數位訊號處理概論
Introduction to Digital Signal Processing: HW1, Quiz1, Topic2 TA Review

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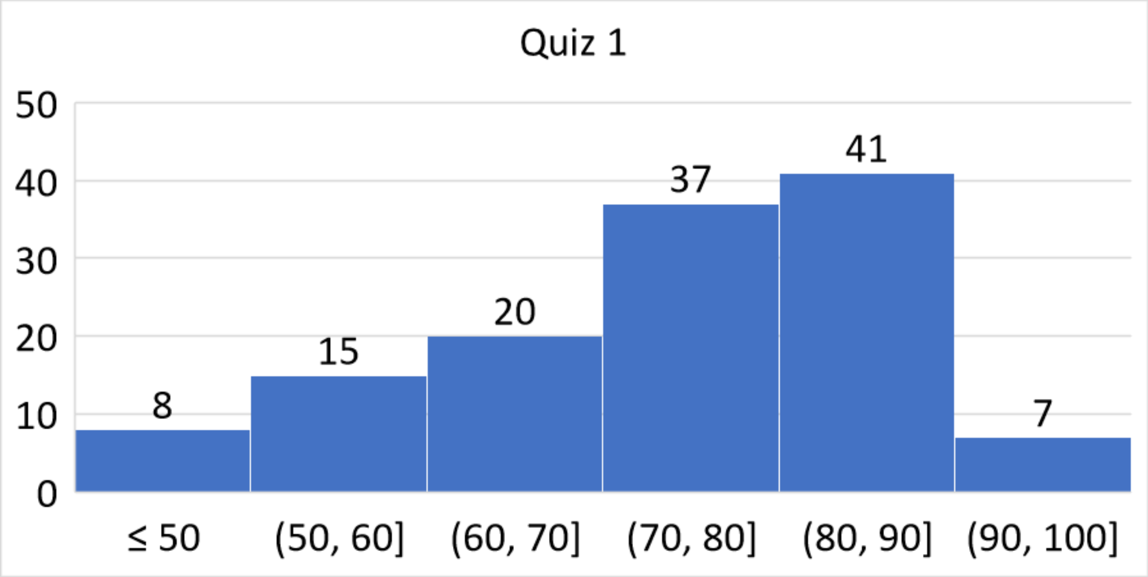
Remind

- HW 2:
  - Due time: **2022/03/30 at 13:20**
  - A4 papers and hand in the homework in class
  - Please provide detailed answers or explanations in English

- Quiz 2:
  - time: **2022/03/30 13:20-15:10**
  - scope: everything we cover in topic 2
  - in eeclas 206 and 208
  - 1 A4 cheat sheet is allowed, and printed from iPad is not accepted.



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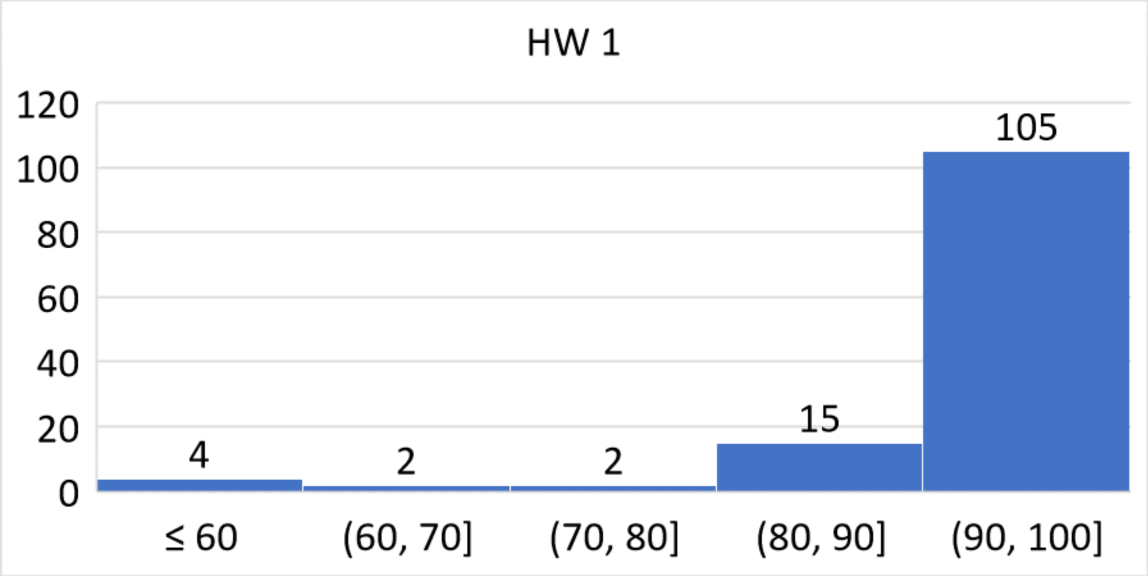

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HW1-1

1. Determine whether the following systems are (1) stable, (2) causal, (3) linear, (4) time invariant. (Brief explanation is needed.) (20%)

   (a) $T(x[n]) = x[n^2]$

   (b) $T(x[n]) = e^{x[n]}$

   (c) $T(x[n]) = (\cos \pi n)x[n]$

   (d) $T(x[n]) = x[n] + 3u[n+1]$

   (e) $T(x[n]) = x[n]\sum_{k=0}^{\infty}\delta[n-k]$

	Stable	Causal	Linear	TI
(a)	O	X	O	X
(b)	O	O	X	O
(c)	O	O	O	X
(d)	O	O	X	X
(e)	O	O	O	X

HW1-2

2. A causal LTI system is described by the difference equation (15%)

$$y[n] - 5y[n-1] + 6y[n-2] = 2x[n-1]$$

(a) Determine the homogeneous response of the system, i.e., the possible outputs if $x[n] = 0$ for all n.

(b) Determine the impulse response of the system.

(c) Determine the step response of the system.

(a) The homogeneous difference equation:

$$y[n] - 5y[n-1] + 6y[n-2] = 0$$

Taking the Z-transform,

$$1 - 5z^{-1} + 6z^{-2} = 0$$

$$(1 - 2z^{-1})(1 - 3z^{-1}) = 0.$$

The homogeneous solution is of the form

$$y_h[n] = A_1(2)^n + A_2(3)^n.$$

(b) We take the z-transform of both sides:

$$Y(z)[1 - 5z^{-1} + 6z^{-2}] = 2z^{-1}X(z)$$

Thus, the system function is

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{2z^{-1}}{1 - 5z^{-1} + 6z^{-2}} \\ &= \frac{-2}{1 - 2z^{-1}} + \frac{2}{1 - 3z^{-1}}, \end{aligned}$$ 2

where the region of convergence is outside the outermost pole, because the system is causal. Hence the ROC is $|z| > 3$. Taking the inverse z-transform, the impulse response is

$$h[n] = -2(2)^n u[n] + 2(3)^n u[n].$$

(c) Let $x[n] = u[n]$ (unit step), then

$$X(z) = \frac{1}{1 - z^{-1}}$$ 1

and

$$\begin{aligned} Y(z) &= X(z) \cdot H(z) \\ &= \frac{2z^{-1}}{(1 - z^{-1})(1 - 2z^{-1})(1 - 3z^{-1})}. \end{aligned}$$

Partial fraction expansion yields

$$Y(z) = \frac{1}{1 - z^{-1}} - \frac{4}{1 - 2z^{-1}} + \frac{3}{1 - 3z^{-1}}.$$ 2

The inverse transform yields:

$$y[n] = u[n] - 4(2)^n u[n] + 3(3)^n u[n].$$

HW1-3

3. A discrete-time signal $x[n]$ is shown in Figure 3. Please sketch and label carefully each of the following signals: (15%)
   (a) $x[4-n]$
   (b) $x[3n]$
   (c) $x[n]u[2-n]$
   (d) $x[n-1]u[n-3]$
   (e) $x[n-2]\delta[n-2]$

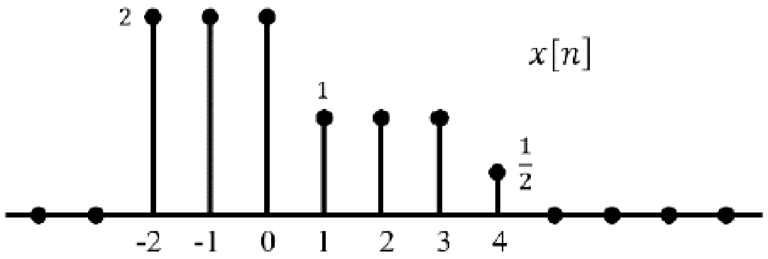
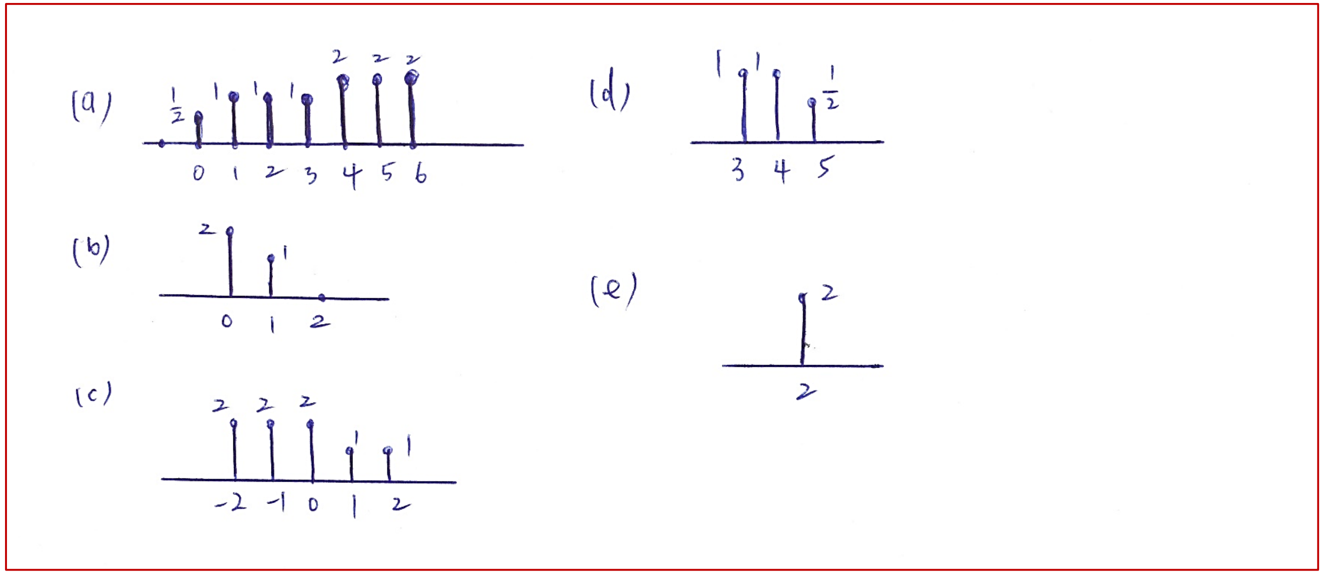


Figure 3: The discrete-time signal $x[n]$



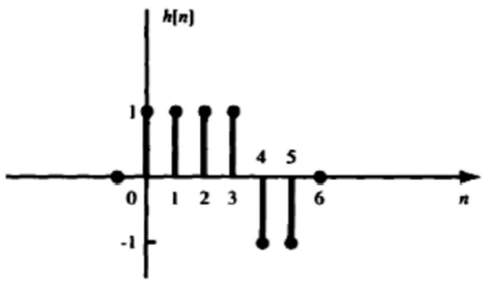
HW1-4

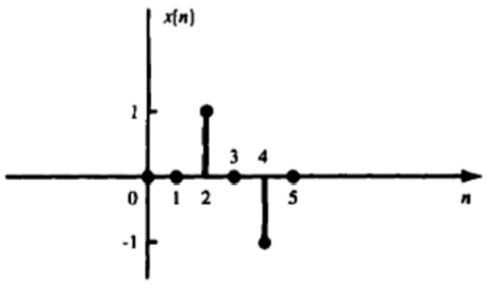
4. Determine whether or not each of the following signal is periodic and if yes then determine its fundamental period. (20%)
   (a) $x[t] = [\sin(4t - 1)]^2$ Periodic with fundamental period of π/4
   (b) $x[n] = \cos(4n + \pi/4)$ Aperiodic
   (c) $x[n] = (-1)^n \cos(2\pi n/7)$ Periodic with fundamental period of 14
   (d) $x[n] = ne^{j\pi n}$ Aperiodic

HW1-5

Some other method 7pts

5. The impulse response $h[n]$ of a discrete-time LTI system, Determine and sketch the output $y[n]$ of this system to the input $x[n]$ **without using the convolution technique**. (10%)





$$h[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] - \delta[n-4] - \delta[n-5],$$

$$x[n] = \delta[n-2] - \delta[n-4]$$

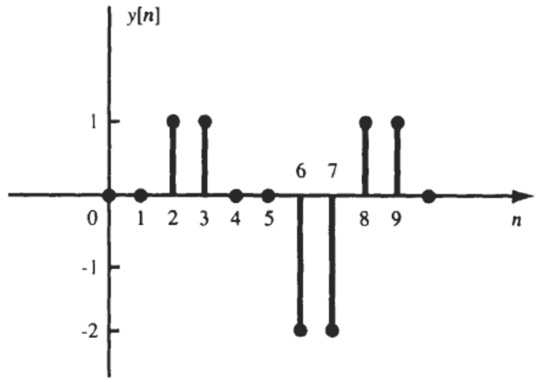
$$x[n] * h[n] = x[n] * \{\delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] - \delta[n-4] - \delta[n-5]\}$$

$$= x[n] + x[n-1] + x[n-2] + x[n-3] - x[n-4] - x[n-5]$$

$$y[n] = \delta[n-2] - \delta[n-4] + \delta[n-3] - \delta[n-5] + \delta[n-4] - \delta[n-6] + \delta[n-5] - \delta[n-7] - \delta[n-6] + \delta[n-8] - \delta[n-7] + \delta[n-9]$$

$$= \delta[n-2] + \delta[n-3] - 2\delta[n-6] - 2\delta[n-7] + \delta[n-8] + \delta[n-9]$$

$$y[n] = \{0,0,1,1,0,0,-2,-2,1,1\}$$

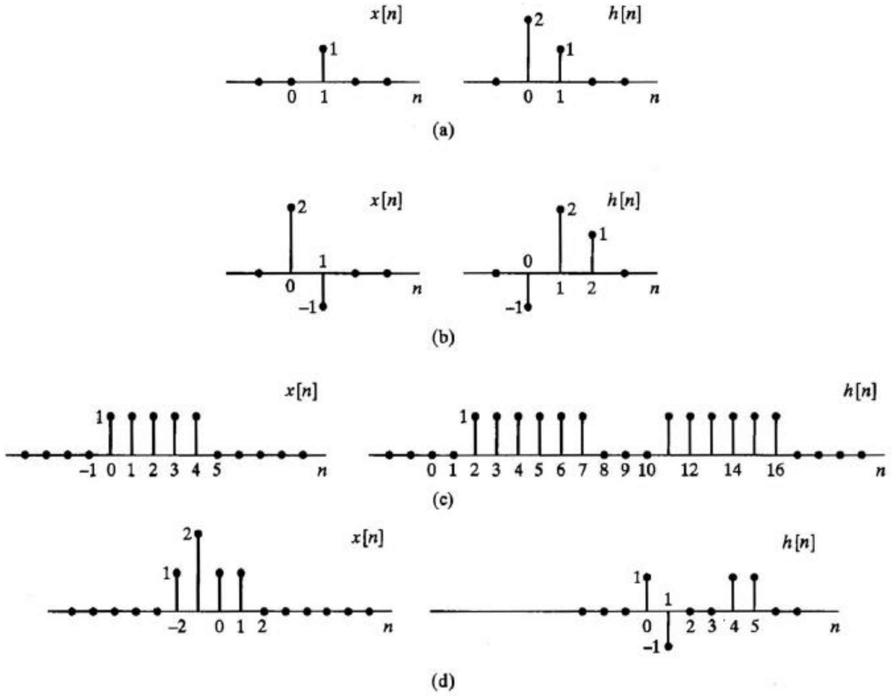


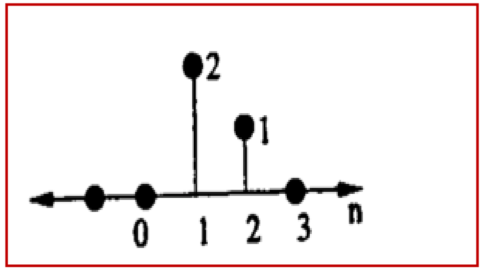
5pt

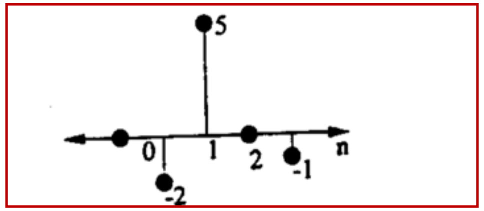
HW1-6

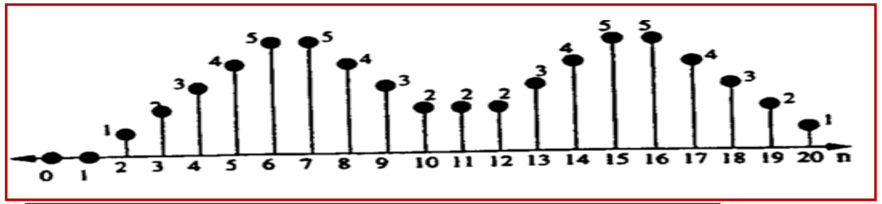
If not plotted some points has been deducted

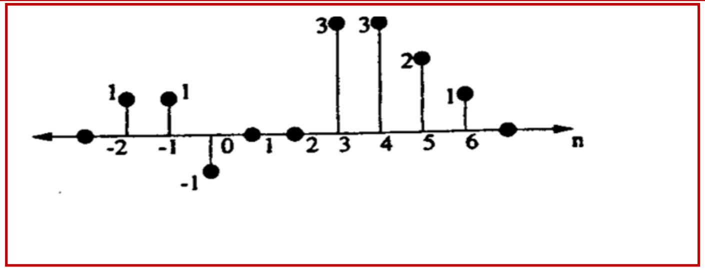
6. For each pairs sequences, use discrete convolution to find the response to the input $x[n]$ of the LTI system with impulse response $h[n]$. (20%)











Quiz1-1

1. **(10 pts.)** Please brief answer the following questions:
   (a) Why we need to check whether the system is causal, stable, linear or time invariant? [4pt]
   (b) Linearity makes it possible to characterize a system in terms of the responses $h_k[n]$ to the shifted impulses $\delta[n-k]$ for all k, whereas time-invariance implies that $h_k[n] = h[n-k]$. For every system of the combination of linearity and time-invariance, (1) what is the complete and unique characterization? [2pt] (2) what are the two characteristics can its output be determined by? [4pt]

(a) Good for system analysis, practical implementation (Ref: Feb23-2022 p4)
(b) (1) Impulse response sequence (2) Convolution of impulse response and input sequences (Ref: Feb24-2022 p10)

Quiz1-2

2. **(10 pts.)** A downsampler is a system,

$$y[n] = \mathcal{H}\{x[n]\} = x[nM],$$

that is used to sample a discrete-time signal $x[n]$ by a factor of M. Test the downsampler for linearity and time invariance.

Linear, time-varying (Ref: Feb23-2022 p10)

Quiz1-3

3. **(20 pts.)** Consider the following difference equation:

$$y[n] - \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = 3x[n].$$

(a) Determine the general form of the homogeneous solution to this difference equation. [5 pt]

(b) Both a causal and an acausal LTI system are characterized by this difference equation. Find the impulse responses of the two systems. [5 pt]

(c) Show that the causal LTI system is stable and the acausal LTI system is unstable. [5 pt]

(d) Find a particular solution to the difference equation when $x[n] = (1/2)^n u[n]$. [5 pt]

(a) The homogeneous solution $y_h[n]$ solves the difference equation when $x[n] = 0$. It is in the form $y_h[n] = \sum A(c)^n$, where the c's solve the quadratic equation

$$c^2 - \frac{1}{4}c + \frac{1}{8} = 0$$

So for $c = 1/2$ and $c = -1/4$, the general form for the homogeneous solution is:

$$y_h[n] = A_1(\frac{1}{2})^n + A_2(-\frac{1}{4})^n$$

(b) Taking the z-transform of both sides, we find that

$$Y(z)(1 - \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}) = 3X(z)$$

and therefore

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{3}{1 - 1/4z^{-1} - 1/8z^{-2}} \\ &= \frac{3}{(1 + 1/4z^{-1})(1 - 1/2z^{-1})} \\ &= \frac{1}{1 + 1/4z^{-1}} + \frac{2}{1 - 1/2z^{-1}} \end{aligned}$$ 1

The causal impulse response corresponds to assuming that the region of convergence extends outside the outermost pole, making

$$h_c[n] = ((-1/4)^n + 2(1/2)^n)u[n]$$ 2

The anti-causal impulse response corresponds to assuming that the region of convergence is inside the innermost pole, making

$$h_{ac}[n] = -((-1/4)^n + 2(1/2)^n)u[-n-1]$$ 2

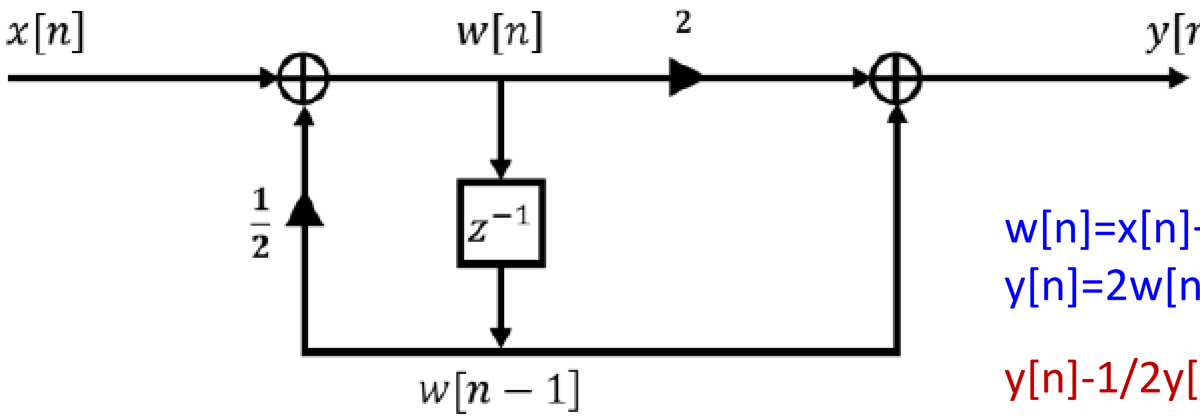
(c) $h_c[n]$ is absolutely summable, while $h_{ac}[n]$ grows without bounds. 寫錯一個 -3 points

(d)

$$\begin{aligned} Y(z) &= X(z)H(z) \\ &= \frac{1}{1 - \frac{1}{2}z^{-1}} \cdot \frac{1}{(1 + \frac{1}{4}z^{-1})(1 - \frac{1}{2}z^{-1})} \\ &= \frac{1/3}{1 + 1/4z^{-1}} + \frac{2}{(1 - 1/2z^{-1})^2} + \frac{2/3}{1 - 1/2z^{-1}} \\ y[n] &= \frac{1}{3}(\frac{1}{4})^n u[n] + 4(n+1)(\frac{1}{2})^{n+1}u[n+1] + \frac{2}{3}(\frac{1}{2})^n u[n] \end{aligned}$$

Quiz1-4

4. **(10 pts.)** Write out the input-output equation for the system.



w[n]=x[n]+1/2w[n-1]
y[n]=2w[n]+w[n-1] → 3 points

y[n]-1/2y[n-1]=x[n]+2x[n-1]

Quiz1-5

5. **(5 pts.)** Suppose the output response of the system is $y[n]$, write the total recursive response of the system in terms of zero-input zero-state response of the system?

(Ref: Feb24-2022 p13-14)

$$y[n] = a^{n+1}y[-1] + h[n]x[0] + h[n-1]x[1] + \cdots + h[0]x[n].$$

$$y[n] = a^{n+1}y[-1] + h[n]x[0] + h[n-1]x[1] + \cdots + h[0]x[n]. \quad (2.82)$$

We see that the output $y[n]$ for $n \geq 0$, depends both on the input $x[n]$ for $n \geq 0$ and the initial condition $y[-1]$. The value of $y[-1]$ summarizes the response of the system to past inputs applied for $n < 0$.

If we set $x[n] = 0$ for $n \geq 0$, we obtain

$$y[n] = a^{n+1}y[-1] + h[n]x[0] + h[n-1]x[1] + \cdots + h[0]x[n]. \quad (2.82)$$

$$y_{zi}[n] = a^{n+1}y[-1], \quad n \geq 0 \quad (2.83)$$

which is known as the *zero-input response* of the system. If we assume that $y[-1] = 0$, that is the system is initially at rest or at zero-state, the output is given by

$$y_{zs}[n] = \sum_{k=0}^{n} h[k]x[n-k], \quad (2.84)$$

which is called the *zero-state response* of the system. Therefore, the total response of the recursive system is given by

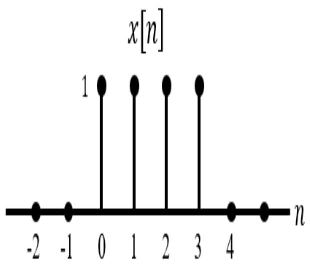
$$y[n] = \underbrace{a^{n+1}y[-1]}_{\text{zero-input response}} + \underbrace{\sum_{k=0}^{n} h[k]x[n-k]}_{\text{zero-state response}} = y_{zi}[n] + y_{zs}[n]. \quad (2.85)$$

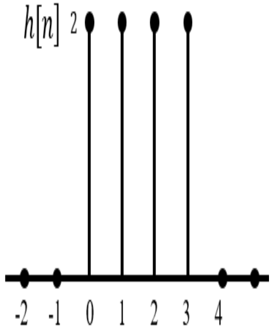
Quiz1-6

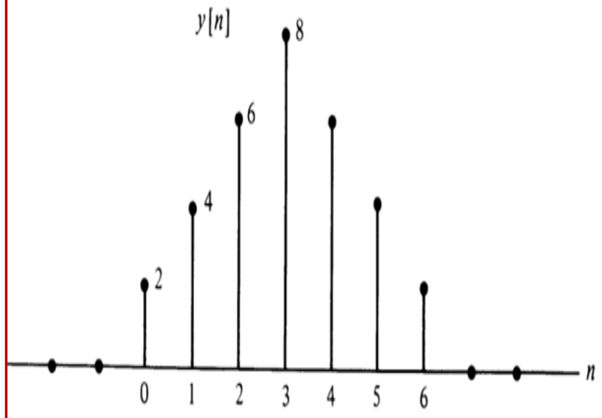
6. **(15 pts.)**

(a) Determine the discrete-time convolution of $x[n]$ and $h[n]$ for the following two cases. [10pt]

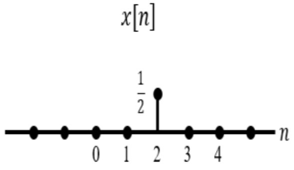
Case (1)

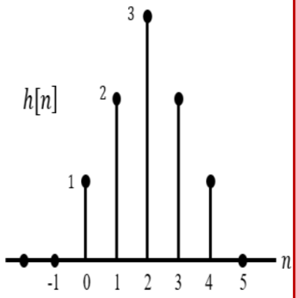


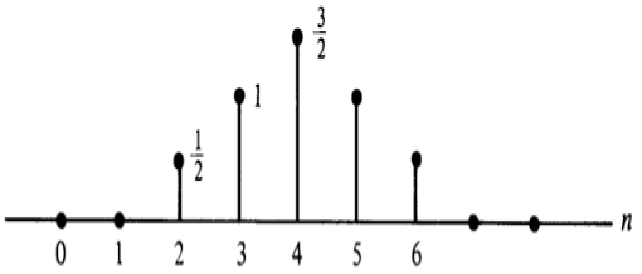




Case (2)







(b) Find the convolution of $x[n] = [1,1,1,1,2,2,2,2]$ with $h[n] = [3,3,0,0,0,0,3,3]$ by using matrix method. [5pt]

$$\begin{bmatrix} y(0)\\ y(1)\\ y(2)\\ y(3)\\ y(4)\\ y(5)\\ y(6)\\ y(7) \end{bmatrix} = \begin{bmatrix} h(0) & h(7) & h(6) & h(5) & h(4) & h(3) & h(2) & h(1)\\ h(1) & h(0) & h(7) & h(6) & h(5) & h(4) & h(3) & h(2)\\ h(2) & h(1) & h(0) & h(7) & h(6) & h(5) & h(4) & h(3)\\ h(3) & h(2) & h(1) & h(0) & h(7) & h(6) & h(5) & h(4)\\ h(4) & h(3) & h(2) & h(1) & h(0) & h(7) & h(6) & h(5)\\ h(5) & h(4) & h(3) & h(2) & h(1) & h(0) & h(7) & h(6)\\ h(6) & h(5) & h(4) & h(3) & h(2) & h(1) & h(0) & h(7)\\ h(7) & h(6) & h(5) & h(4) & h(3) & h(2) & h(1) & h(0) \end{bmatrix} \begin{bmatrix} x(0)\\ x(1)\\ x(2)\\ x(3)\\ x(4)\\ x(5)\\ x(6)\\ x(7) \end{bmatrix}$$

Substituting the values, we get

$$\begin{bmatrix} y(0)\\ y(1)\\ y(2)\\ y(3)\\ y(4)\\ y(5)\\ y(6)\\ y(7) \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 & 0 & 0 & 0 & 0 & 3\\ 3 & 3 & 3 & 3 & 0 & 0 & 0 & 0\\ 0 & 3 & 3 & 3 & 3 & 0 & 0 & 0\\ 0 & 0 & 3 & 3 & 3 & 3 & 0 & 0\\ 0 & 0 & 0 & 3 & 3 & 3 & 3 & 0\\ 0 & 0 & 0 & 0 & 3 & 3 & 3 & 3\\ 3 & 0 & 0 & 0 & 0 & 3 & 3 & 3\\ 3 & 3 & 0 & 0 & 0 & 0 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1\\ 1\\ 1\\ 1\\ 2\\ 2\\ 2\\ 2 \end{bmatrix}$$

$$\begin{bmatrix} y(0)\\ y(1)\\ y(2)\\ y(3)\\ y(4)\\ y(5)\\ y(6)\\ y(7) \end{bmatrix} = \begin{bmatrix} 3\times1+3\times1+3\times1+0\times1+0\times2+0\times2+0\times2+3\times2\\ 3\times1+3\times1+3\times1+3\times1+0\times2+0\times2+0\times2+0\times2\\ 0\times1+3\times1+3\times1+3\times1+3\times2+0\times2+0\times2+0\times2\\ 0\times1+0\times1+3\times1+3\times1+3\times2+3\times2+0\times2+0\times2\\ 0\times1+0\times1+0\times1+3\times1+3\times2+3\times2+3\times2+0\times2\\ 0\times1+0\times1+0\times1+0\times1+3\times2+3\times2+3\times2+3\times2\\ 3\times1+0\times1+0\times1+0\times1+0\times2+3\times2+3\times2+3\times2\\ 3\times1+3\times1+0\times1+0\times1+0\times2+0\times2+3\times2+3\times2 \end{bmatrix} = \begin{bmatrix} 15\\ 12\\ 15\\ 18\\ 21\\ 24\\ 21\\ 18 \end{bmatrix}$$

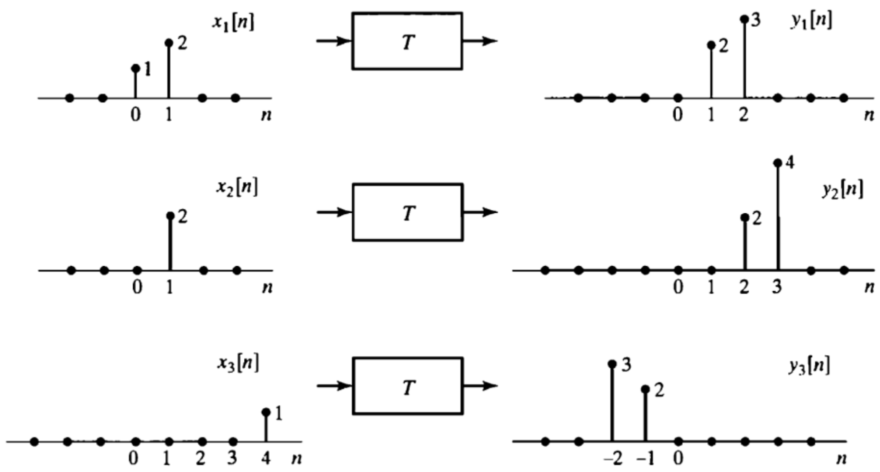
Therefore, the convoluted sum is $y(n)$ = [15, 12, 15, 18, 21, 24, 21, 18].

or

$$\begin{bmatrix} 3\\ 6\\ 6\\ 6\\ 9\\ 12\\ 15\\ 18\\ 12\\ 6\\ 9\\ 12\\ 12\\ 12\\ 6 \end{bmatrix}$$

Quiz1-7

7. **(15 pts.)** The system T in below figure is known to be time-invariant. When the input to the system are $x_1[n]$, $x_2[n]$ and $x_3[n]$, the response of the system are $y_1[n]$, $y_2[n]$ and $y_3[n]$ as shown.
   (a) Determine whether the system T could be linear? [5pt]
   (b) If the input $x[n]$ to the system T is $\delta[n]$, what is the system response $y[n]$? [5pt]
   (c) What are the possible inputs $x[n]$ for which the response of the system T can be determined from the given information alone? [5pt]



(a) Notice that $x_1[n] = x_2[n] + x_3[n+4]$, so if $T\{\cdot\}$ is linear,

$$\begin{aligned} T\{x_1[n]\} &= T\{x_2[n]\} + T\{x_3[n+4]\} \\ &= y_2[n] + y_3[n+4] \end{aligned}$$

From Fig P2.4, the above equality is not true. Hence, the system is NOT LINEAR.

(b) To find the impulse response of the system, we note that

$$\delta[n] = x_3[n+4]$$

Therefore,

$$\begin{aligned} T\{\delta[n]\} &= y_3[n+4] \\ &= 3\delta[n+6] + 2\delta[n+5] \end{aligned}$$

(c) Since the system is known to be time-invariant and not linear, we cannot use choices such as:

$$\delta[n] = x_1[n] - x_2[n]$$

and

$$\delta[n] = \frac{1}{2} x_2[n+1]$$

to determine the impulse response. With the given information, we can only use shifted inputs.

Quiz1-8

8. **(15 pts.)**

(a) State conditions when the signal is said to be periodic or aperiodic in terms of continuous time signal and discrete time signals? Are sinusoids sequence periodic sequence? [5pt] (Ref: Feb16-2022 p16-19)

(b) Examine whether the following signals are periodic or not? Determine the fundamental period of the signal. [10pt]

(1) $x(t) = \cos 10t - \cos(10+\pi)t$ Aperiodic

(2) $x(n) = sin\left(\frac{2\pi}{5}\right)n + cos\left(\frac{2\pi}{7}\right)n$ Periodic with fundamental period of 35

Parseval's Theorem

HW

Use Parseval's theorem to compute the following summation

$$S = \sum_{n=-\infty}^{\infty} \frac{\sin(\pi n/4)}{2\pi n} \frac{\sin(\pi n/6)}{5\pi n}.$$

Continuous-time

Orthogonality property $$\int_{T_0} s_k(t) s_m^*(t)\mathrm{d}t = \int_{T_0} \mathrm{e}^{\mathrm{j}k\Omega_0 t}\mathrm{e}^{-\mathrm{j}m\Omega_0 t}\mathrm{d}t = \begin{cases} T_0, & k = m \\ 0, & k \neq m \end{cases} \quad (4.7)$$

Parseval's relation The average power in one period of $x(t)$ can be expressed in terms of the Fourier coefficients using Parseval's relation (see Problem 6):

$$P_{\mathrm{av}} = \frac{1}{T_0}\int_{T_0} |x(t)|^2 \mathrm{d}t = \sum_{k=-\infty}^{\infty} |c_k|^2. \quad (4.27)$$

Discrete-time

Orthogonality property $$\sum_{n=\langle N\rangle} s_k[n] s_m^*[n] = \sum_{n=\langle N\rangle} \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}kn}\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}mn} = \begin{cases} N, & k = m \\ 0, & k \neq m \end{cases} \quad (4.22)$$

Parseval's relation The average power in one period of $x[n]$ can be expressed in terms of the Fourier series coefficients using the following form of Parseval's relation (see Problem 41):

$$P_{\mathrm{av}} = \frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2 = \sum_{k=0}^{N-1} |c_k|^2. \quad (4.69)$$

DTFT Properties

- Given a sequence $x[n]$ with Fourier transform $X(e^{j\omega})$, determine the Fourier transform of the following sequences in terms of $X(e^{j\omega})$

HW (a) $x_1[n] = 2x[n+2] + 3x[3-n]$

Table 4.4 Operational properties of the DTFT.

	Property	Sequence	Transform
		$x[n]$	$\mathcal{F}\{x[n]\}$
1.	Linearity	$a_1x_1[n] + a_2x_2[n]$	$a_1X_1(e^{j\omega}) + a_2X_2(e^{j\omega})$
2.	Time shifting	$x[n-k]$	$e^{-jk\omega}X(e^{j\omega})$
3.	Frequency shifting	$e^{j\omega_0 n}x[n]$	$X[e^{j(\omega-\omega_0)}]$
4.	Modulation	$x[n]\cos\omega_0 n$	$\frac{1}{2}X[e^{j(\omega+\omega_0)}] + \frac{1}{2}X[e^{j(\omega-\omega_0)}]$
5.	Folding	$x[-n]$	$X(e^{-j\omega})$
6.	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
7.	Differentiation	$nx[n]$	$-j\dfrac{dX(e^{j\omega})}{d\omega}$
8.	Convolution	$x[n] * h[n]$	$X(e^{j\omega})H(e^{j\omega})$
9.	Windowing	$x[n]w[n]$	$\dfrac{1}{2\pi}\int_{2\pi} X(e^{j\theta})W[e^{j(\omega-\theta)}]d\theta$
10.	Parseval's theorem	$\sum_{n=-\infty}^{\infty} x_1[n]x_2^*[n] = \dfrac{1}{2\pi}\int_{2\pi} X_1(e^{j\omega})X_2^*(e^{j\omega})d\omega$	
11.	Parseval's relation	$\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \dfrac{1}{2\pi}\int_{2\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$	

Continuous-time Fourier Transform (CTFT):

$$x(t) = \begin{cases} A, & |t| < \tau \\ 0, & \tau < |t| < T_0/2 \end{cases} \quad (4.37)$$

Repeats with period T_0. The Fourier coefficients are given by

$$c_k = \frac{A\tau}{T_0} \frac{\sin \pi k F_0 \tau}{\pi k F_0 \tau} \triangleq c(kF_0). \quad (4.38)$$

The size of the coefficients c_k depends on the period T_0 and $c_k \rightarrow 0$ as $T_0 \rightarrow \infty$. To avoid this problem, we consider the scaled coefficients

$$c(kF_0)T_0 = A\tau \left. \frac{\sin \pi F \tau}{\pi F \tau} \right|_{F=kF_0}, \quad (4.39)$$

which can be thought of as equally spaced samples of the envelope function. As T_0 increases, the spacing $F = F_0 = 1/T_0$ between the spectral lines decreases. As $T_0 \rightarrow \infty$

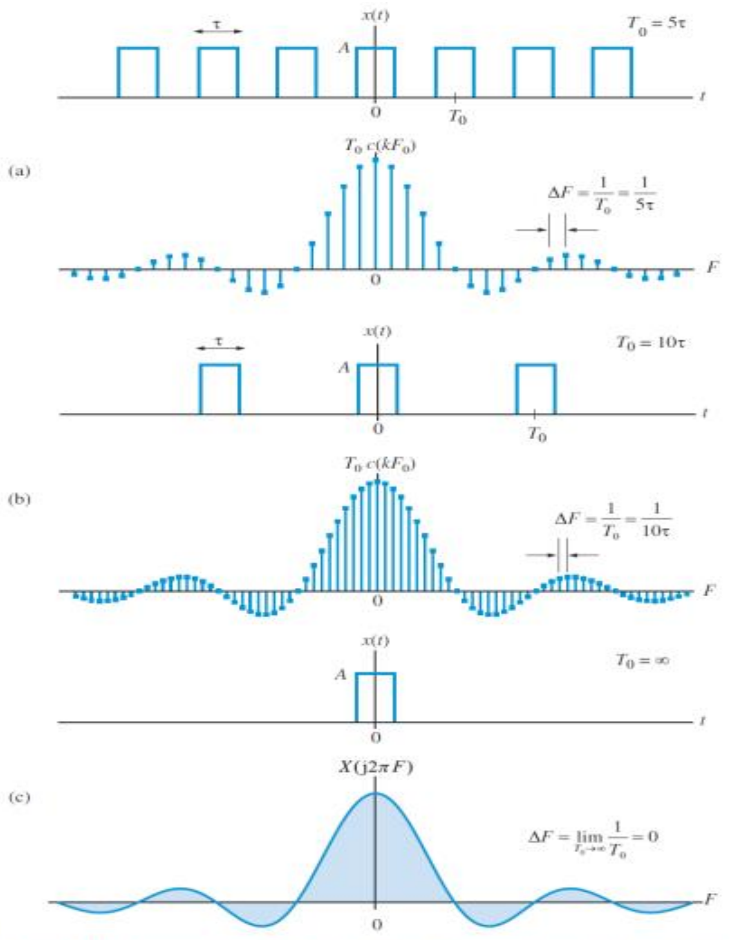


Figure 4.14 Transition from the CTFS to CTFT: (a) the periodic signal $x(t)$ and its scaled CTFS for the fundamental period $T_0 = 5\tau$, (b) the periodic signal $x(t)$ and its scaled CTFS for the fundamental period $T_0 = 10\tau$, and (c) the aperiodic signal $x(t)$ and its CTFT when the period extends to infinity.