11020EE 366000 Introduction to Digital Signal Processing
Final Exam – Answer
June 15, 2022

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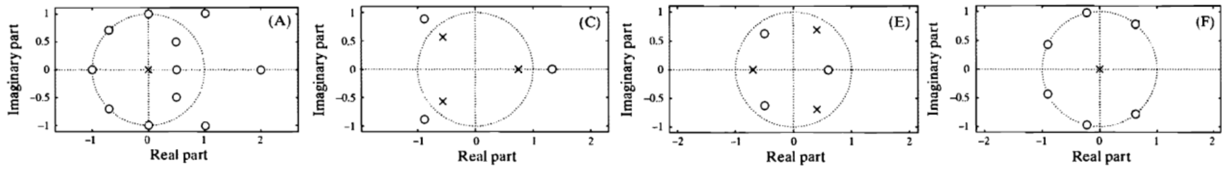
1. **(20 pts.)** Concept MCQs. Ans.debda

1-1. The following diagrams represent systems with pole-zero. Which of the following statement is ***false***?



(a) System C is an IIR system. C, E have poles at places other than the origin and infinity.

(b) Systems A, F are FIR systems. These have poles only at the origin.

(c) Systems A, C, E and F are stable. A causal LTI system is stable if and only if all of its poles lie inside the unit circle.

(d) System E has the shortest (least number of nonzero samples) impulse response. F has the shortest impulse response, with 7 nonzero samples, A has 12 nonzero samples, and the remaining systems are IIR.

(e) Only system E has a stable and causal inverse. This is the only system having all of its zeros inside the unit circle.

1-2. Which of the following is ***false*** about the time signal analysis?

(a) Power spectrum is obtained when we plot $|c_k|^2$ as a function of frequency.

(b) Magnitude voltage spectrum is obtained when we plot $|c_k|$ as a function of frequency.

(c) If an LTI system is described by the difference equation $y[n] = ay[n-1] + bx[n],\ 0 < a < 1$, then b could be $(1-a)$ so that the maximum value of $|H(\omega)|$ is unity.

(d) Energy density spectrum is a continuous spectrum.

(e) $|1 + 2cos\omega|$ is the magnitude of $H(\omega)$ for the 3-point moving average system whose output is given by $y[n] = \frac{1}{3}[x(n+1) + x(n) + x(n-1)]$. $\frac{1}{3}|1 + 2cos\omega|$

Explanation: For a three point moving average system, we can define the output of the system as y(n)=$\frac{1}{3}[x(n+1)+x(n)+x(n-1)] => h(n) = \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$ it follows that H(ω)=$\frac{1}{3}(e^{j\omega}+1+e^{-j\omega}) = \frac{1}{3}(1+2cos\omega)$ =>| H(ω)|=$\frac{1}{3}$|1+2cosω|

1-3. Which of the following is ***false*** about digital to analog converter (DAC)?

(a) The ideal DAC is a noncausal and unstable system, so it is not practically realizable.

(b) The result of DAC converter is a staircase ~~discrete-time signal~~. continuous-time signal

(c) The DAC generates an analog voltage at its output which is determined by the binary word at its input.

(d) DAC converter is done by holding the output voltage of the DAC constant for one sampling period.

(e) An idealized operation of DAC converter that reconstructs a continuous-time signal from its samples.

1-4. Which of the following is ***false*** about aliasing?

(a) The phenomenon of aliasing has a clear meaning in the time domain.

(b) The minimum sampling rate that avoids aliasing in a bandlimited signal.

(c) Under-sampling produces aliasing when reconstructed fall within the Nyquist bandwidth.

(d) A bandpass signal $x_c(t)$ can be reconstructed from its samples without aliasing using a sampling rate in the range $2B \le F_s \le 4B$, where $B =$ ~~Nyquist rate $2F_H$~~. $F_H - F_L$

(e) Perfect reconstruction of a bandlimited image $s_c(x, y)$ from a set of samples $s_c(m\Delta x, n\Delta y)$ without aliasing is possible.

1-5. Which of the following is ***false*** about Discrete Fourier Transform (DFT)?

(a) The DFT is an ~~infinite~~ finite orthogonal transform which provides a unique representation of N consecutive samples x[n].

(b) The DFT does not provide any information about the unavailable samples of the sequence.

(c) The multiplication of two N-point DFTs is equivalent to the circular convolution of the corresponding N-point sequences.

(d) The DFT provides samples of the DTFT of the sequence at a set of equally spaced frequencies.

(e) Appending zeros to a sequence prior to taking its DFT which results in a densely sampled DTFT spectrum.

2. **(15 pts.)** In Figure q2, $H(z)$ is the system function of a causal LTI system.

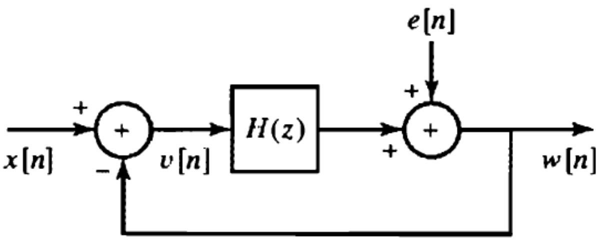


Figure q2

(a) Using z-transform of the signals shown in Figure q2, obtain an expression for $W(z)$ in the form

$$W(z) = H_1(z)X(z) + H_2(z)E(z)$$

(b) For the special case $H(z) = \frac{z^{-1}}{1-z^{-1}}$, determine $H_1(z)$ and $H_2(z)$.

(c) Is the system $H(z)$ stable? Are the systems $H_1(z)$ and $H_2(z)$ stable?

Answer:

(a) After writing the following equalities:

$$V(z) = X(z) - W(z)$$
$$W(z) = V(z)H(z) + E(z)$$

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we solve for W(z):

$$W(z) = \frac{H(z)}{1+H(z)}X(z) + \frac{1}{1+H(z)}E(z)$$

(b)

$$H_1(z) = \frac{H(z)}{1+H(z)} = \frac{\frac{z^{-1}}{1-z^{-1}}}{1+\frac{z^{-1}}{1-z^{-1}}} = z^{-1}$$

$$H_2(z) = \frac{1}{1+\frac{z^{-1}}{1-z^{-1}}} = 1 - z^{-1}$$

錯一個-3

(c) $H(z)$ is not stable due to its pole at $z = 1$, but $H_1(z)$ and $H_2(z)$ are.

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3. **(10 pts.)** Consider the discrete-time system shown in Figure q3-1. Where

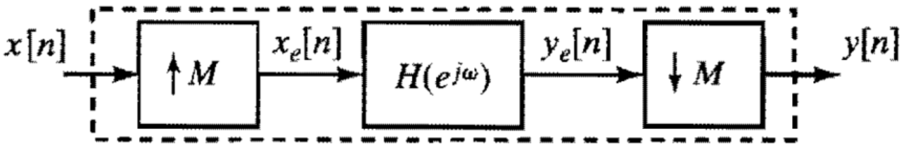


Figure q3-1

(i) M is a n integer.

(ii) $x_e[n] = \begin{cases} x[n/M] & n = kM,\ k \\ 0 & \text{otherwisw.} \end{cases}$ is any integer

(iii) $y[n] = y_e[nM]$.

(iv) $H(e^{j\omega}) = \begin{cases} M & |\omega| \le \frac{\pi}{4} \\ 0 & \frac{\pi}{4} < |\omega| \le \pi. \end{cases}$

(a) We had exercised the assumption of $M = 2$ in Quiz 2. Now, assume that $M = 6$ and $X(e^{j\omega})$, the DTFT of $x[n]$, is real and is as shown in Figure q3-2. When $M = 6$, we have $X_e(e^{j\omega}) = X(e^{j6\omega})$. Then, determine $\boldsymbol{Y_e(e^{j\omega})}$ and $\boldsymbol{Y(e^{j\omega})}$. Be sure to clearly marked with range and the values must be simplified.

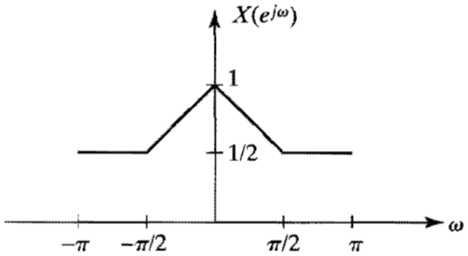
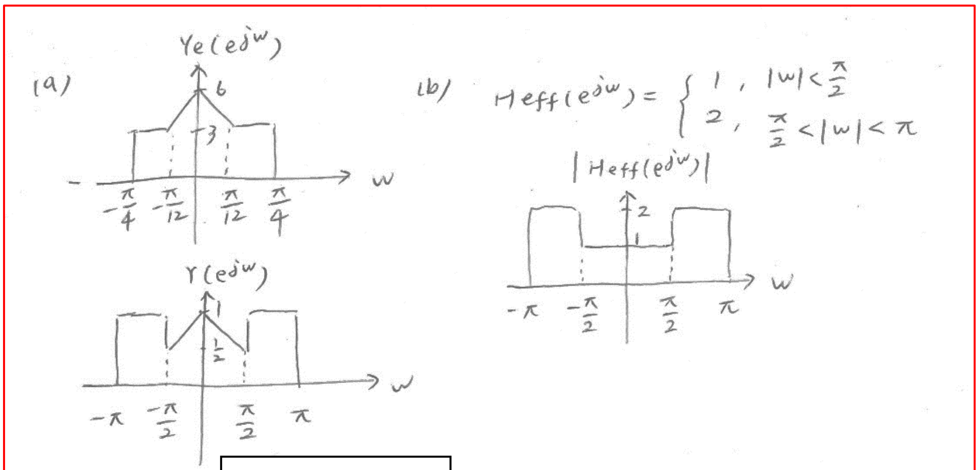


Figure q3-2

(b) For $M = 6$, the overall system is LTI. Determine and sketch the magnitude of the frequency response of the overall system $|H_{eff}(e^{j\omega})|$.

Answer:



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4. **(10 pts.)** Consider the system in Figure q4-1, and answer the following questions.

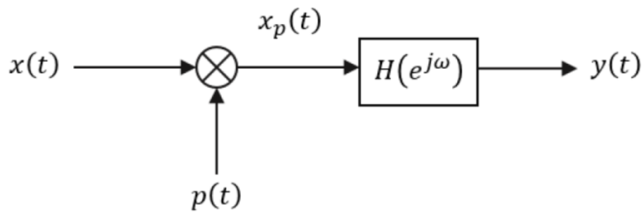


Figure q4-1

The periodic sampling signal $p(t)$, the Fourier Transform of the input signal $x(t)$ are shown:

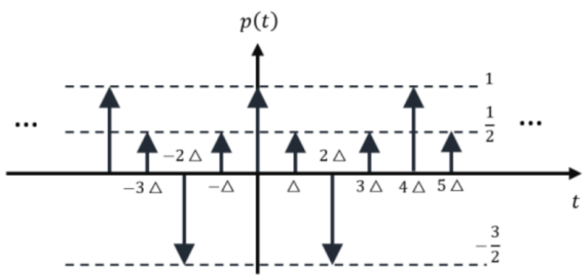


Figure q4-2

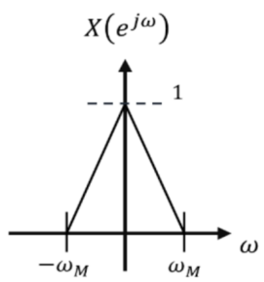


Figure q4-3

(a) To derive the Fourier transform of a continuous signal sampled by impulse train, we first consider the Fourier transform of a simple impulse train $p_{tr}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$ with period T_s, which is known as: $F\{p_{tr}(t)\} = \sum_{k=-\infty}^{\infty} 2\pi C_k \delta(\omega - k\omega_0)$, where C_k are the Fourier Series coefficients and $\omega_0 = 2\pi/T_s$. Show that the formula is true. (Detailed description for each step of the proof is needed.)

(b) For $\Delta < \frac{\pi}{4\omega_M}$, find out the Fourier transform of $p(t)$.

Answer:

(a)

Since $p_{tr}(t)$ is periodic, we can write $p_{tr}(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$ (Fourier Series)

Also, we derive the Fourier transform of $e^{jk\omega_0 t}$:

$$\because F^{-1}\{\delta(\omega - k\omega_0)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - k\omega_0) e^{j\omega t} d\omega = \frac{1}{2\pi} e^{jk\omega_0 t}$$

$$\therefore \mathrm{F}\left\{\frac{1}{2\pi} e^{jk\omega_0 t}\right\} = \delta(\omega - k\omega_0) \Rightarrow \mathrm{F}\{e^{jk\omega_0 t}\} = 2\pi\delta(\omega - k\omega_0)$$

Combine the above results:

$$\mathrm{F}\{p_{tr}(t)\} = \mathrm{F}\left\{\sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}\right\} = \sum_{k=-\infty}^{\infty} C_k\, \mathrm{F}\{e^{jk\omega_0 t}\} = \sum_{k=-\infty}^{\infty} C_k\, 2\pi\delta(\omega - k\omega_0)$$

(b)

Sampling period: $T_s = 4\triangle \Longrightarrow \omega_0 = \frac{2\pi}{T_s} = \frac{\pi}{2\triangle}$

By part (a), we know that $P(\omega) = \sum_{k=-\infty}^{\infty} 2\pi C_k \delta(\omega - k\omega_0)$

$$C_k = \frac{1}{T_s}\int_{T_s} p(t) e^{-jk\omega_0 t} dt = \frac{1}{4\triangle}\left(1 + \frac{1}{2} e^{-jk\frac{\pi}{2\triangle}\triangle} - \frac{3}{2} e^{-jk\frac{\pi}{2\triangle}2\triangle} + \frac{1}{2} e^{-jk\frac{\pi}{2\triangle}3\triangle}\right)$$

$$= \frac{1}{4\triangle}\left(1 - \frac{3}{2} e^{-jk\pi} + \frac{1}{2}\left(e^{-jk\frac{\pi}{2}} + e^{-jk\frac{3\pi}{2}}\right)\right)$$

$$= \frac{1}{4\triangle}\left(1 - \frac{3}{2}(-1)^k + \cos\left(\frac{k\pi}{2}\right)\right)$$

$$P(\omega) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{4\triangle}\left(1 \boxed{-\frac{3}{2}(-1)^k} + \cos\left(\frac{k\pi}{2}\right)\right)\delta(\omega - k\omega_0)$$

5. **(10 pts.)** Let $x[n] = 0.8^n u[n] + (-0.8)^n u[n]$.

(a) Determine the DTFT $\hat{X}(e^{j\omega})$

(b) Let $G[k] = \hat{X}\left(e^{j\frac{2\pi k}{10}}\right)$. Determine $g[n]$ without computing the IDFT.

(a) Solution:
The DTFT $\tilde{X}(e^{j\omega})$ is:

$$\tilde{X}(e^{j\omega}) = \frac{1}{1-0.8e^{-j\omega}} + \frac{1}{1+0.8e^{-j\omega}} = \frac{2}{1-0.8^2e^{-2j\omega}}$$

(b) Solution:
$\omega = \frac{2\pi}{10}k$, $N = 10$, we can conclude $g[n]$ as

$$g[n] = \sum_{\ell=-\infty}^{\infty} x[n-10\ell]$$

6. **(10 pts.)** Determine the system function, magnitude response, and phase response of the following system.

$$y[n] = \frac{1}{4}(x[n] - x[n-1]) + \frac{1}{4}(x[n-3] - x[n-4]).$$

System function: 2pts, magnitude response: 5pts, phase response: 3pts

The system function is:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{4}(1 - z^{-1} + z^{-3} - z^{-4})$$

The frequency response is:

$$\begin{aligned} H(e^{j\omega}) &= \frac{1}{4}(1 - e^{-j\omega} + e^{-3j\omega} - e^{-4j\omega}) \\ &= \frac{1}{4}[(1 - \cos\omega + \cos 3\omega - \cos 4\omega) - j(\sin\omega - \sin 3\omega + \sin 4\omega)] \end{aligned}$$

The magnitude response is:

$$\begin{aligned} |H(e^{j\omega})| &= \frac{1}{4}\sqrt{(1 - \cos\omega + \cos 3\omega - \cos 4\omega)^2 + (\sin\omega - \sin 3\omega + \sin 4\omega)^2} \\ &= \frac{1}{4}\sqrt{4 - 4\cos\omega - 2\cos 2\omega + 4\cos 3\omega - 2\cos 4\omega} \end{aligned}$$

The phase response is:

$$\angle H(e^{j\omega}) = \tan^{-1}\frac{\sin\omega - \sin 3\omega + \sin 4\omega}{1 - \cos\omega + \cos 3\omega - \cos 4\omega}$$

7. **(5 pts.)** Determine whether or not the following signals is periodic. If a signal is periodic, determine its fundamental period: $x[n] = |\cos(0.1\pi n)| + \sin(2\pi n/11)$

Solution:

$$N_1 = \frac{2\pi}{0.1\pi} \times \frac{1}{2} = 10, \quad N_2 = \frac{2\pi}{2\pi/11} = 11, \quad N = 110$$

$x_5[n]$ is periodic with fundamental period $N = 110$.

8. **(15 pts.)** Suppose we have two 4-point sequence $x[n]$ and $h[n]$ as follows:

$$x[n] = \cos\left(\frac{\pi n}{2}\right), \qquad n = 0,1,2,3.$$

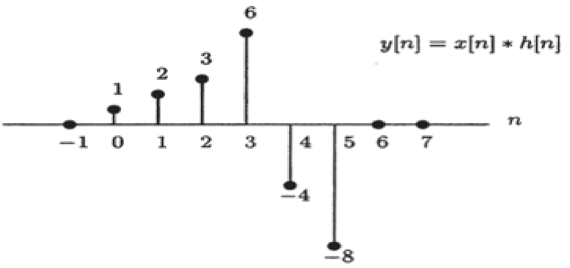
$$h[n] = 2^n, \qquad n = 0,1,2,3.$$

(a) Calculate $y[n] = x[n] \circledast_4 h[n]$ by doing the circular convolution directly.[7pts]

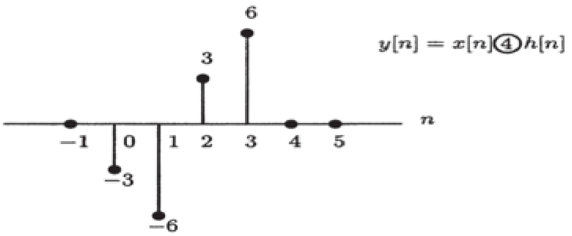
(b) Calculate $y[n]$ of part (a) by multiplying the DFTs of $x[n]$ and $h[n]$ and performing an inverse DFT.[8pts]

Solution: (a)

Remember, circular convolution equals linear convolution plus aliasing. We need $N \geq 3+4-1=6$ to avoid aliasing. Since $N=4$,we expect to get aliasing here. First, find $y[n] = x[n] * h[n]$:



For this problem, aliasing means the last three points ($n = 4,5,6$) will wrap-around on top of the first three points, giving $y[n] = x[n] \circledast_4 h[n]$:



(b)

Using the DFT values we calculated in parts (a) and (b):

$$\begin{aligned} Y[k] &= X[k]H[k] \\ &= 1 + 2W_4^k + 4W_4^{2k} + 8W_4^{3k} - W_4^{2k} - 2W_4^{3k} - 4W_4^{4k} - 8W_4^{5k} \end{aligned}$$

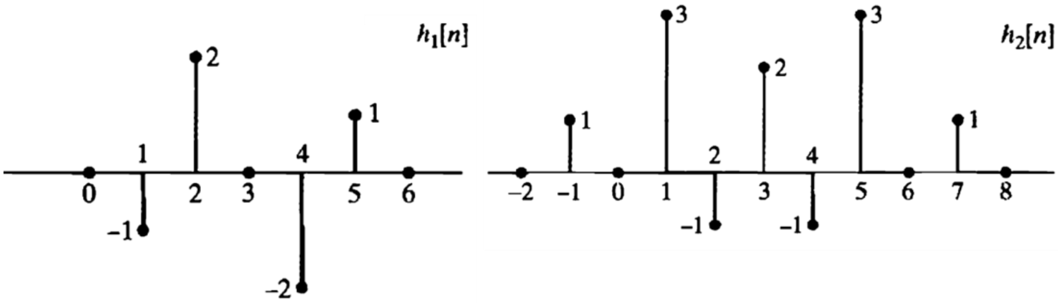
Since $W_4^{4k} = W_4^{0k}$ and $W_4^{5k} = W_4^k$

$$Y[k] = -3 - 6W_4^k + 3W_4^{2k} + 6W_4^{3k}, \qquad 0 \leq k \leq 3$$

Taking the inverse DFT:

$$y[n] = -3\delta[n] - 6\delta[n-1] + 3\delta[n-2] + 6\delta[n-3], \qquad 0 \leq n \leq 3$$

9. **(10 pts.)** Below figures show the impulse response for several different LTI systems. Determine the group delay associated with each system.



Due to the symmetry of the impulse responses, all the systems have generalized linear phase of $\arg[H(e^{j\omega})] = \beta - n_o\omega$ where n_o is the point of symmetry in the impulse response graphs. The group delay is

$$\text{grd}\left[H_i(e^{j\omega})\right] = -\frac{d}{d\omega}\left\{\arg\left[H_i(e^{j\omega})\right]\right\} = -\frac{d}{d\omega}\left\{\beta - n_o\omega\right\} = n_o$$

To find each system's group delay we need only find the point of symmetry n_o in each system's impulse response.

$grd(H_1(e^{j\omega}))$= 3

$grd(H_2(e^{j\omega}))$= 3