EE2020 Partial Differential Equations and Functions of a Complex Variable

2nd Midterm, Fall Semester, 2018

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本次考試滿分 =105，超過則登記為 100 分。

$z = Re^{i\phi}$ $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$

I. 簡答題: 共 45 分。

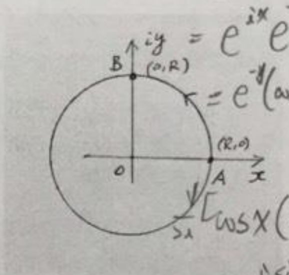
1. [5 pts] The range of $f(x) = \sin x$ is $[-1,1]$ if $x \in \mathbb{R}$. What is the range of $f(z) = \sin z$ when domain of definition is the entire complex plane $z \in \mathbb{C}$?

sin iy

2. [5 pts] Determine whether $f(z) = \mathrm{Re}\{z\}$ is (a) continuous, and (b) differentiable for all $z \in \mathbb{C}$.

$e^{iz} - e^{-iz}$
$= e^{i(x+yi)} - e^{-i(x+yi)}$
$= e^{ix}e^{-y} - e^{-ix}e^{y}$
$= e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)$

3. [5 pts] As the figure on the right shows, let $A = R + 0i$ and $B = 0 + Ri$ be two points on the circle centered at the origin with radius R. Let C be the counter-clockwise path along the circle that begins at A and ends at B.



Calculate $I = \int_C z^2\, dz$. (Hint: define $z = Re^{i\phi}$ and express dz in terms of dϕ.)

$[\cos x(e^{-y} - e^{y}) + i\sin x(e^{-y} + e^{y})]$
$\frac{-i}{2}\cos x(\sinh y)$
$+ \sin x\cosh y$
$\sin x\cosh y - i\cos x\sinh y$

4. [10 pts] Briefly explain what is the *average-value property* of the Laplace equation. Be concise and to-the-point.

5. [10 pts] 一維的 Heat equation $\alpha^2 u_{xx} = u_t$ 裡頭 α^2 越大，代表的物理意義是什麼？（請用 30 字以內解釋）

6. [10 pts] 葉問離開佛山到香港後，發現香港人十分洋化，都喝啤酒。好奇之下，喝完了生平第一罐玻璃瓶裝啤酒。事畢，感嘆歲不我予，對著空瓶吹氣，竟然瓶子發出了聲音。試約略寫出描述這個發聲現象的偏微分方程、及其邊界條件。圖示可。

II. 計算題，共 60 分

7. [15 pts] For any complex number z, denote $z = x + iy$, where x and y are the real and the imaginary part, respectively. Assume that $f(z) = u(x,y) + iv(x,y)$ is analytic for all $z \in \mathbb{C}$. If $u(x,y) = x - 2y$, determine $v(x,y)$ such that $f(0) = 5i$.

8. [15 pts] Find all $z \in \mathbb{C}$ such that $\cos z = 3$.

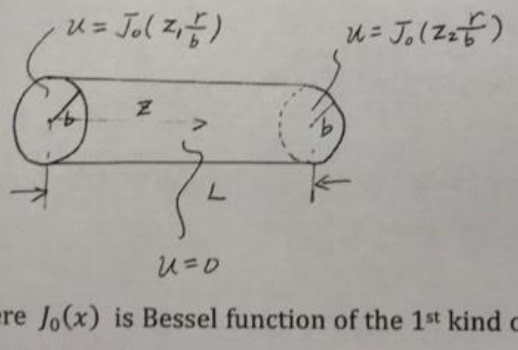
   (Hint: start from $\cos z = \frac{e^{iz}+e^{-iz}}{2} = 3$, let $w = e^{iz}$ and determine w first by solving a quadratic equation.)

9. (20 pts) Assume that $u = u(r,\theta,z)$ satisfies the Laplace equation

   independent of θ

   $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}\overset{\to 0}{\cancel{u_{\theta\theta}}} + u_{zz} = 0$, and that u does not vary against the angle θ, so $u_{\theta\theta} = 0$ for all (r,θ,z). Let us denote $u = u(r,z)$. Assume that we are given the boundary conditions:

$$\begin{cases} u(r,0) = J_0(z_1 r/b), & 0 \le r < b \\ u(r,L) = J_0(z_2 r/b), & 0 \le r < b \\ u(b,z) = 0, & 0 \le z \le L, \end{cases}$$



   where $J_0(x)$ is Bessel function of the 1st kind or order zero, and $z_1 > 0$ and $z_2 > 0$ are its first two zero crossings. Find the solution in terms of the Bessel function, cosh and sinh. Explicitly write down the coefficients.

10. (10 pts). Consider the Laplace equation $\nabla^2 u = 0$, where $u = u(r,\theta)$, in the region $1 < r < \infty$; i.e., this region is the plane with a circular hole of radius 1. Let the boundary condition be $u(1,\theta) = \cos\theta - \sin 2\theta$. Assume that u is bounded as $r \to \infty$. Find the solution.