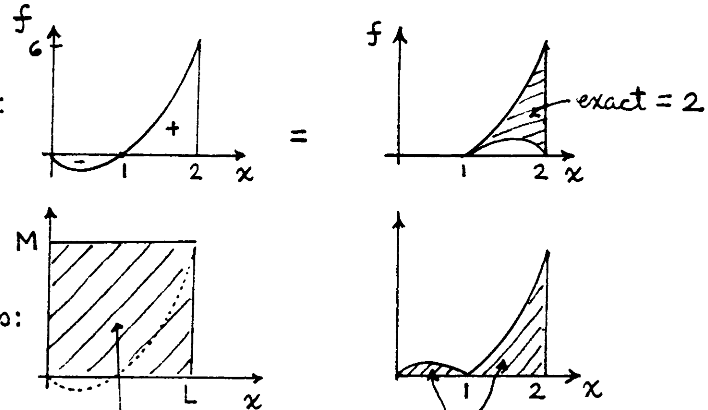
and $L = \sqrt{13}$, so
$$\left|\int_C \frac{\cos z}{z}\,dz\right| \leq \frac{\max|\cos z|}{\min|z|}\sqrt{13} \leq \frac{\cosh 2}{6/\sqrt{13}}\sqrt{13} = \frac{13}{6}\cosh 2.$$

6. A counterexample will suffice. For example, if C is a closed curve of length L then $\oint_C dz = 0$. Thus, with $m=1$, (6.1) gives $0 \geq L$, which is false.

7. (b)

Exact:

Bounds:



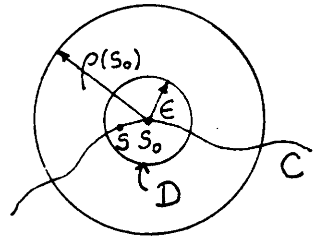
$ML = (6)(2) = 12$

$\int_C |f(z)||dz| = \int_0^1 (x-x^3)\,dx + \int_1^2 (x^3-x)\,dx$
$= 2.5$, which is sharper than the ML bound.

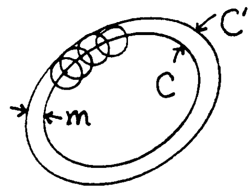
Section 23.3

1. Cauchy's theorem says, essentially, that if $f(z)$ is analytic inside C, then $\oint_C f(z)\,dz = 0$; it does <u>not</u> say that if $f(z)$ is not analytic inside C then $\oint_C f(z)\,dz \neq 0$. That is, the theorem does not contain a converse.

2. Cauchy's theorem, 23.3.1, calls for D to be simply connected, but the D in this exercise is not.

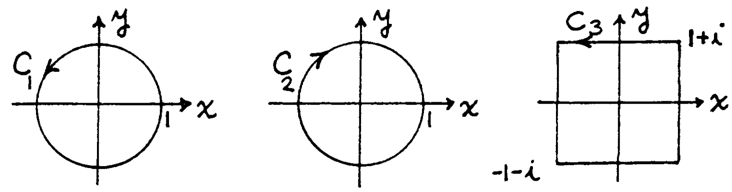


3. If f is analytic on C, then at each point on C there is a disk of radius $\rho(s)$ throughout which f is analytic, where s is arclength from some initial point on C to that point. We wish to show that $\rho(s)$ is continuous. If $|s-s_0|<\epsilon$, then s must fall in the disk D and, clearly, $\rho(s)$ is at least $\rho(s_0)-\epsilon$; i.e., $\rho(s)\geq\rho(s_0)-\epsilon$, or,
$$\rho(s_0)-\rho(s)\leq\epsilon.$$
Using the same argument, with s_0 and s switched, gives
$$\rho(s)-\rho(s_0)\leq\epsilon,$$
so $|\rho(s)-\rho(s_0)|\leq\epsilon$ for all s's such that $|s-s_0|<\delta(\epsilon)=\epsilon$, for $\epsilon>0$ arbitrarily small. Thus, $\rho(s)$ is a continuous function of s. Since C is rectifiable, $0\leq s\leq L<\infty$. It is known from the Calculus that if $\rho(s)$ is continuous on a closed interval then it has an absolute minimum (and maximum) on the interval. That minimum cannot be zero because then f would not be analytic at that point, as assumed. Let the minimum $\rho(s)$ be m. Then f is analytic in the domain D bounded by C' and hence $\oint_C f(z)\,dz=0$ by Cauchy's theorem.



Actually, we should have begun by distinguishing two cases: (i) If $f(z)$ is analytic for all z then $\rho(s)=\infty$ is not continuous – but no matter since if $f(z)$ is analytic for all z then Cauchy's theorem gives $\oint_C f(z)\,dz=0$. (ii) If $f(z)$ is not analytic for all z, then $\rho(s)<\infty$ for all s and the above argument applies.

4.



(a) $\oint_{C_1}\operatorname{Re}z\,dz=\oint_C x(dx+i\,dy)=\int_0^{2\pi}\cos\theta(-\sin\theta+i\cos\theta)\,d\theta=0+\pi i=\pi i$

(b) $\oint_{C_1}\operatorname{Im}z\,dz=\oint_{C_1}y(dx+i\,dy)=\int_0^{2\pi}\sin\theta(-\sin\theta+i\cos\theta)\,d\theta=-\pi+0i=-\pi$

(c) $\oint_{C_3}\operatorname{Im}z\,dz=\oint_{C_3}y\,dx+iy\,dy=\oint_{C_3}y\,dx+i\cancel{\frac{y^2}{2}}\Big|_0^0=\int_1^{-1}1\,dx+0+\int_{-1}^{1}-1\,dx+0=-4$

(d) $\oint_{C_3}\dfrac{dz}{z^2-3}=0$ by Cauchy's theorem since the singular points, $\pm\sqrt{3}$, are

(e) $\oint_{C_1} dz/z^4 = 0$ according to the "important little integral" result in Example 2.

(f) $\oint_{C_1} \frac{dz}{z(z-2)} = -\frac{1}{2}\oint_{C_1} \frac{dz}{z} + \frac{1}{2}\oint_{C_2}\frac{dz}{z-2} = -\frac{1}{2}(2\pi i) + 0 = -\pi i$

by Cauchy's theorem (for the 0); by "important little integral" (for the $2\pi i$)

(g) $\oint_{C_2} \frac{dz}{z(z+5)} = \frac{1}{5}\oint_{C_2}\frac{dz}{z} - \frac{1}{5}\oint_{C_2}\frac{dz}{z+5} = \frac{1}{5}(-2\pi i) - \frac{1}{5}(0) = -2\pi i/5$

Cauchy theorem (for the 0)

(h) $\oint_{C_1} e^{\sin z}\,dz = 0$ by Cauchy's theorem since $e^{\sin z}$ is analytic everywhere.

(i) $\oint_{C_2} \sin(\cos z)\,dz = 0$ by Cauchy's theorem since $\sin(\cos z)$ is analytic everywhere.

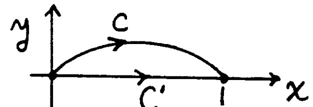
(j) $\oint_{C_3} \frac{dz}{|z|} = \oint_{C_3}\frac{dx+i\,dy}{\sqrt{x^2+y^2}} = \cancel{\int_{-1}^{1}\frac{i\,dy}{\sqrt{1+y^2}}} + \cancel{\int_{1}^{-1}\frac{dx}{\sqrt{x^2+1}}} + \cancel{\int_{1}^{-1}\frac{i\,dy}{\sqrt{1+y^2}}} + \cancel{\int_{-1}^{1}\frac{dx}{\sqrt{x^2+1}}} = 0$, not by Cauchy's theorem (which does not apply), but by "chance" cancellations.

(k) $\oint_{C_1} \bar{z}\,dz = \int_0^{2\pi} e^{-i\theta}(ie^{i\theta}d\theta) = 2\pi i$

(l) $\oint_{C_3} \bar{z}\,dz = \oint_{C_3}(x-iy)(dx+i\,dy) = \int_{-1}^{1}(1-iy)i\,dy + \int_{1}^{-1}(x-i)dx + \int_{1}^{-1}(-1-iy)i\,dy + \int_{-1}^{1}(x+i)dx = 8i$

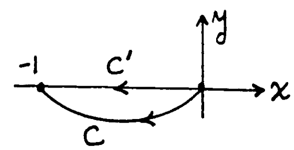
5. No, the conditions of Theorem 23.3.2 are not met because $\bar{z} = x-iy$ is not analytic (anywhere, in fact). In fact, Exercises 4(k) and 4(l), above, show that the results are different for the two paths.

6. (a) z^{20} is analytic everywhere, so we can deform the path to a straight line on the x-axis:

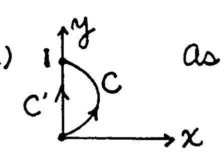


$$\int_C z^{20}dz = \int_{C'} z^{20}dz = \int_0^1 x^{20}dx = 1/21$$

(b) as in (a), $\int_C z^{20}dz = \int_0^{-1} x^{20}dx = -1/21$

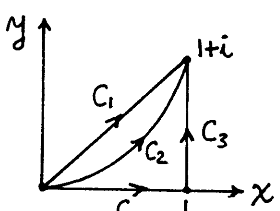


(c)



as in (a), $\int_C z^{20}dz = \int_{C'} z^{20}dz = \int_0^1 (iy)^{20}\,i\,dy = i/21$

7.



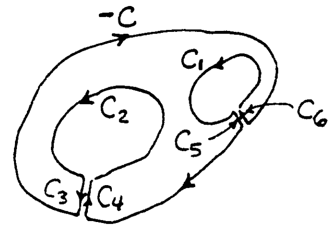
$\int_{C_2} \bar{z}\,dz = \int_{C_2}(x-iy)(dx+i\,dy) = \int_0^1 (x-ix^2)(1+2xi)dx = 1+\frac{i}{3}$,

$\int_{C_1} \bar{z}\,dz = \int_{C_1}(x-iy)(dx+i\,dy) = \int_0^1 (1-i)(1+i)x\,dx = 1$,

$\int_{C_3} \bar{z}\,dz = \int_{C_3}(x-iy)(dx+i\,dy) = \int_0^1 x\,dx + \int_0^1 (1-iy)i\,dy = 1+i$.

No violation (of course; the theorem is true and cannot be contradicted).

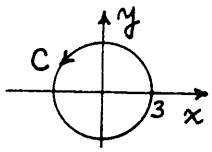
8. Introduce slits so that the slit domain D′ is simply connected (i.e., has no holes): The slit-contour integrals C_3 and C_4 cancel, as do C_5, C_6, so



$$\oint_{C_1+C_5+(-C)+C_4+C_2+C_3+C_6} f\,dz = 0 \text{ by Cauchy's theorem}$$

or, $\oint_{C_1} f\,dz + \cancel{\int_{C_5} f\,dz} + \oint_{-C} f\,dz + \cancel{\int_{C_4} f\,dz} + \oint_{C_2} f\,dz + \cancel{\int_{C_3} f\,dz} + \cancel{\int_{C_6} f\,dz} = 0$

or, $\oint_{C_1} f\,dz - \oint_C f\,dz + \oint_{C_2} f\,dz = 0$, so $\oint_C f\,dz = \oint_{C_1} f\,dz + \oint_{C_2} f\,dz$

9. 

Let us use partial fractions in each case.

(a) $\oint_C \frac{dz}{z(z-1)} = -\oint_C \frac{dz}{z} + \oint_C \frac{dz}{z-1} = -2\pi i + 2\pi i$ per (16) $= 0$

(b) $\oint_C \frac{dz}{z(z-5)} = -\frac{1}{5}\oint_C \frac{dz}{z} + \frac{1}{5}\oint \frac{dz}{z-5} = -\frac{1}{5}(2\pi i) + \frac{1}{5}(2\pi i)$ per (16) $= 0$

(c) $\oint_C \frac{z\,dz}{z^2+1} = \frac{1}{2}\oint_C \frac{dz}{z+i} + \frac{1}{2}\oint_C \frac{dz}{z-i} = \frac{1}{2}(2\pi i) + \frac{1}{2}(2\pi i)$ per (16) $= 2\pi i$

(d) $\oint_C \frac{z\,dz}{z^2-3z+2} = 2\oint_C \frac{dz}{z-2} - \oint_C \frac{dz}{z-1} = 2(2\pi i) - 1(2\pi i)$ per (16) $= 2\pi i$

(e) $\oint_C \frac{dz}{z^3(z^2-1)} = \oint_C \left(\frac{A}{z} + \frac{B}{z^2} + \frac{C}{z^3} + \frac{D}{z+1} + \frac{E}{z-1}\right) dz$

$= -\oint_C \frac{dz}{z} + 0\oint_C \frac{dz}{z^2} - \oint_C \frac{dz}{z^3} + \frac{1}{2}\oint_C \frac{dz}{z+1} + \frac{1}{2}\oint \frac{dz}{z-1}$

$= -2\pi i + 0 - 0 + \frac{1}{2}(2\pi i) + \frac{1}{2}(2\pi i)$ per (16)

$= 0$

(f)
$$\begin{array}{r|l} & z^2+z+1 \\ z-1 & z^3-1 \\ & \underline{z^3-z^2} \\ & z^2-1 \\ & \underline{z^2-z} \\ & z-1 \\ & z-1 \end{array}$$

and $z^2+z+1=0$ gives $z = \frac{-1\pm\sqrt{1-4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} \equiv z_\pm$ for short.

(or, we could get the 3 roots of $z^3-1=0$ as the 3 cube roots of 1)

Then, $\frac{z}{z^3-1} = \frac{z}{(z-1)(z-z_+)(z-z_-)} = \frac{A}{z-1} + \frac{B}{z-z_+} + \frac{C}{z-z_-}$ gives $A = \frac{1}{3}$,

$B = -(1+\sqrt{3}i)/6$, $C = -(1-\sqrt{3}i)/6$

Now, the roots $1, z_+, z_-$ are on the unit circle so all lie inside C. Thus,

$$\oint_C \frac{z\,dz}{z^3-1} = A\oint_C \frac{dz}{z-1} + B\oint_C \frac{dz}{z-z_+} + C\oint_C \frac{dz}{z-z_-}$$
$$= A(2\pi i) + B(2\pi i) + C(2\pi i) \quad \text{per (16)}$$
$$= \left(\frac{1}{3} - \frac{1+\sqrt{3}i}{6} - \frac{1-\sqrt{3}i}{6}\right)2\pi i = 0.$$

Section 23.4

1. If $F_1'(z) = f(z)$ and $F_2'(z) = f(z)$ then, with $G(z) \equiv F_1(z) - F_2(z)$, $G'(z) = f(z) - f(z) = 0$. If $G(z) = u + iv$ and $G'(z) = u_x + iv_x = v_y - iu_y = 0$ gives $u_x = v_x = u_y = v_y = 0$ so $u(x,y) = $ constant and $v(x,y) = $ constant. Thus, $F_1(z)$ and $F_2(z)$ differ by at most a constant (i.e., a complex constant).

2. (a) $z^2/2$, $z^2/2 + 6$, $z^2/2 + 1 - 4i$
   (b) $z^6/6$, $z^6/6 - 14i$, $z^6/6 + 5.73$
   (c) $(e^{2z} - z^2)/2$, $(e^{2z} - z^2)/2 - 14.3 - 2i$, $(e^{2z} - z^2)/2 + 10^5$
   (d) $\sin(z-2)$, $\sin(z-2) - 2 + i$, $\sin(z-2) - 4.13 + 6.75i$

3. (a) $\int_0^i z\,dz = z^2/2\Big|_0^i = -1/2$
   (d) $\int_i^0 \cos 3z\,dz = \frac{\sin 3z}{3}\Big|_i^0 = 0 - \frac{\sin 3i}{3} = -\frac{i}{3}\sinh 3$
   (e) $\int_{1-i}^{1+i} z e^z\,dz = (z-1)e^z\Big|_{1-i}^{1+i} = ie^{1+i} + ie^{1-i} = ie(e^i + e^{-i}) = 2ie\cos 1$
   (g) $\int_0^{3i} z e^{z^2}\,dz = \frac{1}{2}\int_0^{3i} e^{z^2}(2z\,dz) = \frac{1}{2}e^{z^2}\Big|_0^{3i} = \frac{1}{2}(e^{-9} - 1)$

4. $$I = \int_{1-i}^{1+i} \frac{dz}{z(z-1)} = \int_{1-i}^{1+i}\frac{dz}{z-1} - \int_{1-i}^{1+i}\frac{dz}{z} = \left(\log(z-1) - \log z\right)\Big|_{1-i}^{1+i}$$
$$= \log i - \log(1+i) - \log(-i) + \log(1-i)$$
$$= i\left(\frac{\pi}{2} + 2m\pi\right) - \left[\cancel{\ln\sqrt{2}} + i\left(\frac{\pi}{4} + 2n\pi\right)\right] - i\left(-\frac{\pi}{2} + 2p\pi\right) + \left[\cancel{\ln\sqrt{2}} + i\left(-\frac{\pi}{4} + 2q\pi\right)\right]$$
$$= i\left(\frac{\pi}{2} + 2r\pi\right) \quad \text{for } r = 0, \pm 1, \pm 2, \dots$$

5. Let $\tan^{-1} z \equiv t$. Then $z = \tan t = \frac{\sin t}{\cos t} = \frac{1}{i}\frac{e^{it} - e^{-it}}{e^{it} + e^{-it}}$ so

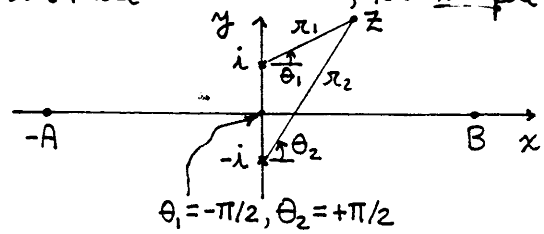
$iz(e^{it} + e^{-it}) = e^{it} - e^{-it}$, $iz(\varphi^2 + 1) = \varphi^2 - 1$ where $\varphi = e^{it}$, $\varphi^2 = \frac{1+iz}{1-iz} = \frac{i-z}{i+z}$,

$e^{i2t} = \frac{i-z}{i+z}$, $i2t = \log\left(\frac{i-z}{i+z}\right)$, $t = \tan^{-1} z = \frac{1}{2i}\log\frac{i-z}{i+z}$, as in Exercise 14(b) of Sec 21.4.

Thus, $I = \int_C \frac{dz}{z^2+1} = \lim_{\substack{A\to\infty \\ B\to\infty}} \tan^{-1} z \Big|_{-A}^{B}$

$= \lim_{\substack{A\to\infty \\ B\to\infty}} \frac{1}{2i} \log\left(\frac{i-z}{i+z}\right)\Big|_{-A}^{B}$

Use these branch cuts, for example:



$= \frac{1}{2i} \lim_{\substack{A\to\infty \\ B\to\infty}} \left[\log\left(\frac{z-i}{z+i}\right) + \log(-1)\right]\Big|_{-A}^{B}$

$= \frac{1}{2i} \lim_{\substack{A\to\infty \\ B\to\infty}} \left[\log(z-i) - \log(z+i) + \log(-1)\right]\Big|_{-A}^{B}$

$= \frac{1}{2i} \lim_{\substack{A\to\infty \\ B\to\infty}} \left[\ln r_1 + i\theta_1 - \ln r_2 - i\theta_2 + \log(-1)\right]\Big|_{-A}^{B}$

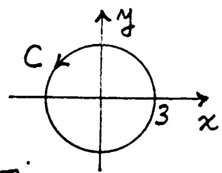
$= \frac{1}{2i} \lim_{\substack{A\to\infty \\ B\to\infty}} \left[\ln \frac{r_1}{r_2} + i(\theta_1 - \theta_2) + \log(-1)\right]\Big|_{-A}^{B}$

As $A, B \to \infty$, $r_1/r_2 \to 1$. At B, $\theta_1 - \theta_2 \to (0-0) = 0$. At A, $\theta_1 - \theta_2 \to (-\pi-\pi) = -2\pi$, and the constant $\log(-1)$ term cancels between the two limits, so

$$I = \frac{1}{2i}\Big((\overset{0}{\cancel{\ln 1}} + i0 + \cancel{\log(-1)}) - (\overset{0}{\cancel{\ln 1}} - i2\pi + \cancel{\log(-1)})\Big) = \frac{2\pi i}{2i} = \pi. \checkmark$$

Section 23.5

1. In each case C is

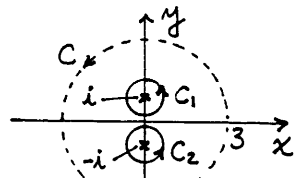


(a) $I = 2\pi i \cos z\big|_{z=0} = 2\pi i$

(b) $I = 2\pi i \sin z\big|_{z=0} = 0$.

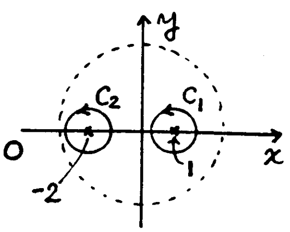
(c) $I = \oint_C \underbrace{\left(\frac{1}{z-5}\right)}_{\text{analytic within } C} \frac{dz}{z} = 2\pi i \left(\frac{1}{z-5}\right)\Big|_{z=0} = -2\pi i/5$

(d) $I = \oint_{C_1} \underbrace{\left(\frac{z^2-1}{z+i} e^z\right)}_{\text{analytic in } C_1} \frac{dz}{z-i} + \oint_{C_2} \underbrace{\left(\frac{z^2-1}{z-i} e^z\right)}_{\text{analytic in } C_2} \frac{dz}{z+i}$



$= 2\pi i \left(\frac{z^2-1}{z+i} e^z\right)\Big|_{z=i} + 2\pi i \left(\frac{z^2-1}{z-i} e^z\right)\Big|_{z=-i} = 2\pi i \left[\left(\frac{-2}{2i}\right)e^{i} + \left(\frac{-2}{-2i}\right)e^{-i}\right] = -4\pi i \sin 1$

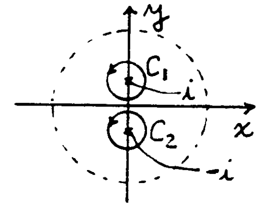
(e) $I = \oint_{C_1} \left(\frac{z+1}{(z+2)^3}\right) \frac{dz}{z-1} + \oint_{C_2} \left(\frac{z+1}{z-1}\right) \frac{dz}{(z+2)^3}$



$= 2\pi i \frac{z+1}{(z+2)^3}\Big|_{z=1} + \frac{2\pi i}{2!} \frac{d^2}{dz^2}\left(\frac{z+1}{z-1}\right)\Big|_{z=-2} = 2\pi i\left(\frac{2}{27} - \frac{2}{27}\right) = 0$

(f) $I = \oint_C \frac{e^{2z}}{z^5}\,dz = \frac{2\pi i}{4!}\frac{d^4}{dz^4}(e^{2z})\Big|_{z=0} = 4\pi i/3$

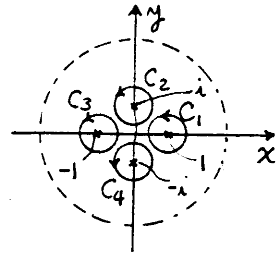
(g) $I = \oint_{C_1} \frac{\sinh 3z}{(z+i)^2}\frac{dz}{(z-i)^2} + \oint_{C_2}\frac{\sinh 3z}{(z-i)^2}\frac{dz}{(z+i)^2}$



$= 2\pi i \frac{d}{dz}\left(\frac{\sinh 3z}{(z+i)^2}\right)\Big|_{z=i} + 2\pi i\frac{d}{dz}\left(\frac{\sinh 3z}{(z-i)^2}\right)\Big|_{z=-i}$

$= 2\pi i(-3\cos 3+\sin 3)/4 + 2\pi i(-3\cos 3+\sin 3)/4 = \pi i(\sin 3 - 3\cos 3)$

(h) $z^4-1=0$ has the roots ± 1 and $\pm i$, so



$I = \oint_{C_1}\frac{(z+2)(z-1)}{z^4-1}\frac{dz}{z-1} + \oint_{C_2}\frac{(z+2)(z+1)}{z^4-1}\frac{dz}{z+1}$

$+ \oint_{C_3}\frac{(z+2)(z-i)}{z^4-1}\frac{dz}{z-i} + \oint_{C_4}\frac{(z+2)(z+i)}{z^4-1}\frac{dz}{z+i}$

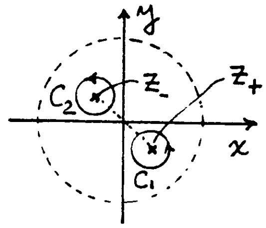
$= 2\pi i\left[\frac{(3)(1)}{4} + \frac{(1)(1)}{-4} + \frac{(i+2)(1)}{4i^3} + \frac{(-i+2)(1)}{4(-i)^3}\right] = 0$

(i) $\cos(z/2)=0$ at $z=\pm\pi, \pm 3\pi, \ldots$, which are outside of C, so

$I = \oint_C \frac{e^{z^2}}{\cos(z/2)}\frac{dz}{z} = 2\pi i\,\frac{e^{z^2}}{\cos(z/2)}\Big|_{z=0} = 2\pi i$

(j) $I = 0$

(k) $z^2+i=0$ at $z=\sqrt{-i} = \pm\left(\frac{1-i}{\sqrt{2}}\right) \equiv z_\pm$ so



$I = \oint_{C_1}\left(\frac{z^3}{z-z_-}\right)\frac{dz}{z-z_+} + \oint_{C_2}\left(\frac{z^3}{z-z_+}\right)\frac{dz}{z-z_-}$

$= 2\pi i\frac{z_+^3}{z_+-z_-} + 2\pi i\frac{z_-^3}{z_--z_+} = 2\pi i\left(\frac{z_+^3}{2z_+}+\frac{z_-^3}{2z_-}\right) = \pi i(z_+^2+z_-^2) = \pi i(-i-i) = 2\pi$

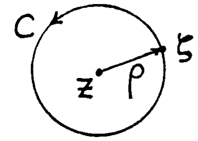
2. If $n=0,1,2,\ldots$ then $(z-a)^n$ is analytic for all z so $I = \oint_C (z-a)^n\,dz = 0$.
If $n=-1$, Cauchy's integral formula gives

$$I = \oint_C \frac{dz}{z-a} = 2\pi i(1) = 2\pi i,$$

and if $n=-2,-3,\ldots$ then the generalized Cauchy integral formula gives

$$I = \oint_C \frac{dz}{(z-a)^m} = \frac{2\pi i}{(m-1)!}\underbrace{\frac{d^{(m-1)}}{dz^{(m-1)}}(1)\Big|_{z=a}}_{0} = 0, \quad \text{where } m \text{ is } -n.$$

3. (a) (22) says $\oint_C \frac{f(\zeta)}{(\zeta-z)^{n+1}}\,d\zeta = \frac{2\pi i}{n!}f^{(n)}(z)$,



so "ML" bound gives

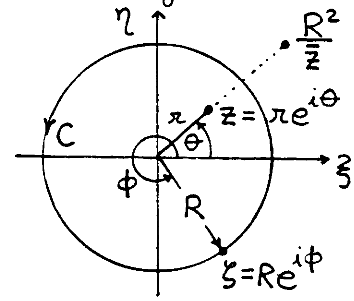
$$\left|\frac{2\pi i}{n!}f^{(n)}(z)\right| = \left|\oint_C \frac{f(\zeta)}{(\zeta-z)^{n+1}}\,d\zeta\right| \le \frac{M}{\rho^{n+1}}2\pi\rho, \text{ or, } |f^{(n)}(z)| \le \frac{n!M}{\rho^n}.$$

(b) Let $n=0$. Then (3.1) is $|f(z)| \le M$, which gives no information.
Let $n=1$. Then (3.1) is $|f'(z)| \le M/\rho$. Since ρ is arbitrarily large, it follows that $|f'(z)|$ is arbitrarily small. Thus $f'(z)=0$ — for each z, so $f(z) =$ constant.

(c) On imaginary axis $\sin z = \sin iy = i \sinh y$ is unbounded.

(d) Suppose $P(z)$ is nonzero everywhere. Then surely $f(z) = 1/P(z)$ is analytic for all z and is therefore at most a constant. But $1/P(z)$ is not a constant (unless $n=0$ of course), so $P(z)$ must not be nonzero everywhere.

4. (a) The term $-\frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - R^2/\bar{z}}\,d\zeta$ can be inserted in (4.1), to give (4.2), because it is 0 by Cauchy's theorem since $f(\zeta)$ is analytic inside and on C and $R^2/\bar{z} = (R^2/r)e^{i\theta}$ lies outside of C since $R^2/r = (R/r)R > R$. Then (4.2) gives



$$= d(Re^{i\phi}) = iRe^{i\phi}\,d\phi = i\zeta\,d\phi$$

$$f(z) = \frac{1}{2\pi i}\oint_C \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - R^2/\bar{z}}\right) f(\zeta)\,d\zeta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\zeta}{\zeta - z} - \frac{\bar{z}\zeta}{\zeta\bar{z} - R^2}\right) f(\zeta)\,d\phi$$

$$\frac{\bar{z}\zeta}{\zeta\bar{z} - R^2} = \frac{\bar{z}}{\bar{z} - \frac{R^2}{Re^{i\phi}}} = \frac{\bar{z}}{\bar{z} - Re^{-i\phi}} = \frac{\bar{z}}{\bar{z} - \bar{\zeta}}$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\zeta}{\zeta - z} + \frac{\bar{z}}{\bar{\zeta} - \bar{z}}\right) f(\zeta)\,d\phi$$

$$\frac{\zeta}{\zeta - z} + \frac{\bar{z}}{\bar{\zeta} - \bar{z}} = \frac{\zeta\bar{\zeta} - \cancel{\zeta\bar{z}} + \cancel{\bar{z}\zeta} - z\bar{z}}{(\zeta - z)(\bar{\zeta} - \bar{z})} = \frac{R^2 - r^2}{|\zeta - z|^2}$$

$$= \frac{R^2 - r^2}{\zeta\bar{\zeta} - \zeta\bar{z} - \bar{\zeta}z + z\bar{z}} = \frac{R^2 - r^2}{R^2 - 2Rr\cos(\phi - \theta) + r^2}$$

and equating real parts gives (4.4).

(b) The term $\oint_C \frac{f(\zeta)\,d\zeta}{\zeta - \bar{z}}$ can be inserted in (4.1), to give (4.5), because it is 0 by Cauchy's theorem since $f(\zeta)$ is analytic inside and on C and $\bar{z}$ lies outside of C. Next,

$$\left|\int_{\frown} \frac{f(\zeta)}{\zeta - z}\,d\zeta\right| \le \frac{M\pi R}{R - r} \sim \pi M \to 0 \text{ as } R \to \infty.$$

Similarly,

$$\left|\int_{\frown} \frac{f(\zeta)}{\zeta - \bar{z}}\,d\zeta\right| \le \frac{M\pi R}{R - r} \sim \pi M \to 0 \text{ as } R \to \infty.$$

Thus, letting $R \to \infty$ in (4.5) does give (4.6), namely,

$$f(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\left(\frac{1}{\xi - z} - \frac{1}{\xi - \bar{z}}\right) f(\xi)\,d\xi = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{z-\bar{z}}{(\xi - x - iy)(\xi - x + iy)}\, f(\xi)\,d\xi$$

or, $u(x,y) + i v(x,y) = \dfrac{1}{2\pi i}\displaystyle\int_{-\infty}^{\infty}\frac{i2y}{(\xi - x)^2 + y^2}\left[u(\xi,0) + i v(\xi,0)\right] d\xi$

and equating real parts gives (4.7).