107-1 Partial Differential Equations and Complex Variables 1st Midterm

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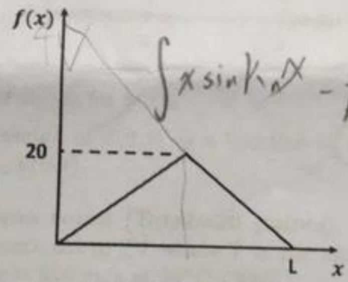
2018.10.23

1 Same heat equation, different boundary conditions. [**Total=30 points**]
Assume that we are solving the following heat equation for $u = u(x,t)$,

$$\alpha^2 u_{xx} = u_t,$$
$$u(0,t) = 0 = u(L,t),\ t > 0$$
$$u(x,0) = f(x),\ 0 < x < L \text{ as shown in the figure.}$$



$\frac{n\pi}{L} \times \frac{L}{5}$

$\int x \sin k_n x - \frac{1}{k_n} x \cos k_n x \Big|_0^{\frac{L}{5}} + \frac{1}{k_n}\int \cos k_n x$

$\frac{1}{k_n^2} \sin k_n x \Big|_0^{\frac{L}{5}}$

$\sin \frac{n\pi}{L} \times \frac{L}{5}$

$\left(\frac{L}{n\pi}\right)^2 \times \frac{4}{L}$

i [**5 pts**] What is the steady-state solution $u^{(\infty)}(x) = \lim_{t\to\infty} u(x,t)$?

ii [**15 pts**] Calculate the exact solution $u(x,t)$ for $t > 0$ in terms of Fourier Series. Find a general expression for the coefficients.

iii [**5 pts**] If the boundary condition changes to $u_x(0,t) = 0$ and without changing the initial condition, would the steady-state solution change? if not, does the temperature change faster or slower than in (ii)? why?

iv [**5 pts**] If the boundary condition changes to $u_x(0,t) = 0 = u_x(L,t)$, calculate the steady-state solution.

$\frac{4}{L}\int_{\frac{L}{5}}^{L} 40 \sin k_n$

$\frac{160}{L}$

$x \sin k_x \quad \frac{4}{L} \times \frac{40}{L} \times \frac{1}{k_n} \cos k_n$

$\frac{n\pi}{L} \times L$

$\cos n\pi -$

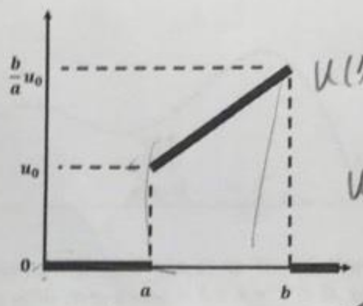
2 **Heat equation over the infinite line [Total=20 points].** Assume that we are now solving the heat equation on the entire real line $-\infty < x < \infty$,

$$\alpha^2 u_{xx} = u_t,\ t > 0$$

with the initial condition $u(x, 0) = f(x)$ defined as follows,

$$f(x) = \begin{cases} \frac{u_0}{a}x, & 0 < a \le x \le b \\ 0, & \text{otherwise.} \end{cases}$$

$\int f(x')\ t(x'-x)$



$u(x,t) = \int_{-\infty}^{+\infty} f(x') \frac{1}{2\alpha\sqrt{t}\sqrt{\pi}} e^{-\frac{(x-x')^2}{4\alpha^2 t}} dx'$

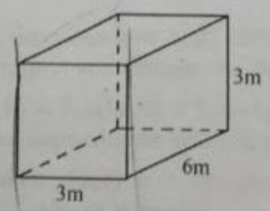
$u(0,t) = \int_{-\infty}^{+\infty} f(0)$

i **[10 pts]** Calculate $u(0, t)$ for $t > 0$.

ii **[10 pts]** Make a sketch of $u(0, t)$ as a function of t. Specifically, marks $u(0, 0)$ and $\lim_{t\to\infty} u(0, t)$.

3 **Resonance of a sauna room [Total=20 points].** The speed of sound is approximately proportional to $T^{\frac{1}{2}}$ where T is the absolute temperature in Kelvin. Assume that $c = 346$ m/s at 27°C (300K).

$V_s \propto T^{\frac{1}{2}}$



K

WX

$W_n = 2\pi f$

$\frac{1}{9} + \frac{1}{9} + \frac{1}{36}$

$\frac{4+4+1}{36}$

9

$\frac{1}{2} \times \pi \times$

i **[10 pts]** Assume now that $c = 360$ m/s, and you are standing in a sauna room of size $3 \times 6 \times 3$ m with the door closed. Calculate the lowest resonance frequency in Hz.

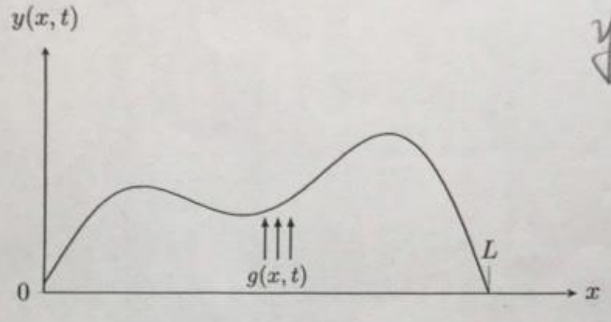
ii **[10 pts]** Here we assume $c^2(u_{xx} + u_{yy} + u_{zz}) = u_{tt}$ and $u(x, y, z) = U(x, y, z)T(t)$. Write down $U(x, y, z)$ corresponding to the lowest resonance frequency. Here you can assume that $u(x, y, z, t) = 0$ at the boundaries.

4 **Forced string with damping [Total=30 points].** Assume that we have a damped string subject to external force in the y-direction $g(x,t)$. The equation can be written as

$$c^2 y_{xx} = y_{tt} + a y_t - g(x,t),$$

where $0 < a < 2\pi\frac{c}{L}$. Further, we assume that the boundary conditions are $y(0,t) = 0 = y(L,t)$.

$y(x,0)$



i **[5 pts]** Briefly explain that, if $g(x,t)$ is "well-behaved", e.g. continuous and differentiable with respect to x for any $t > 0$, then we can write

$$g(x,t) = \sum_{n=1}^{\infty} G_n(t) \cdot \sin\left(n\frac{\pi}{L}x\right).$$

u

ii **[10 pts]** Assume that

$$y(x,t) = \sum_{n=1}^{\infty} b_n(t) \cdot \sin\left(n\frac{\pi}{L}x\right).$$

Find the O.D.E that describes how $b_n(t)$ change in-time (assuming that time-derivation and infinite summation can interchange).

iii **[10 pts]** Assume $y(x,0) = 0$, $y_t(x,0) = 0$, and $g(x,t) = g_0 \sin\left(n\frac{\pi}{L}x\right)\cos(\omega t)$, where $\omega > 0$. Find the approximate solution $y(x,t)$ when $t \gg 1/a$.

iv **[5 pts]** In general, when $g(x,t) = 0$, can $y(x,t)$ be written as $y(x,t) = F(x - ct) + G(x + ct)$? why or why not?

$s + \frac{a}{2}$

$\frac{1}{(s+\frac{a}{2})^2 + \sqrt{K_n^2 - \frac{a^2}{4}}}$