數位訊號處理概論
Introduction to Digital Signal Processing: HW3, Quiz3, Topic4 TA Review

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Remind

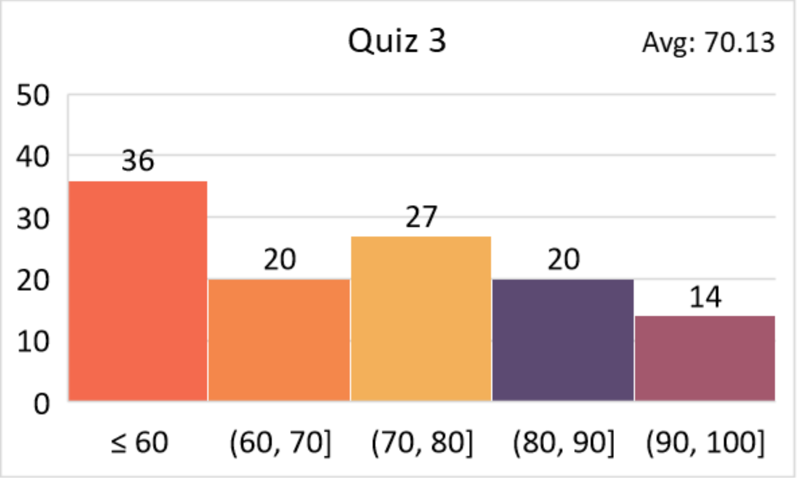
- HW 4:
  - due time: **2022/05/25 at 13:20**
  - A4 papers and hand in the homework in class
  - Please provide detailed answers or explanations in English

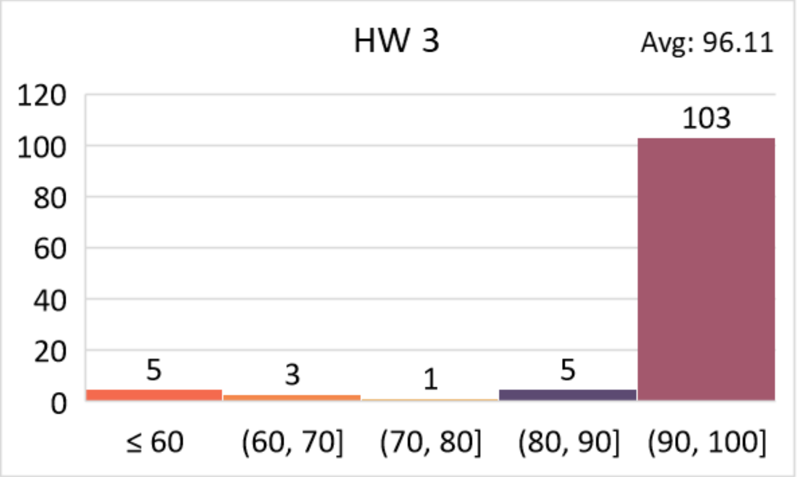
- Quiz 4:
  - time: **2022/05/25 13:20-15:10**
  - scope: everything we cover in topic 4
  - in eeclas 206 and 208
  - 1 A4 cheat sheet is allowed, and printed from everywhere is not accepted.

- 目前作業繳交及考試都依照往常方式進行, 如有更新會再通知同學

Quiz 3 & HW 3

- 調整分數方式:
  - Please take a photo and explain in as much detail as possible
  - **Email both TAs**
    - 簡婉軒: wschien@gap.nthu.edu.tw
    - Shreya: shreya@gapp.nthu.edu.tw
  - Please explain in English for HW q4-q7 and Quiz q3-q6
  - Due time: **2022/05/17 11:59 pm**





HW3-1

1. Let $x_c(t)$ be periodic with fundamental period T_0. It is sampled with $F_s = N/T_0$ to produce a periodic sequence $x[n]$ with fundamental period N. Show that the DTFS coefficients, $\tilde{c}_k$, of $x[n]$ are given by the aliasing of the CTFS coefficients, $\tilde{c}_k$, of $x_c(t)$ with respect to N, that is, (20%)

$$\tilde{c}_k = \sum_{\ell=-\infty}^{\infty} c_{k-\ell N}, k = 0, \pm 1, \pm 2, \cdots$$

Proof:

$$x_c(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi}{T_0}kt}$$

$$x_c(nT) = x_c(nT_0/N) = \sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi}{T_0}k\frac{nT0}{N}} = \sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi}{N}kn}$$

$$x[n] = \sum_{k=0}^{N-1} \tilde{c}_k e^{j\frac{2\pi}{N}kn}$$

Since we have $x[n] = x_c(nT)$, we require that

$$\sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi}{N}kn} = \sum_{k=0}^{N-1} \tilde{c}_k e^{j\frac{2\pi}{N}kn}$$

$$\sum_{k=0}^{N-1} \tilde{c}_k e^{j\frac{2\pi}{N}kn} = \sum_{k=0}^{N-1}\left(\sum_{\ell=-\infty}^{\infty} c_{k-\ell N}\right) e^{j\frac{2\pi}{N}kn} = \sum_{\ell=-\infty}^{\infty}\left(\sum_{k=0}^{N-1} c_{k-\ell N} e^{j\frac{2\pi}{N}kn}\right)$$

$$= \sum_{\ell=-\infty}^{\infty}\left(\sum_{k=0}^{N-1} c_{k-\ell N} e^{j\frac{2\pi}{N}kn} e^{j\frac{2\pi}{N}(-\ell N)}\right)$$

$$= \sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi}{N}kn}$$

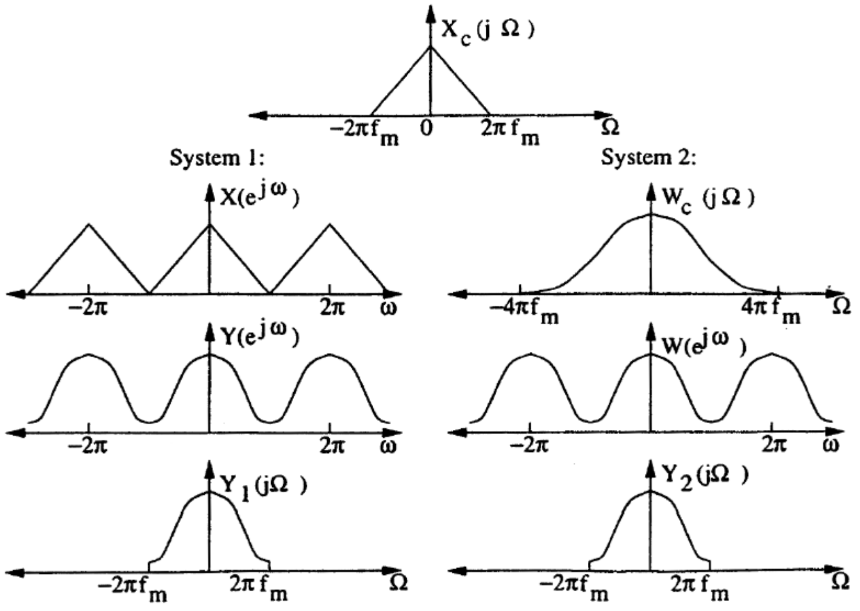
Hence, we prove that

$$\tilde{c}_k = \sum_{\ell=-\infty}^{\infty} c_{k-\ell N},\ k = 0, \pm 1, \pm 2, \ldots$$

HW3-2

2. Consider the two signal-processing systems in Figure q2-1, where the C/D and D/C converters are ideal. The mapping $g[x] = x^2$ represents a memoryless nonlinear device. (20%)
   (a) For the two systems in the figure, sketch the signal spectra at points 1, 2, and 3 when the sampling rate is selected to be $1/T = 2f_m$ Hz and $x_c(t)$ has the Fourier transform shown in Figure q2-2.
   (b) Is $y_1(t) = y_2(t)$? Explain your answer.
   (c) Is $y_1(t) = x^2(t)$? Explain your answer.
   (d) Consider System 1, and let $x(t) = A\,cos(30\pi t)$. Let the sampling rate be $1/\mathrm{T} = 40$ Hz. Is $y_1(t) = x_c^2(t)$? Explain why or why not.

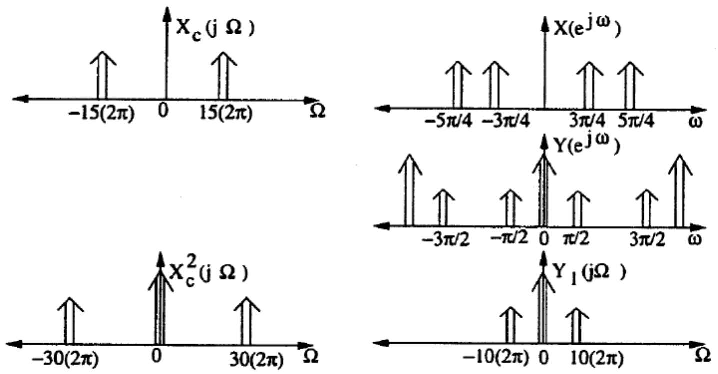
(a) Consider the following plots.



(b) Yes, Convolution is a linear process. Aliasing is a linear process. Periodic convolution is equivalent to convolution followed by aliasing.

(c) No, System 2 at Step 1 shows $X_c^2(j\Omega)$. This is clearly not $Y_1\ (j\Omega)$. $Y_1\ (j\Omega)$ is an aliased version of $X_c\ (j\Omega)$.

(d) No,



HW3-3

3. Consider a continuous-time signal (10%)

$$x_c(t) = 10 + 3sin(20\pi t + \frac{\pi}{3}) + 5cos(40\pi t).$$

It is sampled at $t = 0.01n$ to obtain $x[n]$, which is then applied to an ideal DAC to obtain another continuous-time signal $y_r(t)$.

(a) Determine $x[n]$ and graph its samples along with the signal $x_c(t)$ in one plot (choose few cycles of the $x_c(t)$ signal).

(b) Determine $y_r(t)$ as a sinusoidal signal. Graph and compare it with $x_c(t)$.

寫出算式和畫出圖, 缺一個-2

(a) Solution:

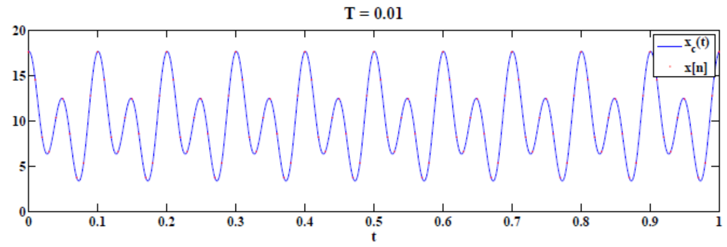
The sampled sequence is:

$$x[n] = x_c(0.01n) = 10 + 3\sin(0.2\pi n + \pi/3) + 5\cos(0.4\pi n)$$

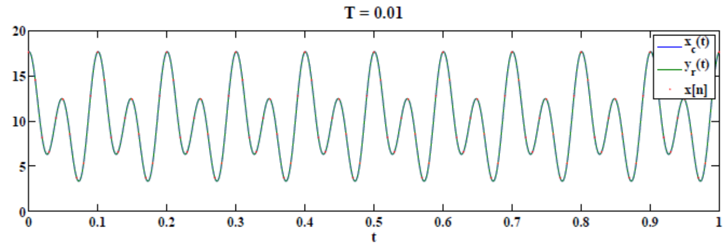
(b) Solution:

The reconstructed signal is: 需寫出$y_r(t) = ?$

$$y_r(t) = 10 + 3\sin(20\pi t + \pi/3) + 5\cos(40\pi t)$$



(a)



(b)

4. $h_c(t)$ Denotes the impulse response of an LTI-continues-time filter and $h_d[n]$ the impulse response of an LTI-discrete-time filter. (15%)

(a) Given

$$h_c(t) = \begin{cases} e^{-at}, t \geq 0 \\ 0, t < 0 \end{cases}$$

Where a is a positive real constant, determine the continues-time filter frequency response and sketch its magnitude.

(b) If $h_d[n] = Th_c(nT)$ with $h_c(t)$ in part (a), determine the discrete-time filter frequency response and sketch its magnitude.

(c) For a given value of a, determine, as the function of T, the minimum magnitude of the discrete-time filter frequency response.

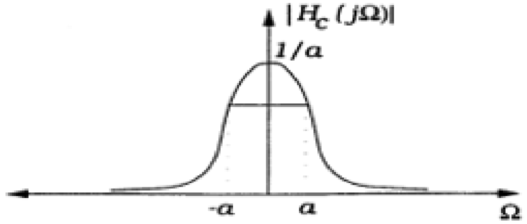
Answer

(a) The Fourier transform of the filter impulse response

$$\begin{aligned} H_c(j\Omega) &= \int_{-\infty}^{\infty} h_c(t)e^{-j\Omega t}\,dt \\ &= \int_{0}^{\infty} a^{-at}e^{-j\Omega t}\,dt \\ &= \frac{1}{a+j\Omega} \end{aligned}$$

So, we take the magnitude

$$|H_c(j\Omega)| = \left(\frac{1}{a^2+\Omega^2}\right)^{\frac{1}{2}}$$



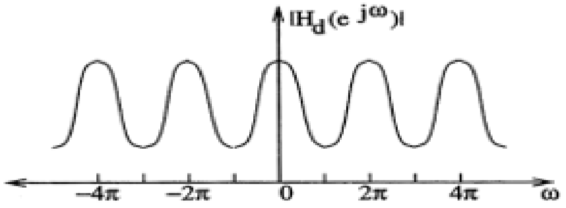
(b) Sampling the filter impulse response in (a), the discrete-time filter is described by

$$\begin{aligned} h_d[n] &= Te^{-anT}u[n] \\ H_d(e^{j\omega}) &= \sum_{n=0}^{\infty} Te^{-anT}e^{-j\omega n} \\ &= \frac{T}{1-e^{-aT}e^{-j\omega}} \end{aligned}$$

Taking the magnitude of this response

$$|H_d(e^{j\omega})| = \frac{T}{(1-2e^{-aT}\cos(\omega)+e^{-2aT})^{\frac{1}{2}}}.$$

Note that the frequency response of the discrete-time filter is periodic, with period 2π.



$$\begin{aligned} |H_d(e^{j\pi})| &= \frac{T}{(1+2e^{-aT}+e^{-2aT})^{\frac{1}{2}}} \\ &= \frac{T}{1+e^{-aT}}. \end{aligned}$$

5. In sampling a bandpass bandlimited signal $x_c(t)$ of bandwidth B Hz with integer band positioning, the ideal reconstruction filter $G_r(j2\pi F)$ of bandwidth B can reconstruct $X_c(t)$) exactly from its samples. Show that the impulse response of this ideal filter is given by the modulated bandlimited interpolation function in below equation. (10%)

$$g_r(t) = \frac{\sin(\pi B t)}{\pi B t} \cos(2\pi F_C t)$$

Where $F_C = (F_H - F_L)/2$

Answer

Solution:

$$g_r(t) = \frac{\sin(\pi B t)}{\pi B t} \cos(2\pi F_c t)$$

The spectra is:

$$G_r(j2\pi F) = \begin{cases} 1, & |F| \in [F_L, F_H] \\ 0, & \text{otherwise} \end{cases}$$

Baseband spectra is:

$$G(j2\pi F) = \begin{cases} T, & |F| \leq B/2 \\ 0, & |F| > B/2 \end{cases}$$

The continuous time signal is

$$g(t) = \frac{\sin(\pi t B)}{\pi t B}$$

We can conclude that

$$G_r(j2\pi F) = G[j2\pi(F + F_c)] + G[j2\pi(F - F_c)]$$

Hence,

$$\begin{aligned} g_r(t) &= \frac{\sin(\pi B t)}{\pi B t}(e^{-j2\pi F_c t} + e^{j2\pi F_c t}) \\ &= \frac{\sin(\pi B t)}{\pi B t} \cos(2\pi F_c t) \end{aligned}$$

where

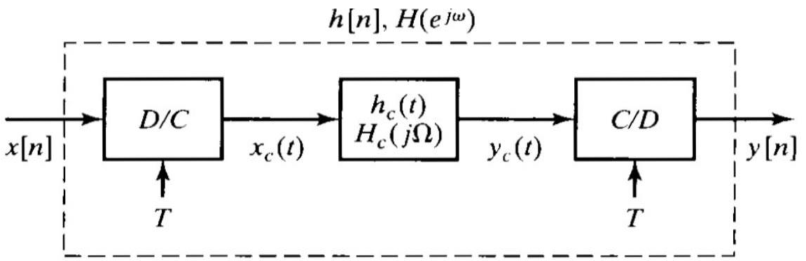
$$F_c = \frac{F_H + F_L}{2}, \qquad B = F_H - F_L$$

6. In the system shown in Figure $h_c(t) = \delta\left(t - \frac{T}{2}\right)$. (15%)

(a) Suppose the input $x[n] = \sin(\pi n/2)$ and T=10 find $y[n]$.

(b) Find $y[n]$ for the same x[n] as in part (a), but have T = 5.

(c) Explain how does the continuous-time LTI system $h_c(t)$ limit the range of the sampling period T that can be used without changing y[n] (in detail) ?



Answer (a)

$$
\begin{aligned}
x_c(t) &= \sin(\frac{\pi}{20}t) \\
y_c(t) &= \sin(\frac{\pi}{20}(t-5)) \\
&= \sin(\frac{\pi}{20}t - \frac{\pi}{4}) \\
y[n] &= \sin(\frac{\pi n}{2} - \frac{\pi}{4})
\end{aligned}
$$

(b) We get the same result as before:

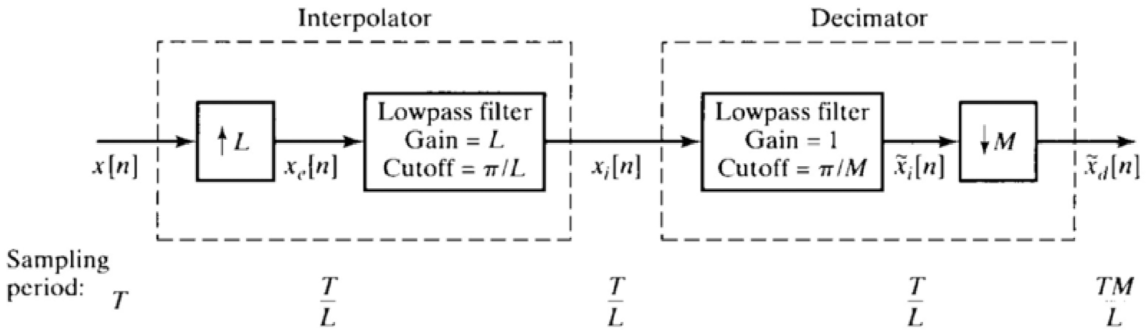
$$
\begin{aligned}
x_c(t) &= \sin(\frac{\pi}{10}t) \\
y_c(t) &= \sin(\frac{\pi}{10}(t-2.5)) \\
&= \sin(\frac{\pi}{10}t - \frac{\pi}{4}) \\
y[n] &= \sin(\frac{\pi n}{2} - \frac{\pi}{4})
\end{aligned}
$$

(c) The sampling period T is not limited by the continuous time system $h_c(t)$.

7. For the below input signal x[n], the upsampling and downsamplig rates L and M for the system in below figure. Determine the corresponding output $\widetilde{x_d}[n]$. (10%)

a) $x[n] = \sin\left(\frac{2\pi n}{3}\right)/\pi n$, L=4, M=3

b) $x[n] = \sin\left(\frac{3\pi n}{4}\right)$, L=3, M=5



Note: for (b), after upasampling and downsampling u will get sin(5pi*n/4) and there will nothing remains in the range so the answer can be 0, but according to textbook the answer can also be –sin(3pi*n/4).

Answer

(a) In the frequency domain,

$$X(e^{j\omega}) = \begin{cases} 1, & |\omega| < 2\pi/3 \\ 0, & 2\pi/3 < |\omega| < \pi \end{cases}$$

After the sampling rate change,

$$\tilde{X}_d(e^{j\omega}) = \begin{cases} 4/3, & |\omega| < \pi/2 \\ 0, & \pi/2 < |\omega| < \pi \end{cases},$$

which leads to

$$x[n] = \frac{4}{3}\frac{\sin(\pi n/2)}{\pi n}$$

(b) Upsampling by 3 and low-pass filtering $x[n] = \sin(3\pi n/4)$ results in $\sin(\pi n/4)$. Downsampling by 5 gives us $\tilde{x}_d[n] = \sin(5\pi n/4) = -\sin(3\pi n/4)$.

Quiz3-1

1. **(25 pts.) True or False.** Write down True or False for each question, and give a short proof to the one(s) you think is true, otherwise, point out the reason why the statement is wrong or providing a counter example. [5 pt for each]
   (a) The error due to under-sampling a continuous signal can be avoided by simply amplifying the signal.
   (b) Let $x[n]$ and $y[n]$ be discrete-time signals. $X(\omega)$ and $Y(\omega)$ is their Fourier transform, respectively. If $y[n] = x[3n]$, then $Y(\omega) = X(\omega/3)$.
   (c) Let $x[n]$ and $y[n]$ be discrete-time signals. $X(\omega)$ and $Y(\omega)$ is their Fourier transform, respectively. Assume that $y[n] = x[n/2]$ when n is even and $y[n] = 0$ when n is odd. Then, $Y(\omega) = X(2\omega)$.
   (d) All practical continuous-time signals are time-limited, and cannot be strictly bandlimited.
   (e) If a signal $x_c(t) = 3 + 2sin(16\pi t) + 10cos(24\pi t)$ is sampled at a rate of $F_s = 20$ Hz to obtain the discrete-time signal $x[n]$. $x_c(t)$ can be recovered from $x[n]$.

(a) False under-sampling produces aliases which when reconstructed fall within the Nyquist bandwidth. Amplification only acts to increase both wanted signal and the distortion, proper bandwidth filtering is the only answer

(b) False because it may cause aliasing when down sampling

沒寫proof or reason: -2

(c) True $Y(\omega) = \sum_{-\infty}^{\infty} x\left[\frac{n}{2}\right] e^{-j\omega n} = \sum_{-\infty}^{\infty} x[k] e^{-2j\omega k} = X(2\omega)$

(d) True (in Apr13-2022.pdf p20)

(e) False

The spectra of the continuous signal $x_c(t)$ is:

$$X_c(j2\pi F) = \begin{cases} 3, & F = 0 \\ \frac{1}{j}, & F = 8 \\ -\frac{1}{j}, & F = -8 \\ 5, & |F| = 12 \\ 0, & \text{otherwise} \end{cases}$$

The spectra of the sampled sequence $x[n]$ is:

$$X(e^{j\omega})\big|_{\omega = 2\pi F/F_s} = F_s \sum_{k=-\infty}^{\infty} X_c[j2\pi(F - kF_s)]$$

The signal $x_c(t)$ can be recovered from $x[n]$ if the sampling rate is (a) F_s = 30 Hz, and can NOT be recovered if the sampling rate is (b) F_s = 20 Hz, (c) F_s = 15 Hz.

Quiz3-2

2. **(20 pts.)** Write abbreviated final answers for following questions.

   (a) Consider a continuous-time signal

   $$x_c(t) = 3cos(2\pi F_1 t + 45^{\circ}) + 3sin(2\pi F_2 t).$$

   (ref to Apr14-2022.pdf p11 and Apr21-2022-TA-Review.pdf p8)

   It is sampled at $t = 0.001n$ to obtain $x[n]$, which is then applied to an ideal DAC to obtain another continuous-time signal $y_r(t)$. For $F_1 = 300$ Hz and $F_2 = 700$ Hz, determine $x[n]$ and determine $y_r(t)$ for the above $x[n]$ as a sinusoidal signal. [10 pt]

   (b) Consider a continuous-time signal

   $$x_c(t) = 10 + 3sin(20\pi t + \pi/3) + 5cos(40\pi t).$$

   (ref to hw3-3)

   It is sampled at $t = 0.05n$ to obtain $x[n]$, which is then applied to an ideal DAC to obtain another continuous-time signal $y_r(t)$. Determine $x[n]$ and determine $y_r(t)$ as a sinusoidal signal. [10 pt]

一個算式5分

(a)

$x[n] = 3cos(0.6\pi n + \pi/4) + 3sin(1.4\pi n)$

$y_r(t) = 3cos(600\pi t + \pi/4) - 3sin(600\pi t)$

(b)

$x_c(t) = 10 + 3sin(\pi n + \pi/3) + 5cos(2\pi n)$

$x_c(t) = 15 + 3sin(20\pi t + \pi/3)$

Quiz3-3

3. **(15 pts.)** An 8-bit ADC has an input analog range of ±5 volts. The analog input signal is

$$x_c(t) = 2\,cos(200\pi t) + 3\,sin(500\pi t).$$

The converter supplies data to a rate of 2048 bits/s. The computer, without processing, supplies these data to an ideal DAC to form the reconstructed signal $y_c(t)$. Determine:

(a) The quantizer resolution (or step). [5 pt.]

(b) The sampling rate, and folding frequency. [5 pt.]

(c) The Nyquist rate, and the reconstructed signal $y_c(t)$. [5 pt.]

Answer

Ref Concepts from Apr13-2022 p21- p25 & Apr14-2022.pdf p19

(a) Solution:

The quantizer resolution is:

$$\frac{10\text{v}}{2^8} = 0.0390625\text{v}$$

(b) Solution:

The sampling rate is:

$$F_s = \frac{2^{11}}{2^3} = 2^8 \text{sam/sec}$$

The folding frequency is $F_s/2 = 2^7$.

(c) Solution:

The Nyquist rate is 500.

The reconstructed signal $y_c(t)$ is:

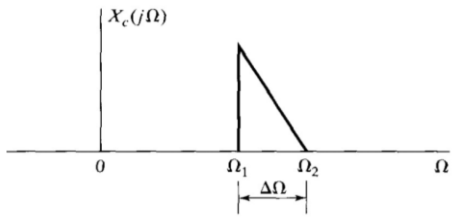
$$y_c(t) = 2\cos(200\pi t) - 3\sin(12\pi t)$$

Quiz3-4

4. **(10 pts.)** A complex-value continuous-time signal $x_c(t)$ has the Fourier transform shown in Figure q4, where $(\Omega_2 - \Omega_1) = \Delta\Omega$. This signal is sampled to produce the sequence $x[n] = x_c(nT)$.

   (a) Sketch the Fourier transform $X(e^{j\omega})$ of the sequence $x[n]$ for $T = \frac{\pi}{\Omega_2}$. [5 pt.]

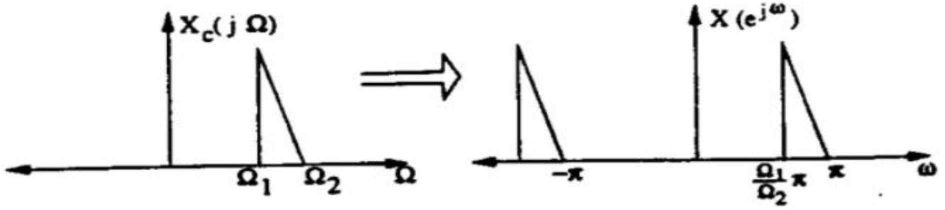
   (b) What is the lowest sampling frequency that can be used without incurring any aliasing distortion i.e., so that $x_c(t)$ can be recovered from $x[n]$? [5 pt]



Answer

ref concept from Apr6-2022.pdf p13

(a) Keeping in mind that after sampling, $\omega = \Omega T$, the Fourier transform of $x[n]$ is



(b) A straight-forward application of the Nyquist criterion would lead to an incorrect conclusion that the sampling rate is at least twice the maximum frequency of $x_c(t)$, or $2\Omega_2$. However, since the spectrum is bandpass, we only need to ensure that the replications in frequency which occur as a result of sampling do not overlap with the original. (See the following figure of $X_s(j\Omega)$.) Therefore, we only need to ensure

$$\Omega_2 - \frac{2\pi}{T} < \Omega_1 \Longrightarrow T < \frac{2\pi}{\Delta\Omega}$$

Quiz3-5

5. **(7 pts.)** A sinusoidal signal $s_c(x,y) = 3\,cos(2.4\pi x + 2.6\pi y)$ is sampled at $\left(F_{s_x}, F_{s_y}\right)$ frequency to obtain the image $s[m,n]$. An ideal reconstruction is used on $f[m,n]$ to obtain the analog sign $s_r(x,y)$. If $F_{s_x} = 2$ sam/meter and $F_{s_y} = 3$ sam/meter, determine $s[m,n]$ and $s_r(x,y)$.

Ref Apr14-2022.pdf p12 to p19

Answer

$$s_c(x,y) = 3\cos(2.4\pi x + 2.6\pi y) = 3\cos(2.4\pi x)\cos(2.6\pi y) - 3\sin(2.4\pi x)\sin(2.6\pi y)$$

$$s[m,n] = 3\cos(0.8\pi m + 1.3\pi n)$$

$$s_r(x,y) = 3\cos(1.6\pi x - 2.6\pi y)$$

6. **(23 pts.)** Show and explain the concepts in detail.

(a) Show that the sampler is a memoryless, linear, time-varying system. [9 pt.]

(b) What is the Nyquist rate and prove why the signal needs to satisfy the Nyquist rate while sampling (detail answer with some plot)? [7 pt.]

(c) Using Parseval's theorem, explain why the amplitude of the Fourier transform changes during downsampling but not during upsampling. [7 pt.]

Quiz3-6

ref Apr6-2022.pdf p13

Answer

a) Proof:

The sampler is:

$$x_{\text{out}}(t) = x_{\text{in}}(nT); \quad nT \le t < (n+1)T, \ \forall n$$

(i) Memoryless. The current system value is only related to the current time index and is not affected by previous system values. Hence, the sampler is memoryless.

(ii) Linearity.

$$a_1 \cdot x_{\text{in1}}(nT) + a_2 \cdot x_{\text{in2}}(nT) = a_1 \cdot x_{\text{out1}}(t) + a_2 \cdot x_{\text{out2}}(t)$$

The S&H system follows the superposition property, and hence is a linear system.

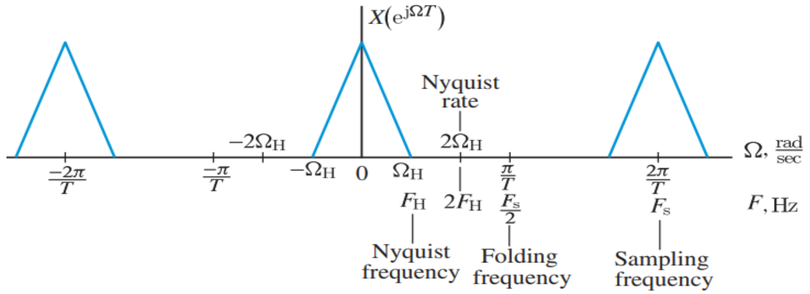
(iii) Time-variance.

$$x_{\text{out}}(t-\tau) \neq x_{\text{out}}(t), \text{if } t-\tau \notin [nT, (n+1)T]$$

Hence, the system is time-varying.

b) The highest frequency F_H, in Hz, present in a bandlimited signal $x_c(t)$ is called the Nyquist frequency. The minimum sampling frequency required to avoid overlapping bands is $2F_H$, which is called the Nyquist rate.

A continuous-time signal $x_c(t)$ with frequencies no higher than F_H can be reconstructed exactly from its samples $x[n] = x_c(nT)$, if the samples are taken at a rate $Fs = 1/T$ that is greater than the Nyquist rate $2F_H$. The spectrum of $x[n]$ is obtained by scaling the spectrum of $x_c(t)$ by Fs and putting copies at all integer multiples of Fs.



c) **Parseval's Theorem:**

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

When we upsample, the added samples are zeros, so the upsampled signal $x_u[n]$ has the same energy as the original $x[n]$:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=-\infty}^{\infty} |x_u[n]|^2,$$

and by Parseval's theorem:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X_u(e^{j\omega})|^2 d\omega.$$

Hence the amplitude of the Fourier transform does not change.

When we downsample, the downsampled signal $x_d[n]$ has less energy than the original $x[n]$ because some samples are discarded. Hence the amplitude of the Fourier transform will change after downsampling.

Some topics to consider in Quiz 4 preparation:

- Circular Convolution
- DFT conversion
- DFS conversion
- DFT Properties
- Different Windowing methods