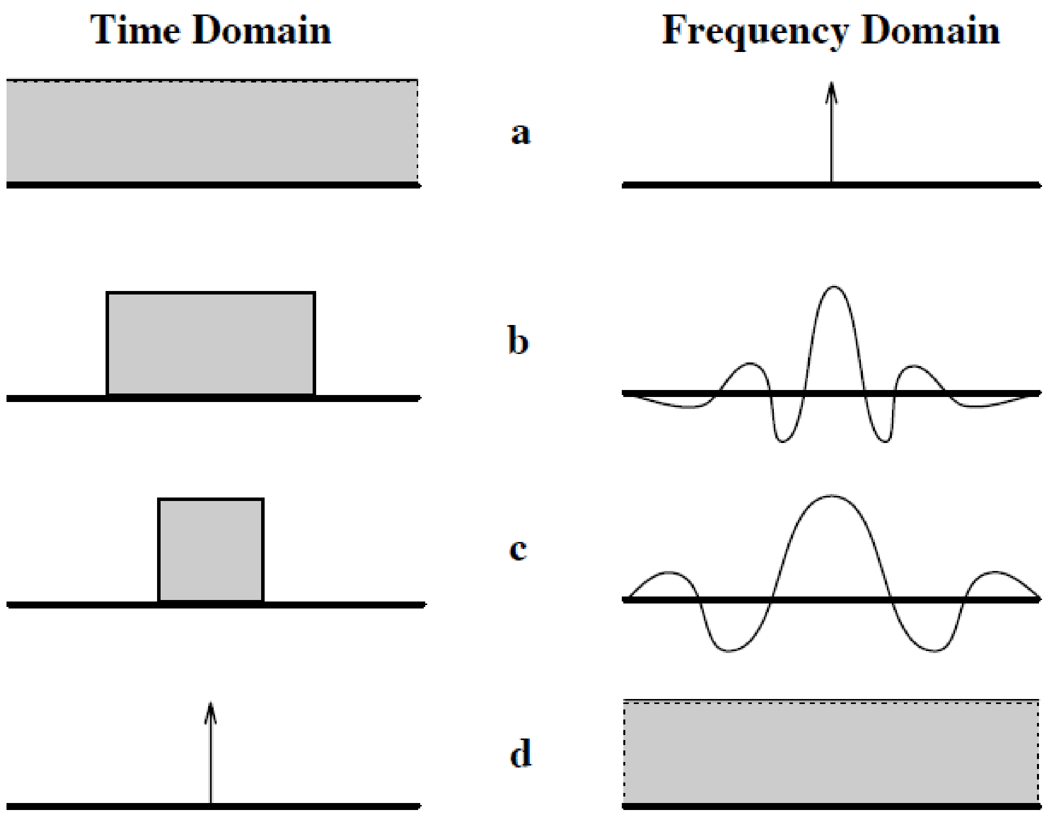
Do you know
Gabor uncertainty principle?

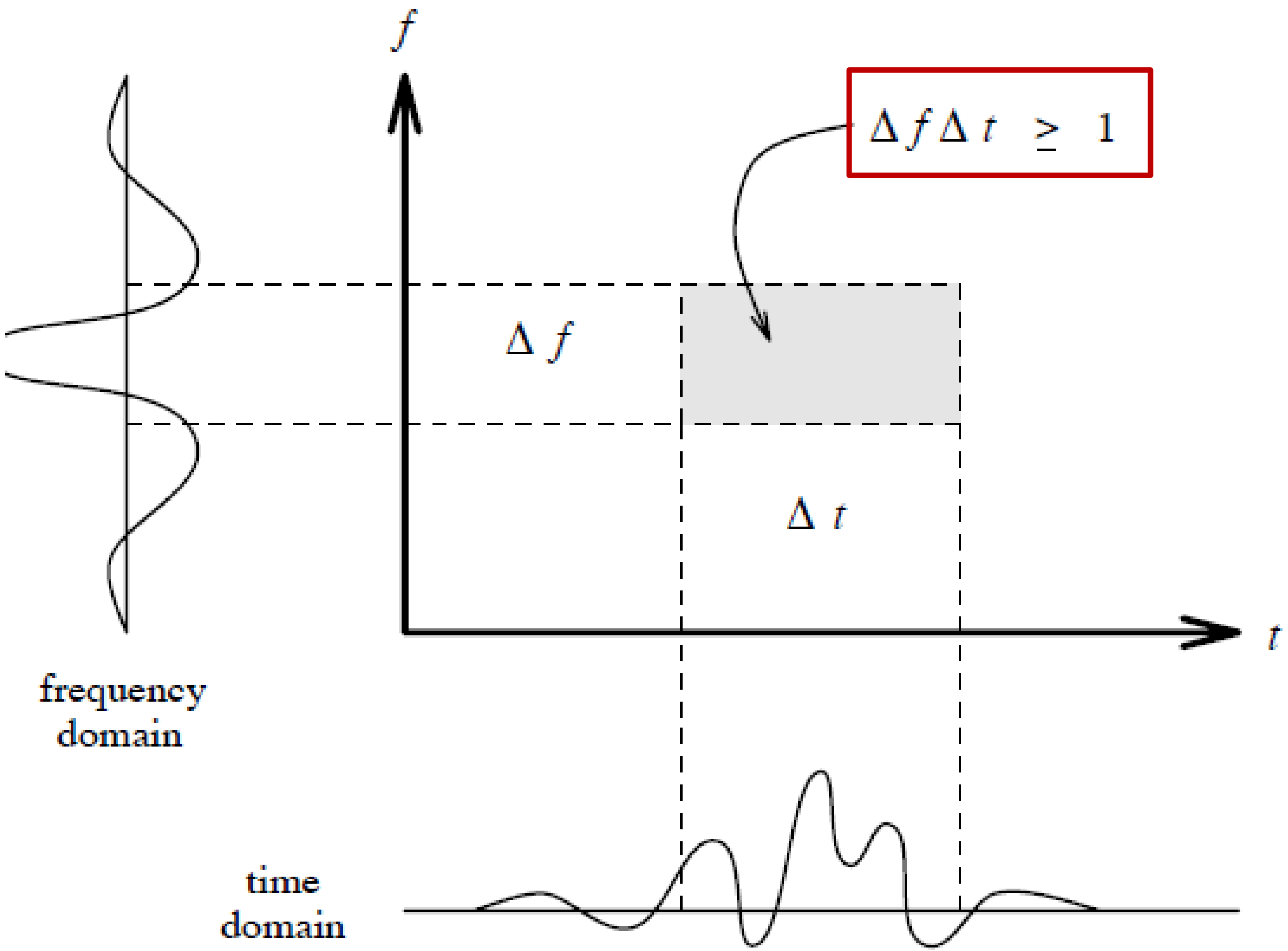


Dennis Gabor
(1900-1979)

Inventor of holography
(Nobel Prize 1971)

Ref:
1. D. Gabor, "Theory of communication," Journal of the Institution of Electrical Engineers - Part III: Radio and Communication Engineering, 1946.
2. B. MacLennan, ""Gabor representation," http://web.eecs.utk.edu/~mclennan/Classes/494-594-UC-F13/handouts/FFC-ch6.pdf

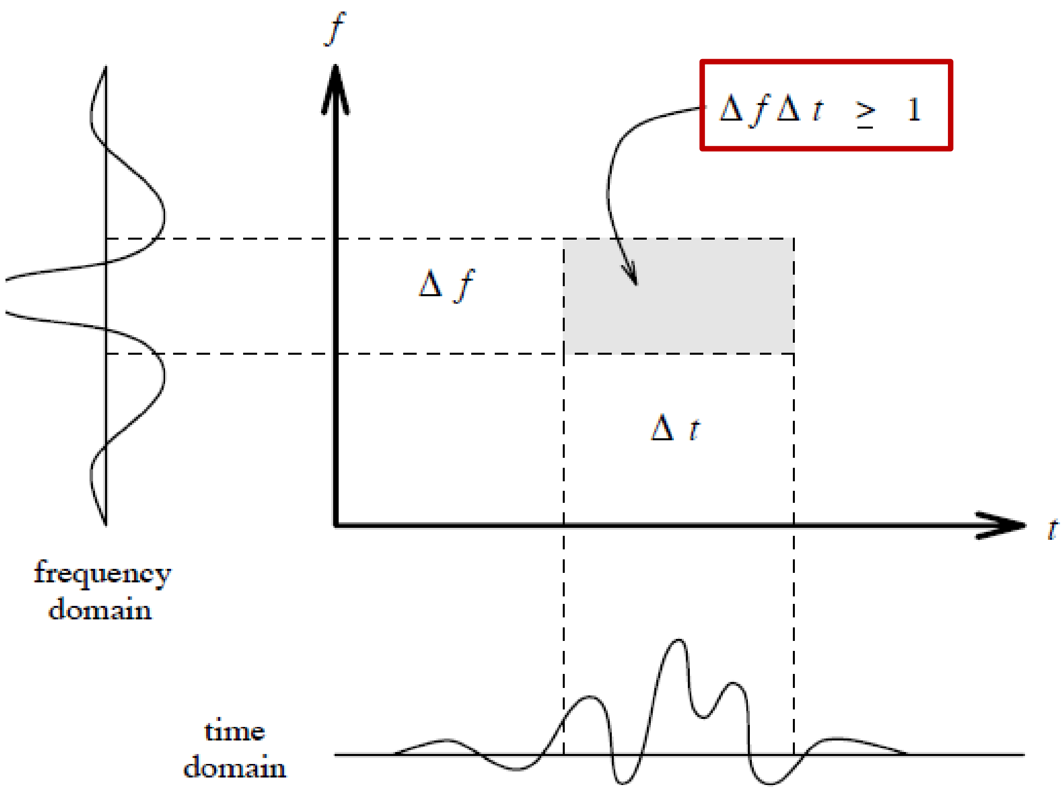
Gabor Uncertainty Principle



Gabor Uncertainty Principle

Given a normalized $f(t)$ s.t. $\int_{-\infty}^{\infty} |f(t)|^2\, dt = 1$ and its Fourier transform $\hat{f}(\xi)$, we can treat $|f(t)|^2$ and $|\hat{f}(\xi)|^2$ as probability density functions.

Define $(\Delta t)^2 = 2\pi \cdot Var(t)$ and $(\Delta f)^2 = 2\pi \cdot Var(f)$.
It can be proved by Cauchy-Schwartz inequality that $\Delta f \cdot \Delta t \geq 1$.



Proof in the textbook

Duration and bandwidth

$$\sigma_t^2 \triangleq \int_{-\infty}^{\infty} t^2 |x_c(t)|^2 \mathrm{d}t \qquad \sigma_\Omega^2 \triangleq \frac{1}{2\pi}\int_{-\infty}^{\infty} \Omega^2 |X_c(\mathrm{j}\Omega)|^2 \mathrm{d}\Omega$$

Schwarz inequality

$$\left|\int_{-\infty}^{\infty} x_{c_1}(t) x_{c_2}(t)\mathrm{d}t\right|^2 \le \int_{-\infty}^{\infty} |x_{c_1}(t)|^2 \mathrm{d}t \int_{-\infty}^{\infty} |x_{c_2}(t)|^2 \mathrm{d}t$$

$$x_{c_1}(t) = t x_c(t),$$

$$x_{c_1}(t) = \frac{\mathrm{d}x_c(t)}{\mathrm{d}t},$$

$$\frac{\mathrm{d}x_c(t)}{\mathrm{d}t} \xleftrightarrow{\text{CTFT}} \mathrm{j}\Omega X_c(\mathrm{j}\Omega)$$

$$\int_{-\infty}^{\infty} t x_c(t) \frac{\mathrm{d}x_c(t)}{\mathrm{d}t}\mathrm{d}t = t\frac{x_c^2(t)}{2}\Big|_{-\infty}^{\infty} - \frac{1}{2}\int_{-\infty}^{\infty} x_c^2(t)\mathrm{d}t = -\frac{1}{2}$$

$$\frac{1}{4} \le \int_{-\infty}^{\infty} t^2 |x_c(t)|^2 \mathrm{d}t \ \frac{1}{2\pi}\int_{-\infty}^{\infty} \Omega^2 |X_c(\mathrm{j}\Omega)|^2 \mathrm{d}\Omega$$

⇩

$$\sigma_t \sigma_\Omega \ge \frac{1}{2}$$

Implication

- Frequency- and time-domain signals can't be very band-limited at the same time.
  - Frequency-accurate signals will be time-inaccurate, and vice versa

- This is not a constraint. Instead, this is the nature of time-frequency analysis.

Other Uncertainty Principles

- Uncertainty principle (quantum mechanics)
  - Wave function of momentum (p) is the Fourier Transform of that of position (x)

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}$$

- Entropic uncertainty principle
  - Entropy for absolute squares of wave functions
  - Stronger than the above principle

$$H_x + H_p \geq \ln(e\pi)$$

Uncertainty principle applies in many places.
Learn how to compromise and how to deal with it!