數位訊號處理概論
Introduction to Digital Signal Processing: HW2, Quiz2, Topic3 TA Review

TA: 簡婉軒, Shreya

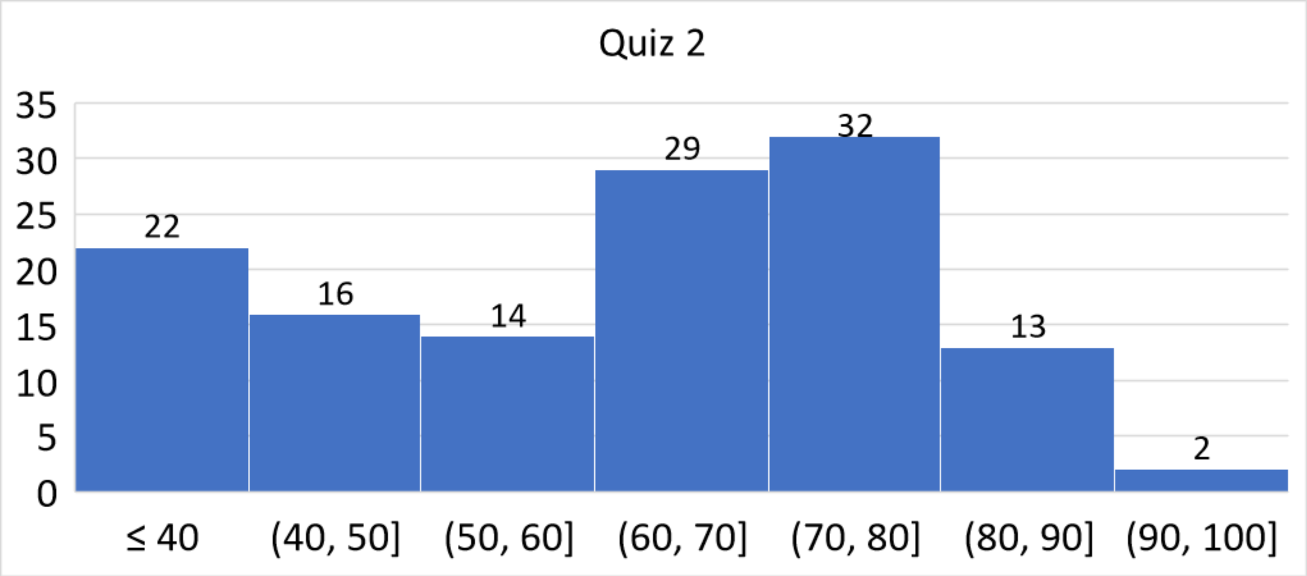
Department of Electrical Engineering

National Tsing Hua University

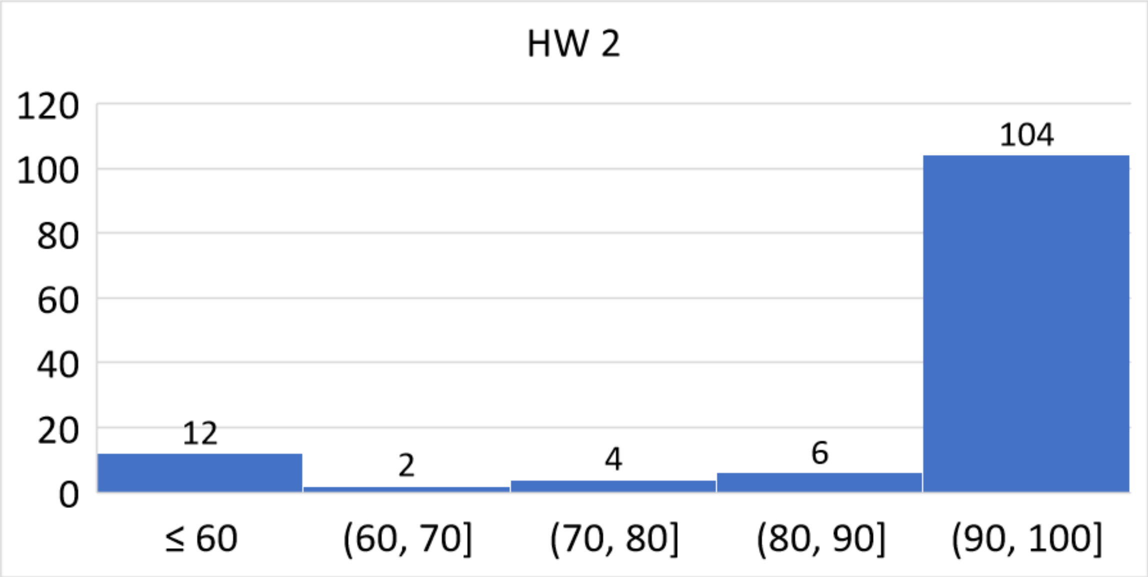
Remind

- HW 3:
  - due time: **2022/04/27 at 13:20**
  - A4 papers and hand in the homework in class
  - Please provide detailed answers or explanations in English

- Quiz 3:
  - time: **2022/04/27 13:20-15:10**
  - scope: everything we cover in topic 3
  - in eeclas 206 and 208
  - 1 A4 cheat sheet is allowed, and printed from everywhere is not accepted.



Avg: 64.89



Avg: 90.11

HW2-1

1. Prove the orthogonality property and use it to prove Parseval's relation. (15%)

   - Orthogonality property:

$$\int_{T_0} s_k(t)s_m^*(t)\,dt = \int_{T_0} e^{jk\Omega_0 t}\,e^{-im\Omega_0 t}dt = \begin{cases} T_0, & k = m \\ 0, & k \neq m \end{cases}$$

   - Parseval's relation:

$$P_{av} = \frac{1}{T_0}\int_{T_0} |x(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |c_k|^2$$

where c_k are Fourier series coefficients

$$P_{\text{av}} = \frac{1}{T_0}\int_{T_0} x(t)x^*(t)dt = \frac{1}{T_0}\int_{T_0}\left(\sum_k c_k \mathrm{e}^{\mathrm{j}k\Omega_0 t}\right)\left(\sum_m c_m \mathrm{e}^{\mathrm{j}m\Omega_0 t}\right)^* dt$$

$$= \frac{1}{T_0}\sum_k\sum_m c_k c_m^* \int_{T_0} \mathrm{e}^{\mathrm{j}k\Omega_0 t}\mathrm{e}^{-\mathrm{j}m\Omega_0 t}$$

$$\int_{T_0} \mathrm{e}^{\mathrm{j}k\Omega_0 t}\mathrm{e}^{-\mathrm{j}m\Omega_0 t} = \begin{cases} 0, & k \neq m \\ T_0, & k = m \end{cases}$$

$$P_{\text{av}} = \frac{1}{T_0}\sum_k T_0 \cdot c_k \cdot c_k^* = \sum_{k=-\infty}^{\infty} |c_k|^2$$

HW2-2

2. Let $x[n]$ and $y[n]$ denote complex sequences and $X(e^{j\omega})$ and $Y(e^{j\omega})$ are their respective Fourier transforms. (15%)
   (a) By using the appropriate operational properties of the DTFT, determine the sequence whose Fourier transform is $X(e^{j\omega})Y^*(e^{j\omega})$, in terms of $x[n]$ and $y[n]$.
   (b) In question 1, we prove Parseval's relation by orthogonal property. Here, try to **proof Parseval's theorem using the result in part (a)**,

$$\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$$

   which is a more general form of Parseval's relation. (Proof by orthogonal property is not accepted.)
   (c) Using Parseval's theorem, determine the numerical value of the sum

$$\sum_{n=-\infty}^{\infty} \frac{\sin(\pi n/4)}{2\pi n}\frac{\sin(\pi n/6)}{5\pi n}$$

(a) The Fourier transform of $y^*[-n]$ is $Y^*(e^{j\omega})$, and $X(e^{j\omega})Y(e^{j\omega})$ forms a transform pair with $x[n]*y[n]$. So

$$G(e^{j\omega}) = X(e^{j\omega})Y^*(e^{j\omega})$$

and

$$g[n] = x[n]*y^*[-n]$$

form a transform pair.

(b)

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})e^{j\omega n}d\omega &= \sum_{n=-\infty}^{\infty}(x[n]*y^*[-n])\,e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} x[k]y^*[k-n]e^{-j\omega n}\end{aligned}$$

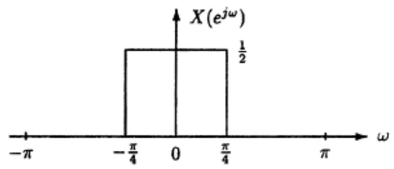
for $n=0$:

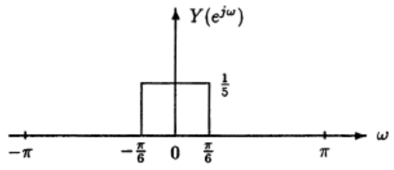
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(e^{j\omega})Y^*(e^{j\omega})d\omega = \sum_{k=-\infty}^{\infty} x[k]y^*[k]$$

(c) Using the result from part (b):

$$\begin{aligned}x[n] &= \frac{\sin(\pi n/4)}{2\pi n} \\ y^*[n] &= \frac{\sin(\pi n/6)}{5\pi n}\end{aligned}$$

We recognize each sequence to be a pulse in the frequency domain:





Substituting into Eq. (P2.77-1):

$$\begin{aligned}\sum_{n=-\infty}^{\infty} x[n]y^*[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega \\ &= \frac{1}{2\pi}\left[\left(\frac{1}{2}\right)\left(\frac{1}{5}\right)\left(\frac{2\pi}{6}\right)\right] \\ &= \frac{1}{60}\end{aligned}$$

HW2-3

3. Given a sequence $x[n]$ with Fourier transform $X(e^{j\omega})$, determine the Fourier transform of the following sequences in terms of $X(e^{j\omega})$: (20%)

(a) $x_1[n] = 2x[n+2] + 3x[3-n]$

(b) $x_2[n] = (1 + x[n])\,cos(0.2\pi n + \pi/6)$

(c) $x_3[n] = (x[n] - x^*[-n])/2$

(d) $x_4[n] = j^n x[n+1] + j^{-n} x[n-1]$

(e) $x_5[n] = 2e^{j0.5\pi(n-2)} x[n+2]$

$$X_1(e^{j\omega}) = 2e^{2j\omega}X(e^{j\omega}) + 3e^{-3j\omega}X(e^{j\omega})$$

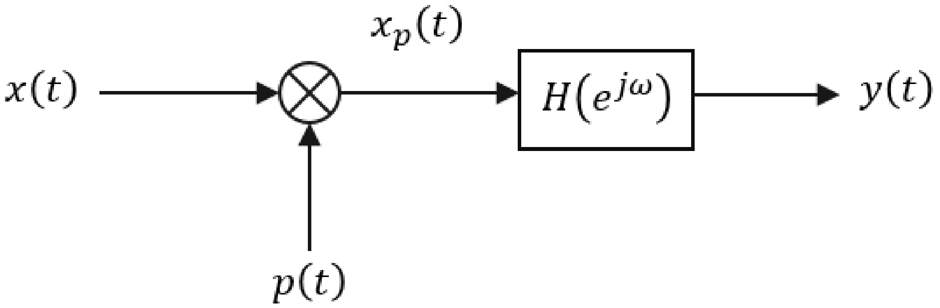
$$X_2(e^{j\omega}) = \frac{1}{2}e^{j\frac{\pi}{6}}\delta(\omega-\frac{\pi}{5}) + \frac{1}{2}e^{-j\frac{\pi}{6}}\delta(\omega+\frac{\pi}{5}) + \frac{1}{2}e^{j\frac{\pi}{6}}X(e^{j(\omega-\frac{\pi}{5})}) + \frac{1}{2}e^{-j\frac{\pi}{6}}X(e^{j(\omega+\frac{\pi}{5})})$$

$$X_3(e^{j\omega}) = \frac{1}{2}X(e^{j\omega}) - \frac{1}{2}X(e^{j\omega})^*$$

$$X_4(e^{j\omega}) = X(e^{j(\omega-\frac{\pi}{2})})e^{j(\omega-\frac{\pi}{2})} + X(e^{j(\omega+\frac{\pi}{2})})e^{-j(\omega+\frac{\pi}{2})}$$

$$X_5(e^{j\omega}) = -2e^{j2(\omega-0.5\pi)}X(e^{j(\omega-0.5\pi)})$$

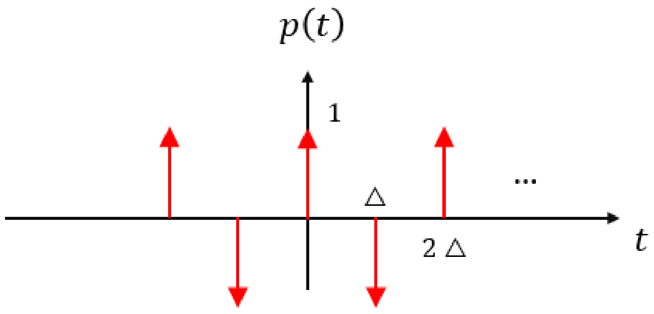
Sampling

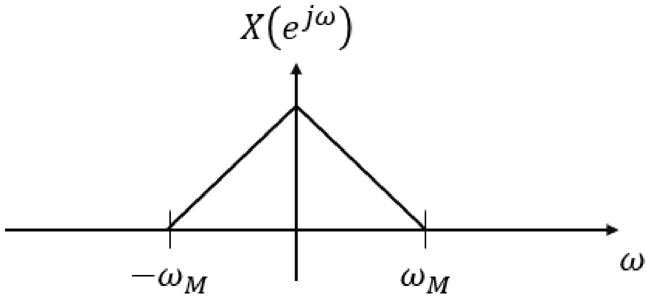


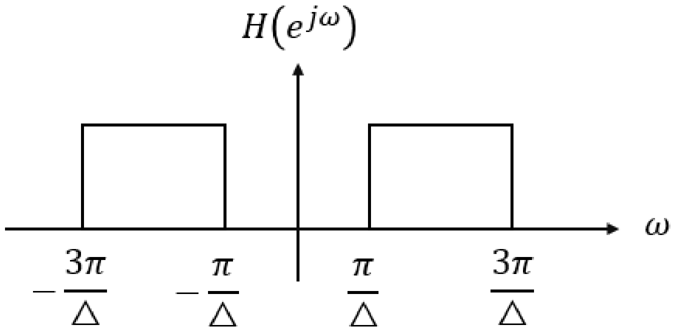
This question would not appear in quiz3, but in final exam

The figure is a system in which the sampling signal is an impulse train with alternating sign.

The sampling signal p(t), the Fourier Transform of the input signal x(t) and the frequency response of the filter are shown:







a) For $\triangle < \frac{\pi}{2\omega_M}$, sketch the Fourier transform of $x_p(t)$ and $y(t)$.

b) For $\triangle < \frac{\pi}{2\omega_M}$, determine a system that will recover $x(t)$ from $x_p(t)$ and another that will recover $x(t)$ from $y(t)$.

Sampling and Reconstruction

Consider a continuous-time signal

$$x_c(t) = 3cos(2\pi F_1 t + 45^{\circ}) + 3sin(2\pi F_2 t).$$

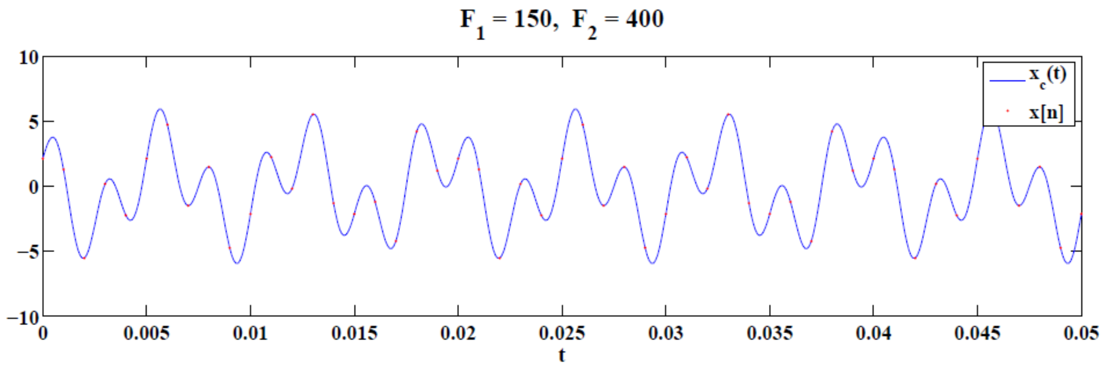
It is sampled at $t = 0.001n$ to obtain $x[n]$, which is then applied to an ideal DAC to obtain another continuous-time signal $y_r(t)$. For $F_1 = 150$ Hz and $F_2 = 400$ Hz,

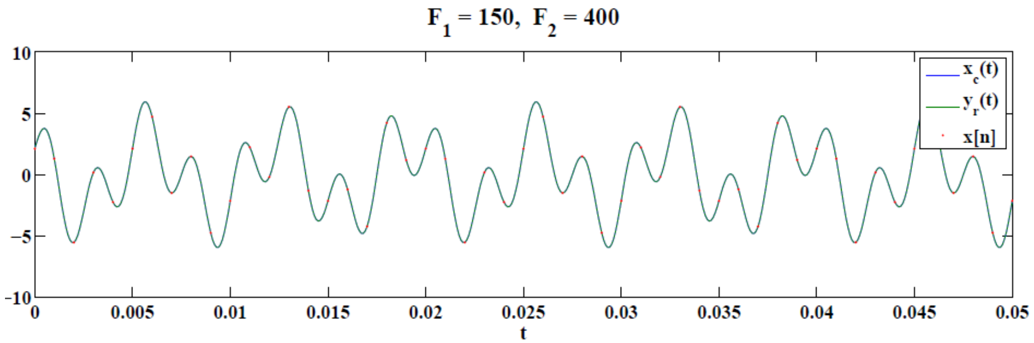
a) Determine $x[n]$ and graph its samples along with the signal $x_c(t)$ in one plot (choose few cycles of the $x_c(t)$ signal).

b) Determine $y_r(t)$ for the above $x[n]$ as a sinusoidal signal. Graph and compare it with $x_c(t)$.

$x[n] = x_c(nT) = 3cos(0.3\pi n + \pi/4) + 3sin(0.8\pi n).$

$y_r(t) = 3cos(300\pi t + \pi/4) + 3sin(800\pi t).$





Other half part of HW

4. Let $x(t) = 2 + 4\sin(3\pi t) + 6\cos(8\pi t + \pi/3)$ and **this is a periodic signal**: (20%)
   (a) Determine the average power P_{av} in x[t].
   (b) Determine the fundamental frequency 0 of $x[t]$.
   (c) Compute the CTFS coefficients C_k and express them in the magnitude-phase format.
   (d) Determine the average power of $x[t]$ from the frequency domain and verify that it equals P_{av} in part (a) above.

4. **(a) Solution:**
The fundamental period of $x(t)$ is $T = 2$.

$$\int_0^2 \sin(3\pi t)dt = \int_0^2 \cos(8\pi t + \pi/3)dt = \int_0^2 \sin(3\pi t)\cos(8\pi t + \pi/3)dt = 0$$

$$\begin{aligned} P_{\text{av}} &= \frac{1}{T}\int_0^2 |x(t)|^2 dt \\ &= \frac{1}{2}\int_0^2 4dt + \frac{1}{2}\int_0^2 16\cos^2(3\pi t - \pi/2)dt + \frac{1}{2}\int_0^2 36\cos^2(8\pi t + \pi/3)dt \\ &= 4 + 8\int_0^2 \frac{1 - \cos(6\pi t - \pi)}{2}dt + 18\int_0^2 \frac{1 - \cos(16\pi t + 2\pi/3)}{2}dt \\ &= 30 \end{aligned}$$

(b) Solution:

$$\Omega_0 = 2\pi \cdot \frac{1}{T} = \pi$$

(c) Solution:

$$x(t) = 2e^{j0\pi t} + 2e^{-j\frac{\pi}{2}}e^{j3\pi t} + 2e^{j\frac{\pi}{2}}e^{-j3\pi t} + 3e^{j\frac{\pi}{3}}e^{j8\pi t} + 3e^{-j\frac{\pi}{3}}e^{-j8\pi t}$$

$$c_0 = 2, c_3 = 2e^{-j\frac{\pi}{2}}, c_{-3} = 2e^{j\frac{\pi}{2}}, c_8 = 3e^{j\frac{\pi}{3}}, c_{-8} = 3e^{-j\frac{\pi}{3}}.$$

(d) Solution:

$$P_{\text{av}} = \sum_{k=-\infty}^{\infty} |c_k|^2 = 30$$

which verifies our computation in part (a).

5. Determine the sequence x[n] corresponding to each of the following Fourier transforms: (20%)

(a) $X(e^{j\omega}) = \delta(\omega) - \delta(\omega - \pi/2) - \delta(\omega + \pi/2)$

(b) $X(e^{j\omega}) = 1, 0 \le |\omega| \le 0.2\pi$ *and* $X(e^{j\omega}) = 0, 0.2\pi < |\omega| \le \pi$

(c) $X(e^{j\omega}) = 2|\omega|/\pi, 0 \le |\omega| \le \pi/2$ *and* $X(e^{j\omega}) = 0, \pi/2 < |\omega| \le \pi$

(d) *with* $\Delta|\omega| > 0$ *and* $\omega_c > \frac{\Delta\omega}{2}, X(e^{j\omega})$ *is given by*

$$X(e^{j\omega}) \begin{cases} 0, & \omega_c - \frac{\Delta\omega}{2} \le |\omega| \le \omega_c + \frac{\Delta\omega}{2} \\ 1. & otherwise \end{cases}$$

(a) Solution:

$$x_1[n] = \frac{1}{2\pi}\left(1 - 2\cos\frac{\pi}{2}n\right)$$

(b) Solution:

$$x_2[n] = \frac{1}{5}\text{sinc}\left(\frac{n}{5}\right)$$

(c) Solution:

$$x_3[n] = \frac{-\text{j}}{n\pi}\cos(\frac{\pi}{2}n) + \frac{2\text{j}}{n^2\pi^2}\sin(\frac{\pi}{2}n)$$

(d) Solution:

$$x_4[n] = \frac{\text{j}}{\pi n}\left\{1 - \cos(\pi n) + \cos[(\omega_c + \frac{\Delta\omega}{2})n] - \cos[(\omega_c - \frac{\Delta\omega}{2})n]\right\}$$

6. Determine the DTFT magnitude and phase spectra of the following signals: (10%)

$$X(e^{j\omega}) = \frac{1-a^2}{(1-ae^{-j\omega})(1-ae^{j\omega})}, \qquad |a|<1$$

a) Find the sequence $x[n]$

b) Calculate $\int_{-\pi}^{\pi} X(e^{j\omega}) \cos(\omega)\, d\omega/2\pi$.

(a) We first perform a partial-fraction expansion of $X(e^{j\omega})$.

$$\begin{aligned} X(e^{j\omega}) &= \frac{1-a^2}{(1-ae^{-j\omega})(1-ae^{j\omega})} \\ &= \frac{1}{1-ae^{-j\omega}} + \frac{ae^{j\omega}}{1-ae^{j\omega}} \\ x[n] &= a^n u[n] + a^{-n} u[-n-1] \\ &= a^{|n|} \end{aligned}$$

(b)

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\cos(\omega)d\omega &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\frac{e^{j\omega}+e^{-j\omega}}{2}d\omega \\ &= \frac{1}{2}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega}d\omega + \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{-j\omega}d\omega\right) \\ &= \frac{1}{2}(x[n-1]+x[n+1]) \\ &= \frac{1}{2}(a^{|n-1|}+a^{|n+1|}) \end{aligned}$$

Quiz answers:

5. **(10 pts.)** Determine sequences corresponding to each of the following Fourier transforms:
   (a) $X_1(e^{j\omega}) = [1 + 5cos(2\omega) + 8cos(4\omega)]e^{-j3\omega}$. [5 pt]
   (b) $X_2(e^{j\omega}) = \omega e^{j(\pi/2-5\omega)}$. [5 pt]

5. (a)

$$X_2(e^{j\omega}) = \left[1 + \frac{5}{2}(e^{j2\omega} + e^{-j2\omega}) + 4(e^{j4\omega} + e^{-j4\omega})\right] e^{-j3\omega}$$

$$x_2[n] = \delta[n-3] + \frac{5}{2}(\delta[n-1] + \delta[n-5]) + 4(\delta[n+1] + \delta[n-7])$$

5. (b)

$$X_5(e^{j\omega}) = \omega e^{j(\pi/2)} e^{-j5\omega} = j\omega e^{-j5\omega}$$

$$x[n] = \frac{1}{2\pi}\int_0^{2\pi} \omega e^{jn\omega} d\omega = -\frac{j}{n}$$

$$x_5[n] = \frac{1}{n-5}$$

6. **(10 pts.)** Determine whether or not each of the following signals is periodic. If periodic, find its fundamental frequency.

(a) $x_1(t) = sin(\sqrt{2}t) + cos(2\sqrt{2}t)$. [5 pt]

(b) $x_2(t) = \frac{1}{3}\left\{sin\left(\frac{t}{11}\right) + cos\left(\frac{t}{79}\right) + sin(\frac{t}{31})\right\}$. [5 pt]

$$T_1 = \frac{2\pi}{\sqrt{2}} = \sqrt{2}\pi, \quad T_2 = \frac{2\pi}{2\sqrt{2}} = \frac{\sqrt{2}}{2}\pi, \quad T = \sqrt{2}\pi$$

$x_2(t)$ is periodic with fundamental period $T = \sqrt{2}\pi$.

(b)

$$T_1 = 22\pi, \quad T_2 = 158\pi, \quad T_3 = 68\pi, \quad T_1 = 22\pi, \quad T = 2{\times}11{\times}79{\times}31\pi$$

$x_3(t)$ is periodic with fundamental period $T = 2 \times 11 \times 79 \times 31\pi = 53,878\pi$.

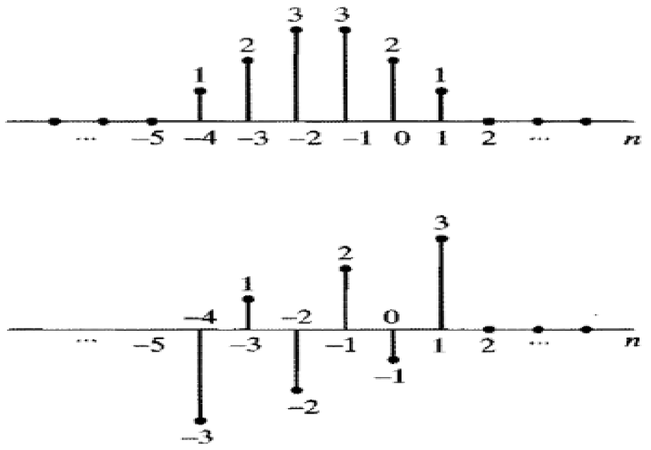


Figure q7

Perform the following calculations without explicitly evaluating $X(e^{j\omega})$.

(a) Evaluate $X(e^{j\omega})|_{\omega=0}$. [1 pt]

(b) Evaluate $X(e^{j\omega})|_{\omega=\pi}$. [2 pt]

(c) Evaluate $\int_{-\pi}^{\pi} X(e^{j\omega})d\omega$. [2 pt]

(d) Determine and sketch the signal (in the time domain) whose Fourier transform is $X(e^{-j\omega})$ [5 pt]

(e) Determine and sketch the signal (in the time domain) whose Fourier transform is $j\text{Im}\{X(e^{j\omega})\}$ [5 pt]

2.44. 1. The Fourier transform of $x[n]$ is given by $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$. Then

$$X(e^{j\omega})\Big|_{\omega=0} = \sum_{n=-\infty}^{\infty} x[n] = 12.$$

2. $X(e^{j\omega})\Big|_{\omega=\pi} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\pi n} = \sum_{n=-\infty}^{\infty} (-1)^n x[n] = -j12.$

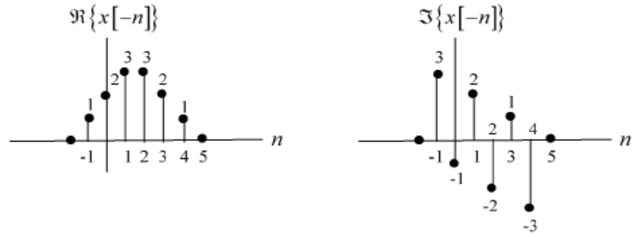
3. The inverse Fourier transform is given by $\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega = x[n]$. Then

$$\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega = 2\pi x[n]$$

$$\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega 0}d\omega = 2\pi x[0]$$

$$\int_{-\pi}^{\pi} X(e^{j\omega})d\omega = 2\pi(2-j).$$

4. If $x[n] \overset{F}{\longleftrightarrow} X(e^{j\omega})$, then $x[-n] \overset{F}{\longleftrightarrow} X(e^{-j\omega})$.



5. If $x[n] \overset{F}{\longleftrightarrow} X(e^{j\omega})$, then $x_o[n] \overset{F}{\longleftrightarrow} j\Im\{X(e^{j\omega})\}$, where $x_o[n] = \frac{1}{2}(x[n] - x^*[-n])$.