

**Homework Assignment #3: Chap. 5-6**  
**Solution**

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**I Paper Assignment (74%)**

1. (6%) Determine the system function, magnitude response, and phase response of the following systems and use the pole-zero pattern to explain the shape of their magnitude response:

(a)  $y[n] = \frac{1}{4}(x[n] + x[n - 1]) - \frac{1}{4}(x[n - 2] + x[n - 3])$

(b)  $y[n] = x[n] - x[n - 4] + 0.6561y[n - 4]$

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(a)

System function:  $H(z) = \frac{Y(z)}{X(z)} = \frac{1}{4}(1 + z^{-1} - z^{-2} - z^{-3})$

Frequency response:

$$H(e^{j\omega}) = \frac{1}{4}[(1 + \cos \omega - \cos 2\omega - \cos 3\omega) + j(-\sin \omega + \sin 2\omega + \sin 3\omega)]$$

Magnitude response:

$$|H(e^{j\omega})| = \frac{1}{4} \sqrt{(1 + \cos \omega - \cos 2\omega - \cos 3\omega)^2 + (-\sin \omega + \sin 2\omega + \sin 3\omega)^2}$$

Phase response:

$$\angle H(e^{j\omega}) = \tan^{-1} \left( \frac{-\sin \omega + \sin 2\omega + \sin 3\omega}{1 + \cos \omega - \cos 2\omega - \cos 3\omega} \right)$$

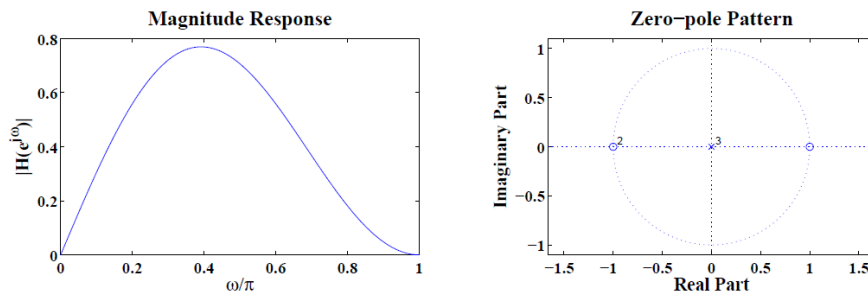


FIGURE 5.20: Magnitude response and pole-zero plot of  $y[n] = \frac{1}{4}(x[n] + x[n - 1]) - \frac{1}{4}(x[n - 2] + x[n - 3])$ .

(b)

System function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-z^{-4}}{1-0.6561z^{-4}} = \frac{(1+z^{-2})(1+z^{-1})(1-z^{-1})}{(1+0.81z^{-2})(1+0.9z^{-1})(1-0.9z^{-1})} = \frac{(z+i)(z-i)(z+1)(z-1)}{(z+0.9i)(z-0.9i)(z+0.9)(z-0.9)}$$

Magnitude response:

$$|H(e^{j\omega})| = \frac{\sqrt{(1-\cos 4\omega)^2 + (\sin 4\omega)^2}}{\sqrt{(1-0.6561 \cos 4\omega)^2 + (0.6561 \sin 4\omega)^2}}$$

Phase response:

$$\angle H(e^{j\omega}) = \tan^{-1}\left(\frac{\sin 4\omega}{1-\cos 4\omega}\right) - \tan^{-1}\left(\frac{0.6561 \sin 4\omega}{1-0.6561 \cos 4\omega}\right)$$

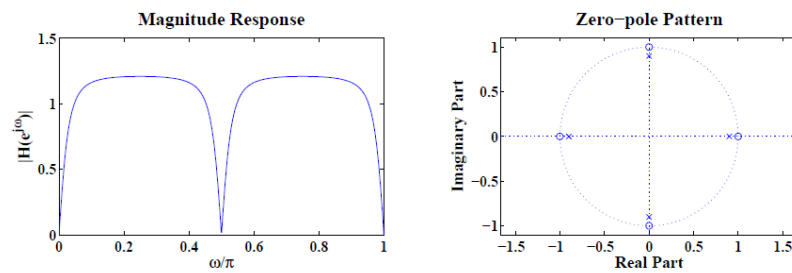


FIGURE 5.137: (a) Magnitude response (b) Pole-zero pattern.

2. (12%) Consider a periodic signal

$$x[n] = \sin(0.1\pi n) + \frac{1}{3} \sin(0.3\pi n) + \frac{1}{5} \sin(0.5\pi n)$$

For each of the following systems, determine if the system imparts (i) no distortion, (ii) magnitude distortion, and/or (iii) phase (or delay) distortion.

(a)  $h[n] = \{1_{n=0}, -2, 3, -4, 0, 4, -3, 2, -1\}$

(b)  $y[n] = 10x[n - 10]$

No distortion means: constant gain, linear phase

(a) Magnitude distortion, Phase distortion

$$\begin{aligned} H(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} \\ &= (1 - 2 \cos \omega + 3 \cos 2\omega - 4 \cos 3\omega + 4 \cos 5\omega - 3 \cos 6\omega + 2 \cos 7\omega - \cos 8\omega) \\ &\quad + j(2 \sin \omega - 3 \sin 2\omega + 4 \sin 3\omega - 4 \sin 5\omega + 3 \sin 6\omega - 2 \sin 7\omega + \sin 8\omega) \end{aligned}$$

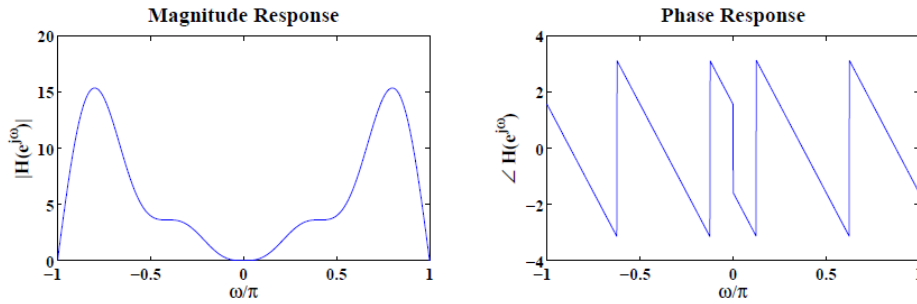


FIGURE 5.67: Magnitude and phase responses of the system.

(b) No distortion

$$H(e^{j\omega}) = 10e^{-10j\omega} \quad |H(e^{j\omega})| = 10, \quad \angle H(e^{j\omega}) = -10\omega$$

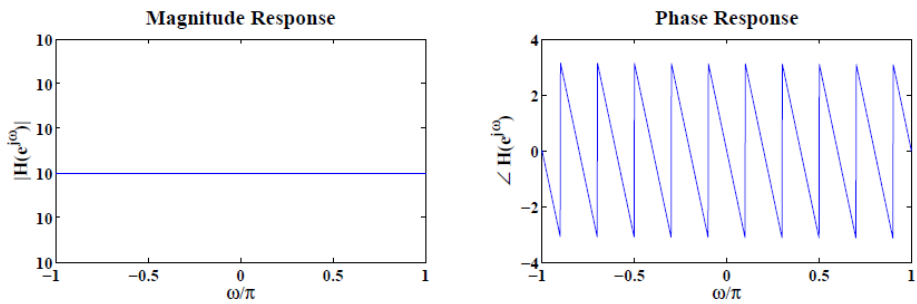


FIGURE 5.69: Magnitude and phase responses of the system.

3. (12%) An economical way to compensate for the droop distortion in S/H DAC is to use an appropriate digital compensation filter prior to DAC.

(a) Determine the frequency response of such an ideal digital filter  $H_r(e^{j\omega})$  that will perform an equivalent filtering given by following  $H_r(j\Omega)$

$$H_r(j\Omega) = \begin{cases} \frac{\Omega T/2}{\sin(\Omega T/2)} e^{j\Omega T/2}, & |\Omega| < \pi/T \\ 0, & \text{otherwise} \end{cases}$$

(b) One low-order FIR filter suggested in Jackson (1996) is

$$H_{FIR}(z) = -\frac{1}{16} + \frac{9}{8}z^{-1} - \frac{1}{16}z^{-2}$$

Compare the magnitude response of  $H_{FIR}(e^{j\omega})$  with that of  $H_r(e^{j\omega})$  above.

(c) Another low-order IIR filter suggested in Jackson (1996) is

$$H_{IIR}(z) = \frac{9}{8 + z^{-1}}$$

Compare the magnitude response of  $H_{IIR}(e^{j\omega})$  with that of  $H_r(e^{j\omega})$  above.

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(a) Solution:

$$g_{SH}(t) = \begin{cases} 1, & 0 \leq t \leq T \\ 0, & \text{otherwise} \end{cases} \xrightarrow{\text{CTFT}} G_{SH}(j\Omega) = \frac{2 \sin(\Omega T/2)}{\Omega} e^{-j\Omega T/2}$$

$$H_r(j\Omega) = \begin{cases} \frac{\Omega T/2}{\sin(\Omega T/2)} \cdot e^{j\Omega T/2}, & |\Omega| < \pi/T \\ 0, & \text{otherwise} \end{cases}$$

The frequency response is:

$$H_r(e^{j\omega}) = \begin{cases} \frac{\omega/2}{\sin(\omega/2)} e^{j\omega/2}, & |\omega| < \pi \\ 0, & \text{otherwise} \end{cases}$$

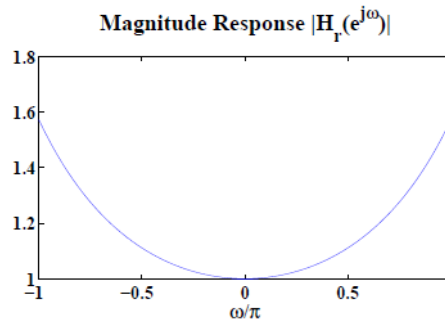


FIGURE 6.21: Magnitude response of ideal digital filter  $H_r(e^{j\omega})$ .

(b) Solution: The magnitude response of  $H_{FIR}(e^{j\omega})$  is:

$$|H_{FIR}(e^{j\omega})| = \sqrt{\left(-\frac{1}{16} + \frac{9}{8} \cos \omega - \frac{1}{16} \cos 2\omega\right)^2 + \left(-\frac{9}{8} \sin \omega + \frac{1}{16} \sin 2\omega\right)^2}$$

$$= \sqrt{\frac{1}{16^2} + \frac{9^2}{8^2} + \frac{1}{16^2} - 4 \times \frac{9}{8} \times \frac{1}{16} \cos \omega + \frac{2}{16^2} \cos 2\omega}$$

(c) Solution: The magnitude response of  $H_{IIR}(e^{j\omega})$  is:

$$|H_{IIR}(e^{j\omega})| = \frac{9}{\sqrt{(8 + \cos \omega)^2 + \sin^2 \omega}} = \frac{9}{\sqrt{1 + 8^2 + 2 \cos \omega}}$$

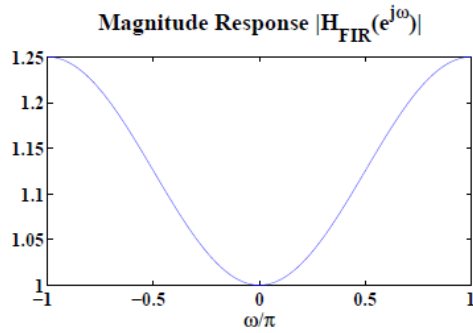


FIGURE 6.22: Magnitude response of low-order FIR filter  $H_{\text{FIR}}(e^{j\omega})$ .

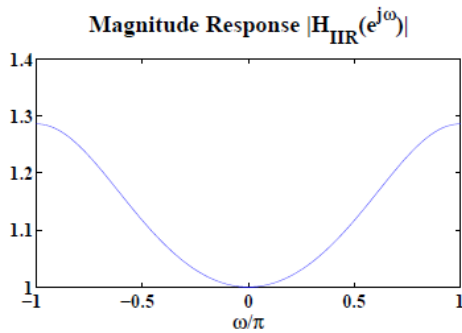


FIGURE 6.23: Magnitude response of low-order IIR filter  $H_{\text{IIR}}(e^{j\omega})$ .

4. (12%) Consider the following continuous-time system

$$H(s) = \frac{s^4 - 6s^3 + 10s^2 + 2s - 15}{s^5 + 15s^4 + 100s^3 + 370s^2 + 744s + 720}$$

- Show that the system  $H(s)$  is a nonminimum phase system.
  - Decompose  $H(s)$  into the product of minimum phase component  $H_{\text{min}}(s)$  and an all pass component  $H_{\text{ap}}(s)$ .
  - Briefly plot the magnitude and phase responses of  $H(s)$  and  $H_{\text{min}}(s)$  in one figure and explain your plots.
  - Briefly plot the magnitude and phase responses of  $H_{\text{ap}}(s)$ .
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(a) Proof:

$$H(s) = \frac{(s - 3)(s - 2 - j)(s - 2 + j)(s + 1)}{(s + 5)(s + 3 - 3j)(s + 3 + 3j)(s + 2 - 2j)(s + 2 + 2j)}$$

Hence, there are three zero on the right-hand plane which proved that the system  $H(s)$  is NOT minimum phase system.

(b) Solution:

$$H(s) = H_{\min}(s) \cdot H_{\text{ap}}(s)$$

$$H_{\min}(s) = \frac{(s + 3)(s + 2 - j)(s + 2 + j)(s + 1)}{(s + 5)(s + 3 - 3j)(s + 3 + 3j)(s + 2 - 2j)(s + 2 + 2j)}$$

$$H_{\text{ap}}(s) = \frac{(s - 3)(s - 2 - j)(s - 2 + j)}{(s + 3)(s + 2 - j)(s + 2 + j)}$$

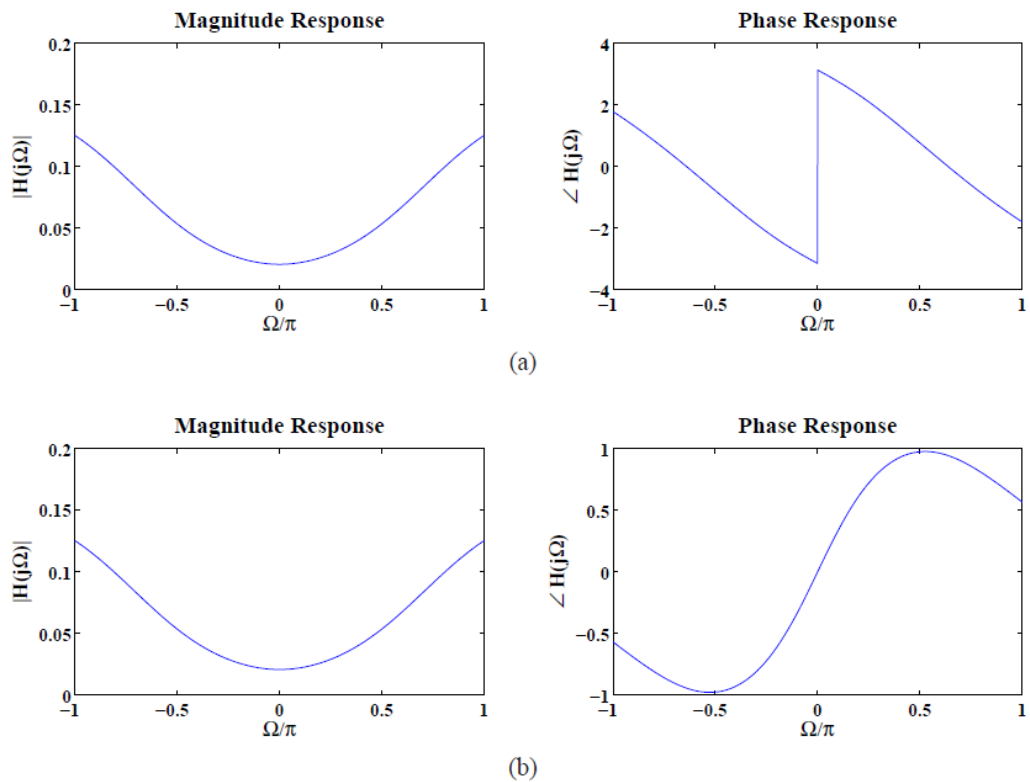


FIGURE 5.47: Magnitude and phase responses of (a)  $H(s)$  and (b)  $H_{\min}(s)$ .

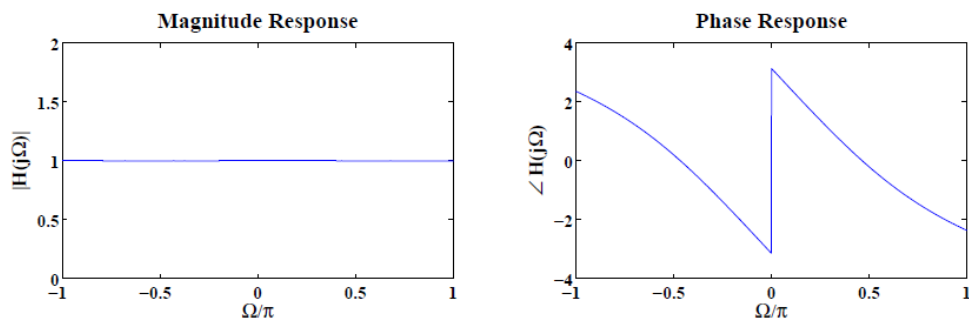


FIGURE 5.48: Magnitude and phase responses of  $H_{ap}(s)$ .

5. (12%) We want to design a second-order IIR filter using pole-zero placement that satisfies the following requirements: (1) the magnitude response is 0 at  $\omega_1 = 0$  and  $\omega_3 = \pi$  (2) The maximum magnitude is 1 at  $\omega_{2,4} = \pm \frac{\pi}{4}$  and (3) the magnitude response is approximately  $\frac{1}{\sqrt{2}}$  at frequencies  $\omega_{2,4} \pm 0.05$
- Determine locations of two poles and two zeros of the required filter and then compute its system function  $H(z)$ .
  - Briefly graph the magnitude response of the filter.
  - Briefly graph phase and group-delay responses.

Determine the response with the definition of Mag/Phase response:

$$|H(e^{j\omega})| = |b_0| \prod_{k=1}^M |1 - z_k e^{-j\omega}| / \prod_{k=1}^N |1 - p_k e^{-j\omega}|,$$

$$\angle H(e^{j\omega}) = \angle b_0 + \sum_{k=1}^M \angle(1 - z_k e^{-j\omega}) - \sum_{k=1}^N \angle(1 - p_k e^{-j\omega}),$$

Or from the Pole/Zero Map

(a) Solution:

$$\text{zeros: } z_1 = e^{j0} = 1, \quad z_2 = e^{j\pi} = -1$$

$$\text{poles: } p_1 = re^{j\frac{\pi}{4}}, \quad p_2 = re^{-j\frac{\pi}{4}}, \quad r \in (0, 1)$$

The system function is:

$$H(z) = b_0 \frac{(1 - z^{-1})(1 + z^{-1})}{(1 - re^{j\frac{\pi}{4}}z^{-1})(1 - re^{-j\frac{\pi}{4}}z^{-1})}$$

The frequency response is:

$$H(e^{j\omega}) = b_0 \frac{(1 - e^{-j\omega})(1 + e^{-j\omega})}{(1 - re^{j\frac{\pi}{4}}e^{-j\omega})(1 - re^{-j\frac{\pi}{4}}e^{-j\omega})}$$

Constrain  $|H(e^{j\omega})|_{\max} = 1$ , we have

$$|b_0| = \frac{(1 - r)\sqrt{1 + r^2}}{\sqrt{2}}$$

Choose  $r = 0.95$ .

(b) See plot below.

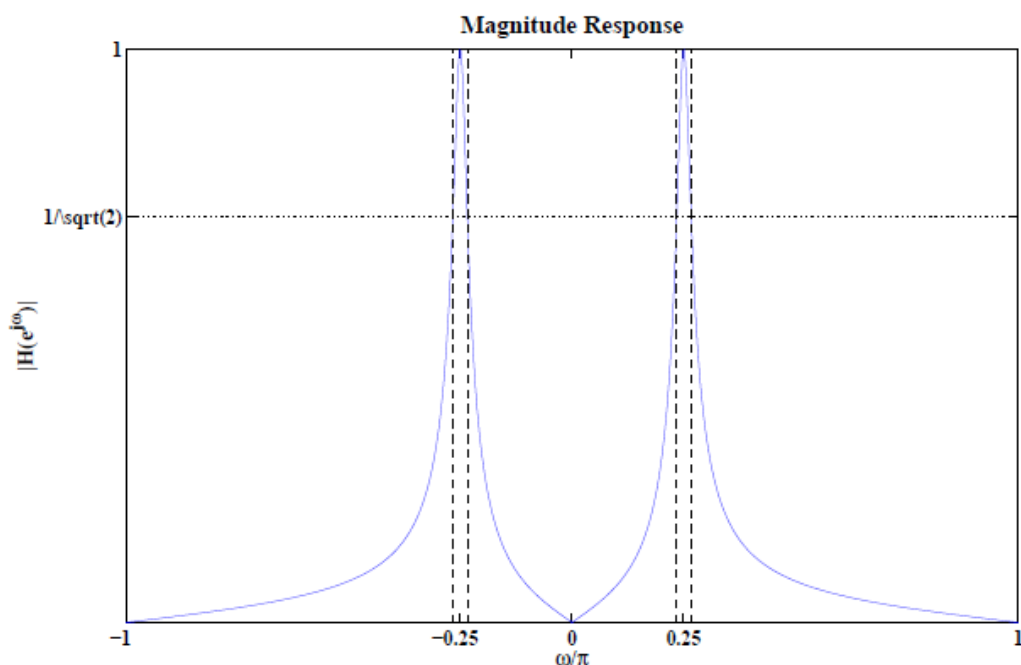


FIGURE 5.96: Magnitude response of the filter.



(c)

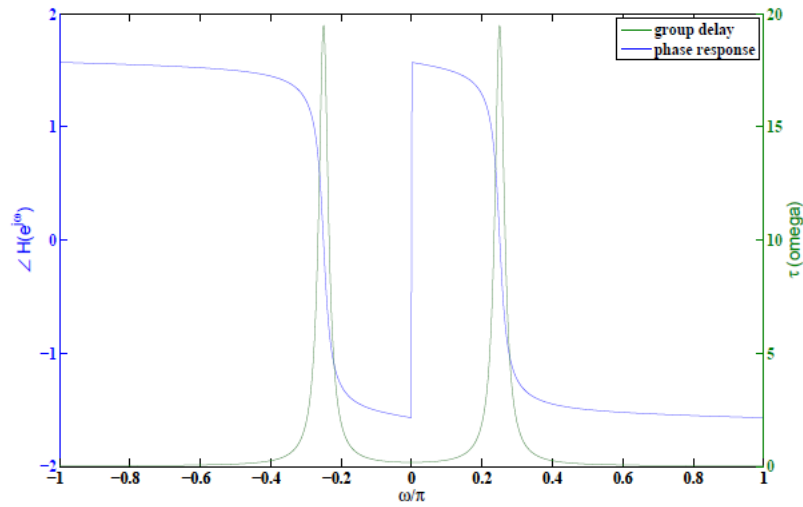


FIGURE 5.97: Phase and group-delay responses of the filter.

6. (10%) The following signals  $x_c(t)$  is sampled periodically to obtain the discrete-time signal  $x[n]$ . For each of the given sampling rates in  $F_s$  Hz or in  $T$  period, (i) determine the spectrum  $X(e^{j\omega})$  of  $x[n]$ ; (ii) plot its magnitude and phase as a function of  $\omega$  in  $\frac{\text{rad}}{\text{sam}}$  and as a function of  $F$  in Hz; and (iii) explain whether  $x_c(t)$  can be recovered from  $x[n]$ .
- (a)  $x_c(t) = 5e^{i40t} + 3e^{-i70t}$ , with sampling period  $T = 0.01, 0.04, 0.1$
- (b)  $x_c(t) = 3 + 2 \sin(16\pi t) + 10 \cos(24\pi t)$ , with sampling rate  $F_s = 30, 20, 15$  Hz.

(a)

$$X(j\Omega) = 10\pi\delta(\Omega - 40) + 6\pi\delta(\Omega + 70)$$

The spectra of the sampled sequence  $x[n]$  is:

$$X(e^{j\omega})|_{\omega=\Omega T} = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_r[j(\Omega - k\Omega_s)]$$

The continuous signal  $x_c(t)$  can be recovered if the sampling interval is (a)  $T = 0.01$ , (b)  $T = 0.04$ , and can NOT be recovered if the sampling interval is (c)  $T = 0.1$ .

$$T=0.01, \quad \Omega_s = 2\pi F_s = 200\pi, \quad \omega_s = \Omega_s/100$$

$$X(e^{j\omega}) = 100 \sum_{k=-\infty}^{\infty} 10\pi\delta(\Omega - 40 - 200\pi k) + 6\pi\delta(\Omega + 70 - 200\pi k)$$

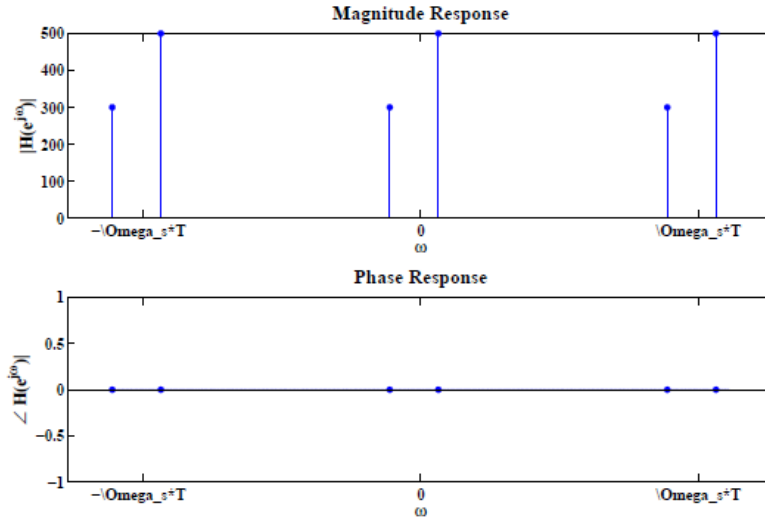


FIGURE 6.30: Magnitude and phase responses of  $X(e^{j\omega})$  as a function of  $\omega$  in  $\frac{\text{rad}}{\text{sam}}$  when the sampling interval is  $T = 0.01$ .

$$T=0.04, \quad \Omega_s = 2\pi F_s = 50\pi, \quad \omega_s = \Omega_s/25$$

$$X(e^{j\omega}) = 25 \sum_{k=-\infty}^{\infty} 10\pi\delta(\Omega - 40 - 50\pi k) + 6\pi\delta(\Omega + 70 - 50\pi k)$$

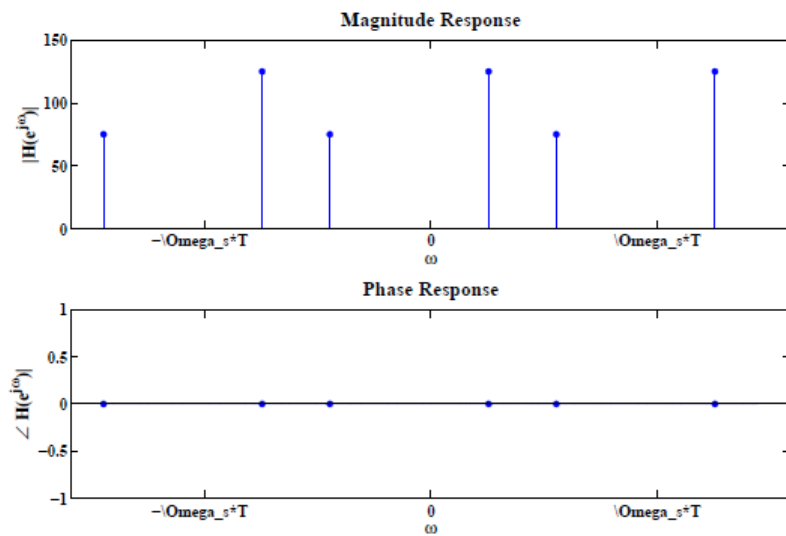


FIGURE 6.31: Magnitude and phase responses of  $X(e^{j\omega})$  as a function of  $\omega$  in  $\frac{\text{rad}}{\text{sam}}$  when the sampling interval is  $T = 0.04$ .

$$T=0.1, \quad \Omega_s = 2\pi F_s = 20\pi, \quad \omega_s = \Omega_s/10$$

$$X(e^{j\omega}) = 10 \sum_{k=-\infty}^{\infty} 10\pi\delta(\Omega - 40 - 20\pi k) + 6\pi\delta(\Omega + 70 - 20\pi k)$$

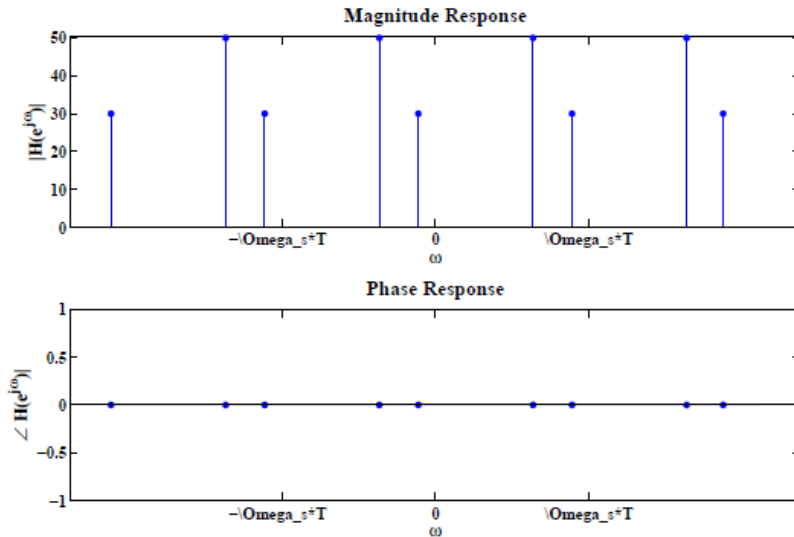


FIGURE 6.32: Magnitude and phase responses of  $X(e^{j\omega})$  as a function of  $\omega$  in  $\frac{\text{rad}}{\text{sam}}$  when the sampling interval is  $T = 0.1$ .

(b)

$$X_c(j\Omega) = 6\pi + \frac{1}{j}2\pi\delta(\Omega - 16\pi) - \frac{1}{j}2\pi\delta(\Omega + 16\pi) + 10\pi\delta(\Omega - 24\pi) + 10\pi\delta(\Omega + 24\pi)$$

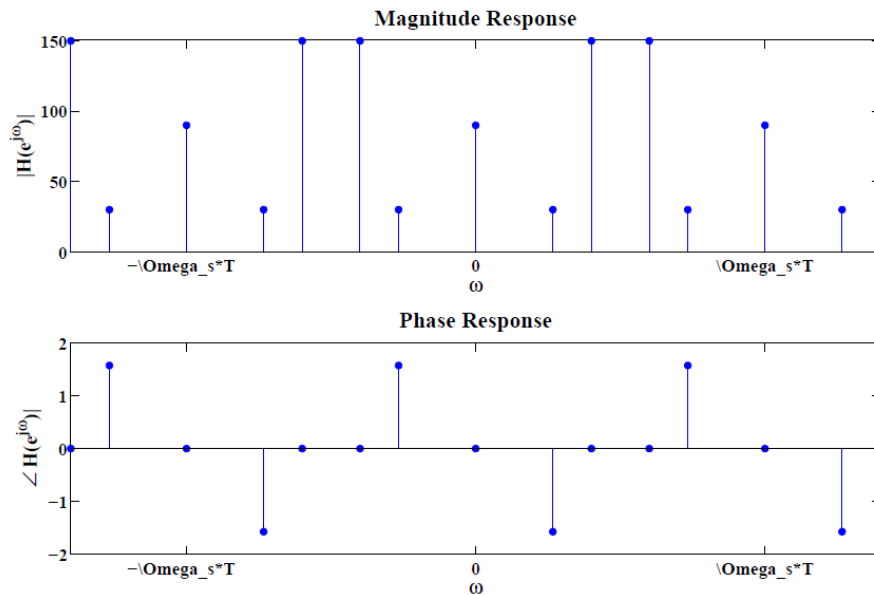
The spectra of the sampled sequence  $x[n]$  is:

$$X(e^{j\omega})\big|_{\omega=2\pi F/F_s} = F_s \sum_{k=-\infty}^{\infty} X_c[j2\pi(F - kF_s)]$$

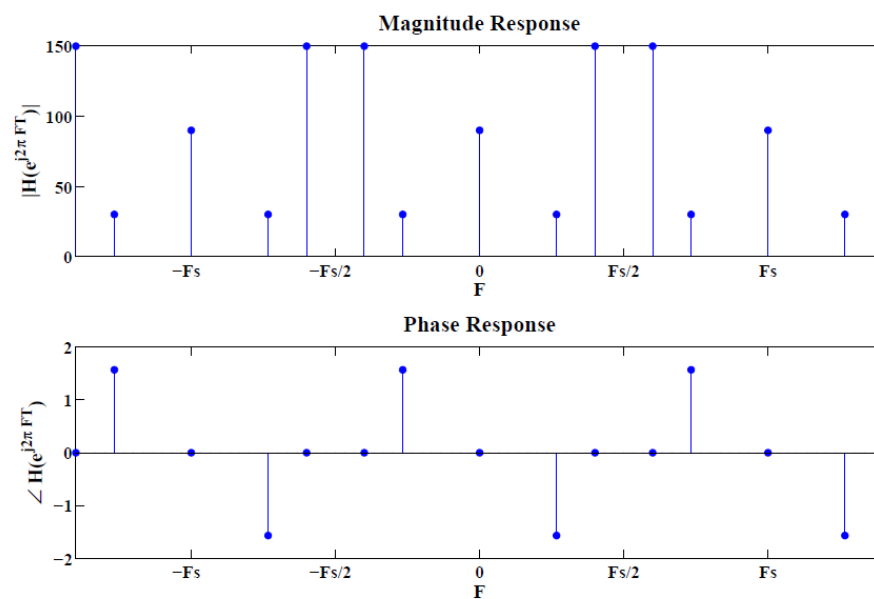
The signal  $x_c(t)$  can be recovered from  $x[n]$  if the sampling rate is (a)  $F_s = 30$  Hz, and can NOT be recovered if the sampling rate is (b)  $F_s = 20$  Hz, (c)  $F_s = 15$  Hz.

$F_s = 30\text{KHz}$

$$X(e^{j\omega}) = 30 \sum_{k=-\infty}^{\infty} X_c(j\Omega - j60\pi k)$$



(a)



(b)

FIGURE 6.48: Spectra of  $X(e^{j\omega})$  as a function of (a)  $\omega$  in  $\frac{\text{rad}}{\text{sam}}$  and (b)  $F$  in Hz when the sample rate is  $F_s = 30\text{KHz}$ .

$F_s = 20\text{KHz}$

$$X(e^{j\omega}) = 20 \sum_{k=-\infty}^{\infty} X_c(j\Omega - j40\pi k)$$

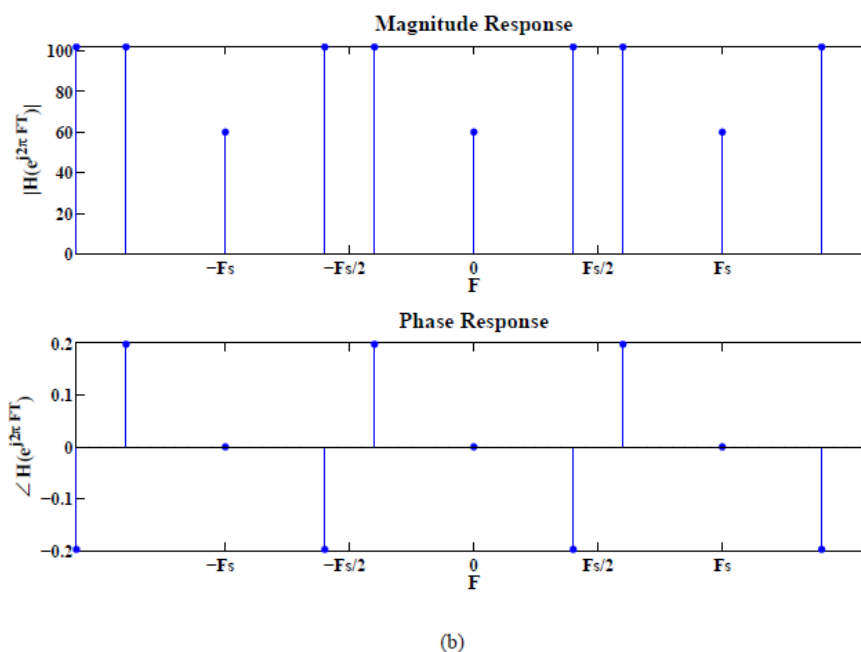
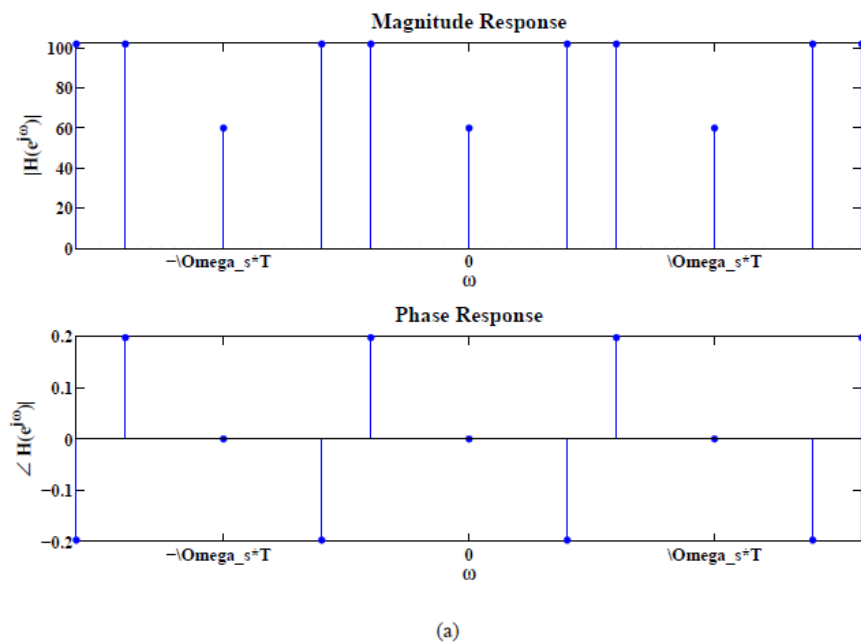


FIGURE 6.49: Spectra of  $X(e^{j\omega})$  as a function of (a)  $\omega$  in  $\frac{\text{rad}}{\text{sam}}$  and (b)  $F$  in Hz when the sample rate is  $F_s = 20\text{ KHz}$ .

$F_s = 15\text{KHz}$

$$X(e^{j\omega}) = 15 \sum_{k=-\infty}^{\infty} X_c(j\Omega - j30\pi k)$$

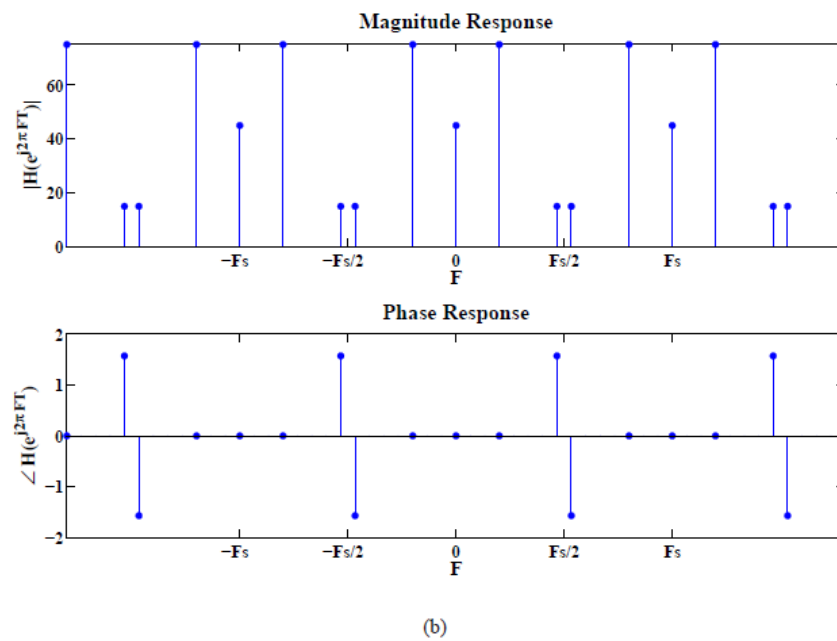
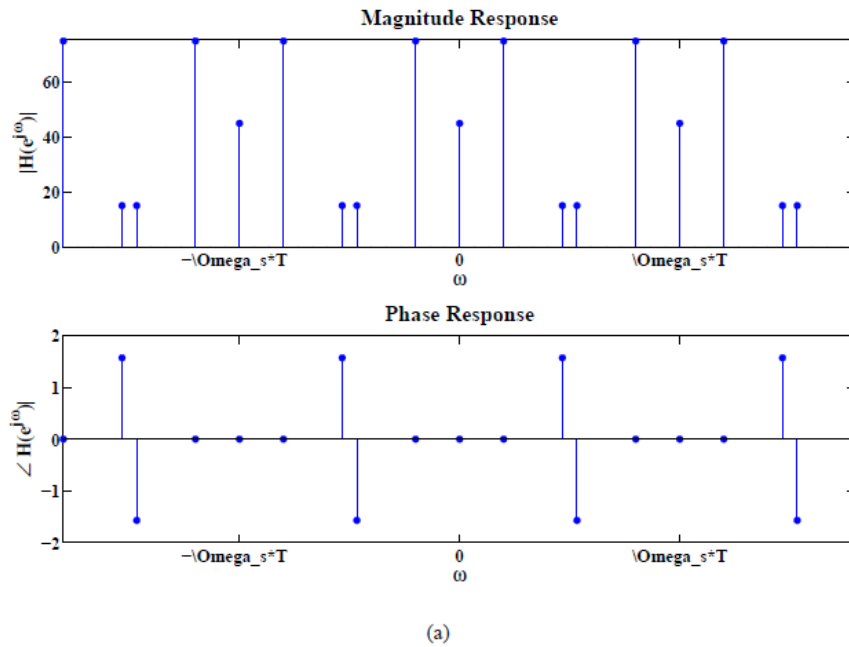


FIGURE 6.50: Spectra of  $X(e^{j\omega})$  as a function of (a)  $\omega$  in  $\frac{\text{rad}}{\text{sam}}$  and (b)  $F$  in Hz when the sample rate is  $F_s = 15\text{KHz}$ .

7. (8%) An 8-bit ADC has an input analog range of  $\pm 5$  volts. The analog input signal is

$$x_c(t) = 2 \cos(200\pi t) + 3 \sin(500\pi t)$$

The converter supplies data to a computer at a rate of 2048 bits/s. The computer, without processing, supplies these data to an ideal DAC to form the reconstructed signal  $y_c(t)$ . Determine:

- (a) the quantizer resolution (or step),  
 (b) the SQNR in dB,  
 (c) the folding frequency and the Nyquist rate,

- (a) Solution:

The quantizer resolution is:

$$\frac{10\text{v}}{2^8} = 0.0390625\text{v}$$

- (b) Solution:

$$e(t) = \frac{\Delta}{2} t$$

$$P_Q = \frac{1}{\Delta} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} e^2(t) dt = \frac{\Delta^2}{12} = \frac{0.039^2}{12} = 1.2675 * 10^{-4}$$

$$x_c(t) = 2 \cos(200\pi t) + 3 \sin(500\pi t)$$

$$P_S = \frac{1}{T} \int_0^T x_c^2(t) dt = 6.5$$

$$\text{SQNR} = \frac{P_S}{P_Q} = \frac{6.5}{1.2675 * 10^{-4}} = 5.128 * 10^4$$

$$\text{SQNR in dB} = 10 \log_{10}(5.128 * 10^4) = 47.09$$

- (c) Solution:

The sampling rate is:

$$F_s = \frac{2^{11}}{2^3} = 2^8 \text{ sam/sec}$$

The folding frequency is  $F_s/2 = 2^7$ .

The Nyquist rate is 500.