(Z-transform)

Linearity	$a_1x_1[n]+a_2x_2[n]$	$a_1X_1(z)+a_2X_2(z)$	At least $R_{x_1}\bigcap R_{x_2}$
Time shifting	$x[n-k]$	$z^{-k}X(z)$	R_x except $z=0$ or ∞
Scaling	$a^n x[n]$	$X(a^{-1}z)$	$\lvert a\rvert R_x$
Differentiation	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x
Conjugation	$x^*[n]$	$X^*(z^*)$	R_x
Real-part	$\mathrm{Re}\{x[n]\}$	$\frac{1}{2}[X(z)+X^*(z^*)]$	At least R_x
Imaginary part	$\mathrm{Im}\{x[n]\}$	$\frac{1}{2}[X(z)-X^*(z^*)]$	At least R_x
Folding	$x[-n]$	$X(1/z)$	$1/R_x$
Convolution	$x_1[n]*x_2[n]$	$X_1(z)X_2(z)$	At least $R_{x_1}\bigcap R_{x_2}$
Initial-value theorem	$x[n]=0$ for $n<0$	$x[0]=\lim_{z\to\infty}X(z)$	

$$X(z)=\sum_{n=-\infty}^{\infty}x[n]z^{-n}$$

$$x[n]=\frac{1}{2\pi j}\oint_C X(z)z^{n-1}\mathrm{d}z$$

$$x[n]=\sum_{k=1}^{N}A_k(p_k)^n \overset{\mathcal{Z}}{\longleftrightarrow} X(z)=\sum_{k=1}^{N}\frac{A_k}{1-p_kz^{-1}}$$

Causality and stability: A LTI system is stable if and only if the ROC of the system function H(z) includes the unit circle |z|=1.

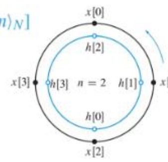
(CTFT) (CTFS) (DTFT) (DTFS) (DFT)

$$y[n]\triangleq h[n]\,ⓝ\,x[n] \qquad y[n]=\sum_{m=0}^{N-1}h[m]x[\langle n-m\rangle_N]$$

$$y[n]=h[n]\,ⓝ\,x[n]\overset{\mathrm{DFT}}{\underset{N}{\longleftrightarrow}}Y[k]=H[k]X[k]$$

$$r_{xy}[\ell]\triangleq\sum_{n=0}^{N-1}x[n]y[\langle n-\ell\rangle_N]$$

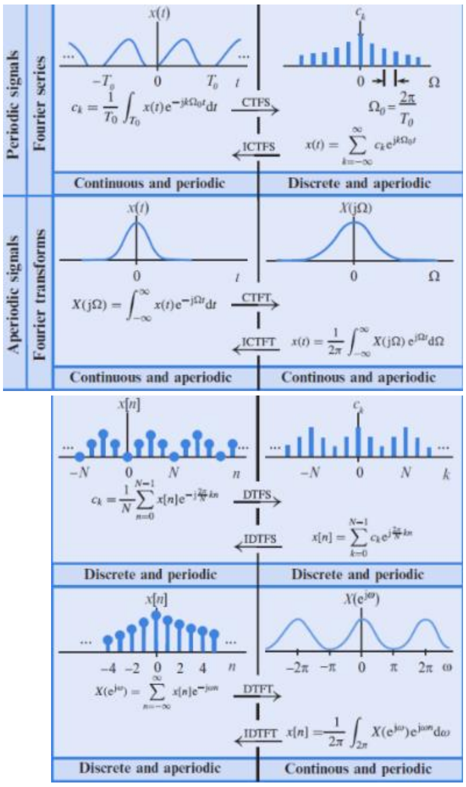
$$r_{xy}[\ell]=x[n]\,ⓝ\,y[\langle -n\rangle_N]\overset{\mathrm{DFT}}{\underset{N}{\longleftrightarrow}}R_{xy}[k]=X[k]Y[\langle -k\rangle_N]$$



$$X[k]=\sum_{n=0}^{N-1}x[n]W_N^{kn}\overset{\mathrm{DFT}}{\underset{N}{\longleftrightarrow}}x[n]=\frac{1}{N}\sum_{k=0}^{N-1}X[k]W_N^{-kn}$$

$$W_N\triangleq \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}}$$

$$2\pi f_0N=2\pi k \qquad f_0=\frac{F_0}{F_s}=\frac{k}{N}=\frac{1/T_0}{1/T}=\frac{T}{T_0}$$



Sequence $x[n]$	Transform $X(e^{j\omega})$
	Complex signals
$x^*[n]$	$X^*(e^{-j\omega})$
$x^*[-n]$	$X^*(e^{j\omega})$
$x_R[n]$	$X_e(e^{j\omega})\triangleq\frac{1}{2}\left[X(e^{j\omega})+X^*(e^{-j\omega})\right]$
$jx_I[n]$	$X_o(e^{j\omega})\triangleq\frac{1}{2}\left[X(e^{j\omega})-X^*(e^{-j\omega})\right]$
$x_e[n]\triangleq\frac{1}{2}(x[n]+x^*[-n])$	$X_R(e^{j\omega})$
$x_o[n]\triangleq\frac{1}{2}(x[n]-x^*[-n])$	$jX_I(e^{j\omega})$

N-point Sequence	N-point DFT
	Complex signals
$x^*[n]$	$X^*[\langle -k\rangle_N]$
$x^*[\langle -n\rangle_N]$	$X^*[k]$
$x_R[n]$	$X^{cce}[k]=\frac{1}{2}(X[k]+X^*[\langle -k\rangle_N])$
$jx_I[n]$	$X^{cco}[k]=\frac{1}{2}(X[k]-X^*[\langle -k\rangle_N])$
$x^{cce}[n]=\frac{1}{2}(x[n]+x^*[\langle -n\rangle_N])$	$X_R[k]$
$x^{cco}[n]=\frac{1}{2}(x[n]-x^*[\langle -n\rangle_N])$	$jX_I[k]$
	Real signals
{Any real $x[n]$	$\begin{cases}X[k]=\tilde{X}^*[\langle -k\rangle_N]\\ X_R[k]=X_R[\langle -k\rangle_N]\\ X_I[k]=-X_I[\langle -k\rangle_N]\\ \lvert X[k]\rvert=\lvert X[\langle -k\rangle_N]\rvert\\ \angle X[k]=-\angle X[\langle -k\rangle_N]\end{cases}$

Sequence $x[n]$	Transform $X(e^{j\omega})$
	Real signals
Any real $x[n]$	$X(e^{j\omega})=X^*(e^{-j\omega})$ $X_R(e^{j\omega})=X_R(e^{-j\omega})$ $X_I(e^{j\omega})=-X_I(e^{-j\omega})$ $\lvert X(e^{j\omega})\rvert=\lvert X(e^{-j\omega})\rvert$ $\angle X(e^{j\omega})=-\angle X(e^{-j\omega})$
$x_e[n]=\frac{1}{2}(x[n]+x[-n])$	$X_R(e^{j\omega})$
Even part of $x[n]$	real part of $X(e^{j\omega})$ (even)
$x_o[n]=\frac{1}{2}(x[n]-x[-n])$	$jX_I(e^{j\omega})$
Odd part of $x[n]$	imaginary part of $X(e^{j\omega})$ (odd)

Property	N-point sequence	N-point DFT
	$x[n], h[n], v[n]$	$X[k], H[k], V[k]$
	$x_1[n], x_2[n]$	$X_1[k], X_2[k]$
Linearity	$a_1x_1[n]+a_2x_2[n]$	$a_1X_1[k]+a_2X_2[k]$
Time shifting	$x[\langle n-m\rangle_N]$	$W_N^{km}X[k]$
Frequency shifting	$W_N^{-mn}x[n]$	$X[\langle k-m\rangle_N]$
Modulation	$x[n]\cos(2\pi/N)k_0n$	$\frac{1}{2}X[\langle k+k_0\rangle_N]+\frac{1}{2}X[\langle k-k_0\rangle_N]$
Folding	$x[\langle -n\rangle_N]$	$X[\langle -k\rangle_N]$
Conjugation	$x^*[n]$	$X^*[\langle -k\rangle_N]$
Duality	$X[n]$	$Nx[\langle -k\rangle]_N]$
Convolution	$h[n]\,ⓝ\,x[n]$	$H[k]X[k]$
Correlation	$x[n]\,ⓝ\,y[\langle -n\rangle_N]$	$X[k]Y[\langle -k\rangle_N]$
Windowing	$v[n]x[n]$	$\frac{1}{N}V[k]\,ⓝ\,X[k]$
Parseval's theorem	$\sum_{n=0}^{N-1}x[n]y^*[n]=\frac{1}{N}\sum_{k=0}^{N-1}X[k]Y^*[k]$	
Parseval's relation	$\sum_{n=0}^{N-1}\lvert x[n]\rvert^2=\frac{1}{N}\sum_{k=0}^{N-1}\lvert X[k]\rvert^2$	

(Distortionless response System)

$$y[n]=Gx[n-n_d], \quad G>0$$ Maintain the "shape"

$$Y(e^{j\omega})=Ge^{-j\omega n_d}X(e^{j\omega}) \qquad \lvert H(e^{j\omega})\rvert=G$$ Constant gain

$$H(e^{j\omega})=\frac{Y(e^{j\omega})}{X(e^{j\omega})}=Ge^{-j\omega n_d}$$

$$\angle H(e^{j\omega})=-\omega n_d$$ Linear phase

Phase delay $\tau_{pd}(\omega)\triangleq-\dfrac{\angle H(e^{j\omega})}{\omega}$ **Group delay** $\tau_{gd}(\omega)\triangleq-\dfrac{d\Psi(\omega)}{d\omega}$

(frequency-domain relationship) & (Sampling theorem)

$$X_c(j\Omega)=\int_{-\infty}^{\infty}x_c(t)e^{-j\Omega t}dt,\qquad \longleftrightarrow \qquad X(e^{j\omega})=\sum_{n=-\infty}^{\infty}x[n]e^{-j\omega n},$$

$$x_c(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X_c(j\Omega)e^{j\Omega t}d\Omega \qquad x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\omega})e^{j\omega n}d\omega,$$

$$x[n]=x_c(nT) \qquad \omega=\Omega T=2\pi FT=2\pi\frac{F}{F_s}=2\pi f$$

$$X(e^{j\omega})=\frac{1}{T}\sum_{k=-\infty}^{\infty}X_c\left(j\frac{\omega}{T}-j\frac{2\pi}{T}k\right)$$

$$X(e^{j\Omega T})=\frac{1}{T}\sum_{k=-\infty}^{\infty}X_c\left(j\Omega-j\frac{2\pi}{T}k\right)$$

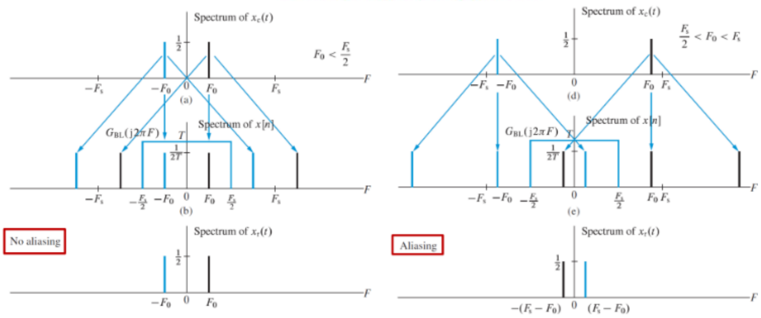
Sampling theorem: Let $x_c(t)$ be a continuous-time bandlimited signal with Fourier transform

$$X_c(j\Omega)=0 \quad \text{for} |\Omega|>\Omega_H. \tag{6.18}$$

Then $x_c(t)$ can be uniquely determined by its samples $x[n]=x_c(nT)$, where $n=0,\pm1,\pm2,\dots$, if the sampling frequency Ω_s satisfies the condition

$$\Omega_s=\frac{2\pi}{T}\geq 2\Omega_H. \tag{6.19}$$

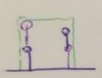
Sampling frequency ≥ Nyquist rate



① Sampling, if ① $F_s \times \frac{1}{2} = F_H$ ② $\frac{1}{2}F_s < F_H$ (aliasing) ③ $\frac{1}{2}F_s > F_H$

↳ Ex. $x_c(t)=\sin(2\pi F_0 t+\theta_s)$

①



$\Rightarrow y_r(t)=\sin(2\pi F_0 t+\theta_s)+\sin(-2\pi F_0 t+\theta_s)$
$= 2\sin(\theta_0)\cos(2\pi F_0 t)$

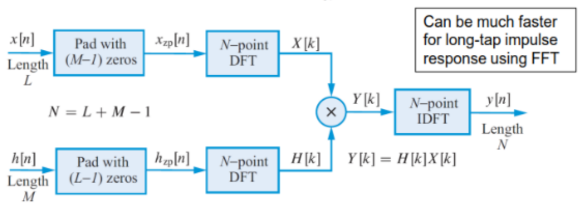
②
$x_c(t)=\sin(2\pi F_0 t+\theta_s)=\sin(2\pi F_0' t)$, $F_0'=F_0+\frac{\theta_s}{2\pi t}$
$\Rightarrow y_r(t)=\sin(2\pi(F_s-F_0')t)=\sin(2\pi(F_s-F_0)t-\theta_s)$

③ $y_r(t)=\sin(2\pi F_0 t+\theta_s)$

Nyquist frequency = Fs/2, Fs: Sampling frequency

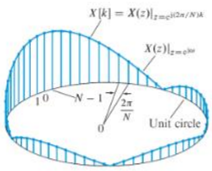
Zero padding + Circular convolution

$$y_{zp}[n]=h_{zp}[n] \circledN x_{zp}[n] \xleftrightarrow[N]{\text{DFT}} Y[k]=H[k]X[k]$$



(Sampling DTFT in frequency domain)

Aperiodic signal $x[n]=\frac{1}{2\pi}\int_0^{2\pi}\tilde{X}(e^{j\omega})e^{-j\omega n}d\omega$



Sampling DTFT as DFT $X[k]\triangleq\tilde{X}(e^{j\frac{2\pi}{N}k})=\sum_{n=-\infty}^{\infty}x[n]e^{-j\frac{2\pi}{N}kn}$

$$X[k]=\sum_{n=0}^{N-1}\left(\sum_{\ell=-\infty}^{\infty}x[n-\ell N]\right)e^{-j\frac{2\pi}{N}kn}$$

IDFT $\tilde{x}[n]=\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}(e^{j\frac{2\pi}{N}k})e^{j\frac{2\pi}{N}kn}=\frac{1}{N}\sum_{k=0}^{N-1}X[k]W_N^{-kn}$

Periodic extension $\tilde{x}[n]\triangleq\sum_{\ell=-\infty}^{\infty}x[n-\ell N]$ (may introduce time-domain aliasing)

IDTFT approximation $x[n]=\tilde{x}[n]p_N[n]$

(Minimum-phase systems)

* Stable ⇒ poles must be inside the unit circle

Minimum Phase Systems: (DT)
A causal stable LTI system H with tranfer function H(z) with all zeros and poles inside the unit circle.

Maximum Phase Systems: (CT)
A causal stable LTI system H w/ transfer function H(z) w/ all zeros outside the unit circle and all poles ~~inside~~ outside the unit circle (for stable).

Minimum Phase Systems (CT)
all poles and zeros of the systems must be strictly inside the left-half "s-plane" ⟺ Causality & Stability

(Quantization Noise)

6.5 EE3660 Intro to DSP, Spring 2020 18

1. signal 可以跑的 swing 大小：$2X_m$
2. Quantization step: Δ
3. ↳ 我們把可以跑的訊號範圍切成 $\frac{2X_m}{\Delta}$ 份，$\frac{2X_m}{\Delta}=2^B$

Quantization Noise

$e_c(t)\triangleq x_q(t)-x_c(t)$, $e_c(t)=\frac{\Delta}{2\tau}t$, $-\tau\leq t\leq\tau$ [using statistical techniques]

quantizer

Quantization error power (assume error is uniformly distributed)

$$P_Q=\frac{1}{2\tau}\int_{-\tau}^{\tau}e_c^2(t)dt=\frac{\Delta^2}{12}$$

average power for sine signal
$x_c=X_m\sin(\frac{2\pi}{T_p}t)$

T_p: spans the range of the quantizer

↳ **Signal power** (assume sinusoidal signals)

$$P_S=\frac{1}{T_p}\int_0^{T_p}X_m^2\sin^2\left(\frac{2\pi}{T_p}\right)dt=\frac{X_m^2}{2}$$

$$x_c(t)=X_m\sin(\tfrac{2\pi}{T_p}t)$$

· $\boxed{\text{Quantization Step}}$ $(\Delta)=(2X_m)/2^B$ for the quantization step of a B-bit quantizer

Signal-to-quantization-noise-ratio $\text{SQNR}\triangleq\frac{P_S}{P_Q}=\frac{3}{2}\times2^{2B}$

$B\equiv$ bit width

$$\text{SQNR(dB)}=10\log_{10}\text{SQNR}=6.02B+1.76$$

(one additional bit adds 6dB)

↳ B 取越大 SQNR 會比較好

(LP FIR filter design using fixed window)

→ $\omega_C=\frac{\omega_P+\omega_S}{2}$, $A=-20\log_{10}\delta$, $\Delta\omega=\omega_S-\omega_P$

→ Choose the window function that provides the smallest stopband attenuation greater than A

→ Choose $\omega_S-\omega_P\geq\Delta\omega$ (of table) and L is odd (for Type I)