EE2020 Partial Differential Equations and Functions of a Complex Variable

Final Exam, Jan. 8, 2019

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Part I. 名詞定義理解題，共 20 分

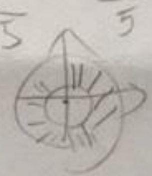
1. (10 points). Consider the region $D = \{z: 1 < |z| < \infty\} \subseteq \mathbb{C}$. (a) Is it open? (b) Is it closed? (c) Is it connected? (d) Is it simply connected? If any of the answers above is 'No', briefly explain the reasons.
2. (5 points). Continuing from above, suppose that $f(z) = z^{-1}$. Is the integral $\int f(z)dz$ in D path-independent? If so, prove it; if not, show a counter-example.
3. (5 points). The generalized Cauchy integral formula has the following form:

   $\oint_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(a)$. Describe the underlying assumptions for the equation to hold.

Part II. 複變計算題：共 50 分

$\frac{3}{5}$ $\frac{2}{5}$

4. (15 points). Consider $f(z) = \frac{1}{(z-2)(z+3)}$.



   (a) [3 pts] Assume that $f(z) = \frac{A}{z-2} + \frac{B}{z+3}$, find the coefficients A and B.

   (b) [7 pts] Find the Laurent series expansion $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ in the region $2 < |z| < 3$.

   (c) [5 pts] Find the Taylor series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$ near $z = 0$ and determine its region of convergence.

$\frac{1}{z}$

5. (15 points). Let $f(z) = z^{-4}e^{z}$, and C be the contour around the circle $|z| = 1$ in the counter-clockwise direction.

   z^{-4}

   $-4z^{-5}$

   $\dfrac{e^{\frac{1}{z}}}{\frac{1}{z^4}}$

   (a) Calculate $\oint_C f(z)\, dz$. **(b)** Calculate $\oint_C \frac{1}{f(z)} dz$.

   (c) Calculate $\oint_C f(1/z)\, dz$.

   $\dfrac{\frac{-1}{z^2} e^{\frac{1}{z}}}{\frac{-4}{z^5}}$

6. (10 points). Consider $f(z) = \frac{\sin z}{(z-\pi/4)^3}$.

   $z^3 e^{\frac{1}{z}}$

$u(t)$

(a) Find the Taylor expansion of $\sin z = \sum_{n=0}^{\infty} a_n \left(z - \frac{\pi}{4}\right)^n$.

(b) Find the residue of $f(z)$ at $z = \frac{\pi}{4}$ by this formula: $c_{-1} =$

$\lim_{z \to \pi/4} \frac{1}{(N-1)!} \frac{d^{N-1}}{dz^{N-1}} f(z) \left(z - \frac{\pi}{4}\right)^N$. Hint: You should choose the right N and the answer should be the same as a_2 in part (a).

$\frac{1}{s+2}$

$u(t)t$

$u(t)$

$\delta(t)e$

$-2u$

7. (10 points). Calculate the inverse Laplace transform $\mathcal{L}^{-1}\{F(s)\}$ where $F(s) = 1/(s+2)^2$ using the formula $f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} F(s) e^{st} ds$. You should choose an appropriate γ, show the appropriate integral contour on the complex plane with part of the contour being a semicircle, and argue that when the radius of the semicircle approaches infinity, the contour integral approaches $\mathcal{L}^{-1}\{F(s)\}$ if $t > 0$.

0

$e^{s_0 t}$ $s - s_0$ $\frac{-1}{(s+2)^2}$ $\frac{1}{s}$ $u(t)$ $u(t)$

Part III. PDE 複習題 (30 分)

$t u(t) e^{-2t}$ $\delta(t) e^{-2t} - 2u(t) e^{-2t}$

8. (20 points). Let $u(x) = 1,\ 0 < x < L$. Assume that $u(x) = \sum_{n=0}^{\infty} a_n \sin \frac{n\pi}{L} x$.
   - **(a)** Show that $a_n = 0$ if n is even.
   - **(b)** Calculate a_n for odd n.
   - **(c)** Describe a problem in PDE for which you will need to perform this expansion to find the solution $u = u(x,t)$.
   - **(d)** Find the solution of your problem in (c).

$t u(t) e^{-2t}$

s

9. (10 points). The expression of the Laplacian operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ in the polar coordinate is $\nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}$. If we are solving the Laplace equation $\nabla^2 u = 0$ in the region $r < R$ with boundary conditions $u(r, \theta) = f(\theta)$ on $r = R$, use the technique of separation of variables to break the PDE into two ordinary differential equations (ODEs). Remarks: just derive the ODEs. You don't need to show the solution here.