Mid term solution

1.

$f(z)=\sin(z)=\sin(a+bi)=\sin(a)\cosh(b)+i\cos(a)\sinh(b)$ +2

$\because$ a、b independent and ~~$\sin(x)$, $x\in\mathbb{R}$~~

$$\begin{cases}\sin(x)\subseteq[-1,1], & x\in\mathbb{R}\\ \cosh(x)\subseteq[1,\infty), & x\in\mathbb{R}\end{cases}$$

so the real part of $\sin(z)\subseteq(-\infty,\infty)$
同理 imaginary part 也是

$\Rightarrow$ $\sin(z)$ 包含整個 complex domain +5

2、(a) 2分

Let $z\in\mathbb{C}$, $z=a+bi$ $a,b\in\mathbb{R}$

$\Rightarrow$ $f(z)=a$, in this function, real part

為 $f(x)=x$ 為 continuous，Imaginary part 為 $f(y)=0$ 也是 continuous，故 $f(z)=a=a+0i$ 也是 continuous

(b) 3分

$$\frac{\partial f(z)}{\partial x}=1\neq\frac{\partial f(z)}{\partial y}=0$$

故非 differentiable

3.

$$\int_C z^2dz=\int_0^{\frac{\pi}{2}}R^2e^{i3\phi}Ri\,d\phi$$ +2

$z=Re^{i\phi}$

$\frac{dz}{d\phi}=ie^{i\phi}\cdot R$

$$=iR^3\int_0^{\frac{\pi}{2}}e^{i3\phi}d\phi$$

$$=\frac{R^3}{3}\left(e^{\frac{3}{2}\pi i}-1\right)=\frac{R^3}{3}(-i-1)$$ +5

4. 10分 對圓周做積分之平均 = 圓心中心值

5. α^2 越大，溫差固定的話，u_t 越大

$\Rightarrow$ 擴散快

六、

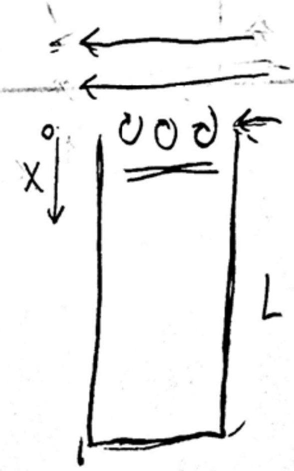
由於 $f(z)$ analytic $\Rightarrow$ 符合 Cauchy-Riemann equation

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \Rightarrow \begin{cases} v_y = 1 \\ v_x = 2 \end{cases}$$

$\Rightarrow$ $v(x,y) = \underline{2x+y+C}$ +10 ，又 $f(0) = v(0,0)i = 5i$

$C=5$ $v(x,y) = \underline{2x+y+5}$ +15

6.



vortex (the direct force to the air column)

$$C^2 \nabla^2 y = \frac{d^2 y}{dt^2} \quad , \; y \text{ is the acoustic wave}$$

$$\left. \frac{dy}{dx} \right|_{x=L} = 0$$

$$\left. \frac{d^2 y}{dx^2} \right|_{x=0} = \frac{F_{vortex}}{m_{effective}}$$

8,

$\frac{w+w^{-1}}{2}=3$ $\quad w^2+1=6w$ $\quad w^2-6w+1=0$ $\quad -\cosh^{-1}x=\ln\frac{1}{x\pm\sqrt{x^2-1}}$

$w=\frac{6\pm\sqrt{36-4}}{2}=3\pm2\sqrt{2}=e^{iz}$ $\quad -y=\ln(3\pm2\sqrt{2})=iz,\ y=-\ln(3\pm2\sqrt{2})i$

$e^{-y}\cos x=3\pm2\sqrt{2} \Rightarrow x=2n\pi$

$e^{-y}\sin x=0 \Rightarrow z=2n\pi-\ln(3\pm2\sqrt{2})i \quad n\in\mathbb{Z}$

-4

N. -2

7.

$R'' + \frac{R'}{r} + k^2 R = 0 \qquad R = \begin{cases} A + B\ln r, & k = 0 \\ E J_0(kr) + F Y_0(kr), & k \neq 0 \end{cases}$

$Z'' - k^2 Z = 0$

④ $\qquad Z \begin{cases} C + Dz, & k = 0 \\ G\cosh(kz) + H\sinh(kz), & k \neq 0 \end{cases}$

$\ln r \to -\infty \quad B = 0 \qquad Y_0(kr)$ diverge $\to F = 0$

② $\quad (C + Dz) + (G\cosh(kz) + H\sinh(kz))J_0(kr)$

$u(b, z) \longrightarrow C + Dz = 0$ and $J_0(kb) = 0$

let $k_n = \frac{z_n}{b} \quad n = 1, 2, 3, \ldots$

②

$u(r, z) \longrightarrow \sum_{n=1}^{\infty}\left[G_n\cosh(k_n z) + H_n(\sinh k_n z)\right]J_0(k_n r)$

③

$u(r, 0) \longrightarrow \sum_{n=1}^{\infty} G_n J_0(k_n r) = J_0\left(\frac{z_1}{b} r\right)$

$u(r, L) \longrightarrow \sum_{n=1}^{\infty}\left[G_n\cosh(k_n L) + H_n\sinh(k_n L)\right]J_0(k_n r) = J_0\left(\frac{z_2}{b} r\right)$

$\langle J_0(k_n r), J_0(\frac{z_1}{b} r)\rangle = 0 \quad \forall n \neq 1$

正交 may 15 $\quad \langle J_0(k_n r), J_0(\frac{z_2}{b} r)\rangle = 0 \quad \forall n \neq 2$

$\Rightarrow G_1 = 1 \quad G_n = 0 \ \forall n \neq 1$

$\sum_{n=1}^{\infty} J_0(k_n r) H_n \sinh(k_n L) = J_0\left(\frac{z_2 r}{b}\right) - J_0\left(\frac{z_1 r}{b}\right)\cosh\left(\frac{z_1}{b} L\right)$

$\langle \phi_1(r), \phi_2(r)\rangle = \int_0^b r\phi_1(r)\phi_2(r)\,dr$

$H_n\sinh\left(\frac{z_n}{b} L\right) = \begin{cases} -\cosh\left(\frac{z_1}{b} L\right), & n = 1 \\ 1, & n = 2 \\ 0, & \text{else} \end{cases}$

$\langle\,\rangle$ no specify $15 - 2 = 13$

[illegible]

[illegible] $\Rightarrow u(r, z) = J_0\left(\frac{z_1 r}{b}\right)\left(\cosh\left(\frac{z_1}{b} z\right) - \frac{\cosh\left(\frac{z_1}{b} L\right)}{\sinh\left(\frac{z_1}{b} L\right)}\sinh\left(\frac{z_1}{b} z\right)\right) + J_0\left(\frac{z_2}{b} r\right)\frac{\sinh\left(\frac{z_2}{b} z\right)}{\sinh\left(\frac{z_2}{b} L\right)}$

[illegible] $\to \langle\,\rangle \leftarrow$ [illegible]

10. $\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$ eq. (1)

Set $u = R(r)\phi(\theta)$

eq. (1) comes to

$$R''\phi + \frac{1}{r}R'\phi + \frac{1}{r^2}R\phi'' = 0$$

$$\Rightarrow r^2R''\phi + rR'\phi + R\phi'' = 0$$

$$\Rightarrow \frac{rR'' + r^2R''}{R} = \frac{-\phi''}{\phi} = k^2$$

$$R'(r) = \begin{cases} A + B\ln r, & k = 0 \\ Cr^k + Dr^{-k}, & k \neq 0 \end{cases}$$

$$\phi(\theta) = \begin{cases} E + F\theta, & k = 0 \\ G\cos k\theta + H\sin k\theta, & k \neq 0 \end{cases}$$

$$u(r,\theta) = (A + B\ln r)(E + F\theta) + (Cr^k + Dr^{-k})(G\cos k\theta + H\sin k\theta)$$

Because u is bounded, when $r \to \infty$

$\ln r \to \infty$, $r^k \to \infty \Rightarrow B = 0$, $C = 0$

$$u(r,\theta) = I + J\theta + D r^{-k}(G'\cos k\theta + H'\sin k\theta)$$

u 2π-periodic in $\theta \Rightarrow J = 0$, $k = n$

$$u(r,\theta) = I + \sum_{n=1}^{\infty} r^{-n}(G_n' \cos n\theta + H_n' \sin n\theta)$$

$$u(1,\theta) = f(\theta) = I + \sum_{n=1}^{\infty} (G_n' \cos n\theta + H_n' \sin n\theta) \quad ; \text{B.C.}$$

$f(\theta) = \cos\theta - \sin 2\theta$

$$\Rightarrow I = \frac{1}{2\pi}\int_{-\pi}^{\pi} (\cos\theta - \sin 2\theta)\, d\theta = 0$$

$$G_n' = \frac{1}{\pi}\int_{-\pi}^{\pi} (\cos\theta - \sin 2\theta) \cos n\theta \, d\theta$$

$$H_n' = \frac{1}{\pi}\int_{-\pi}^{\pi} (\cos\theta - \sin 2\theta) \sin n\theta \, d\theta$$

$$G_n' = \begin{cases} 1 & , n=1 \\ 0 & , n \neq 1 \end{cases} \qquad H_n' = \begin{cases} -1 & , n=2 \\ 0 & , n \neq 2 \end{cases}$$

$$u(r,\theta) = \sum_{n=1}^{\infty} r^{-n}(G_n' \cos n\theta + H_n' \sin n\theta)$$

$$= \frac{1}{r}\cos\theta - \frac{1}{r^2}\sin 2\theta$$