Chap9
Structures for discrete-time systems

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Performance index

- Computation complexity
  - Time cost for software
  - Area cost for hardware
- Memory
  - Storage requirement for software
  - Area cost for hardware
- Quantization error
  - Finite precision arithmetic
  - Rounding and saturation
- Computation path
  - Speed for hardware

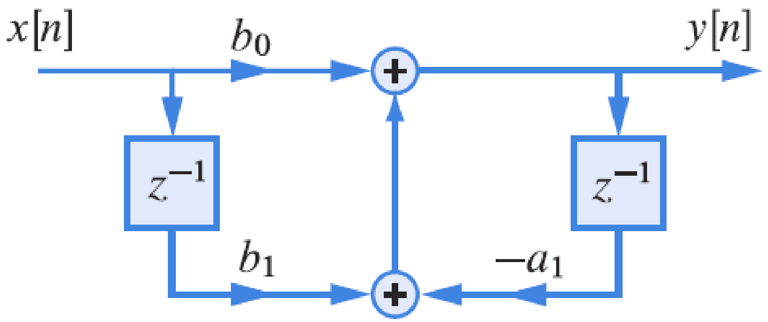
Chap 9 Structures for discrete-time systems

- 9.1 Block diagrams and signal flow graphs
- 9.2 IIR system structures
- 9.3 FIR system structures
- 9.4.1 All-zero lattice structure

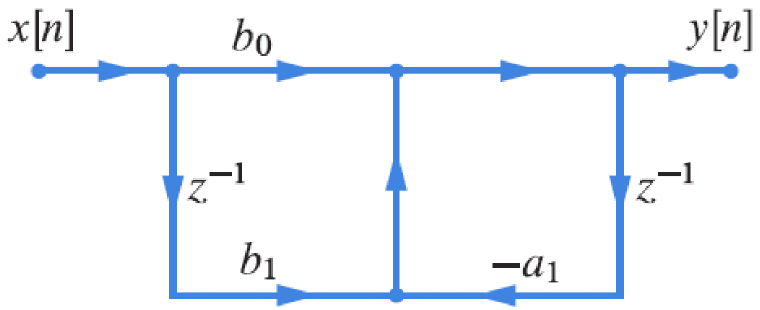
Block diagram vs. Signal flow graph

$$y[n] = b_0 x[n] + b_1 x[n-1] - a_1 y[n-1]$$

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1}}$$



Block diagram
(implementable)

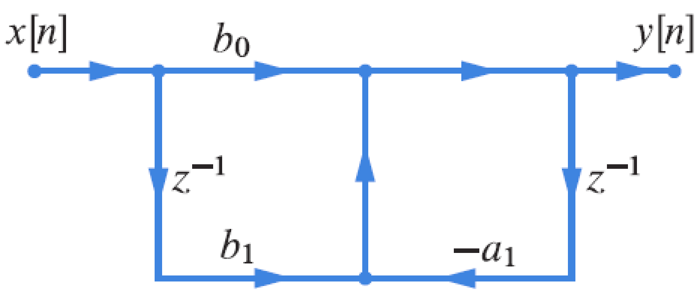


Signal flow graph
(mathematical)

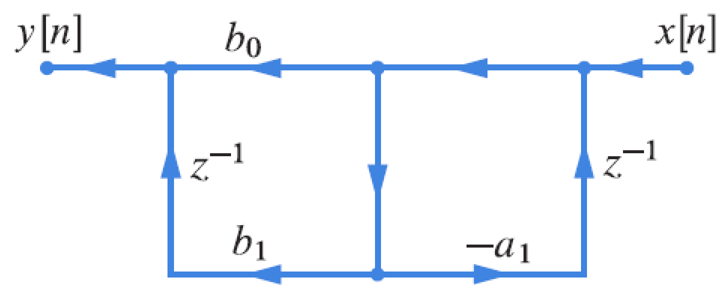
Transposition theorem

Transposed form will be an equivalent structure if

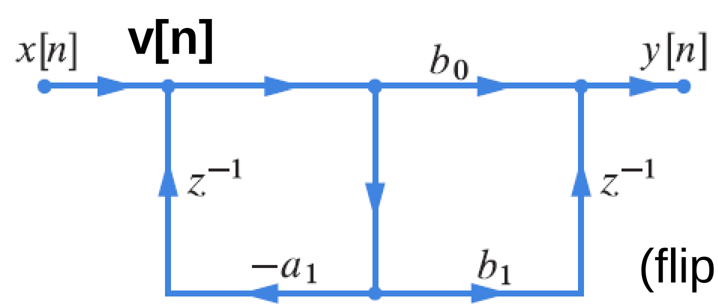
1. Reverse all branch directions
2. Replace branch nodes by summing ones and vice versa
3. Interchange input and output nodes



Normal form



Transposed form



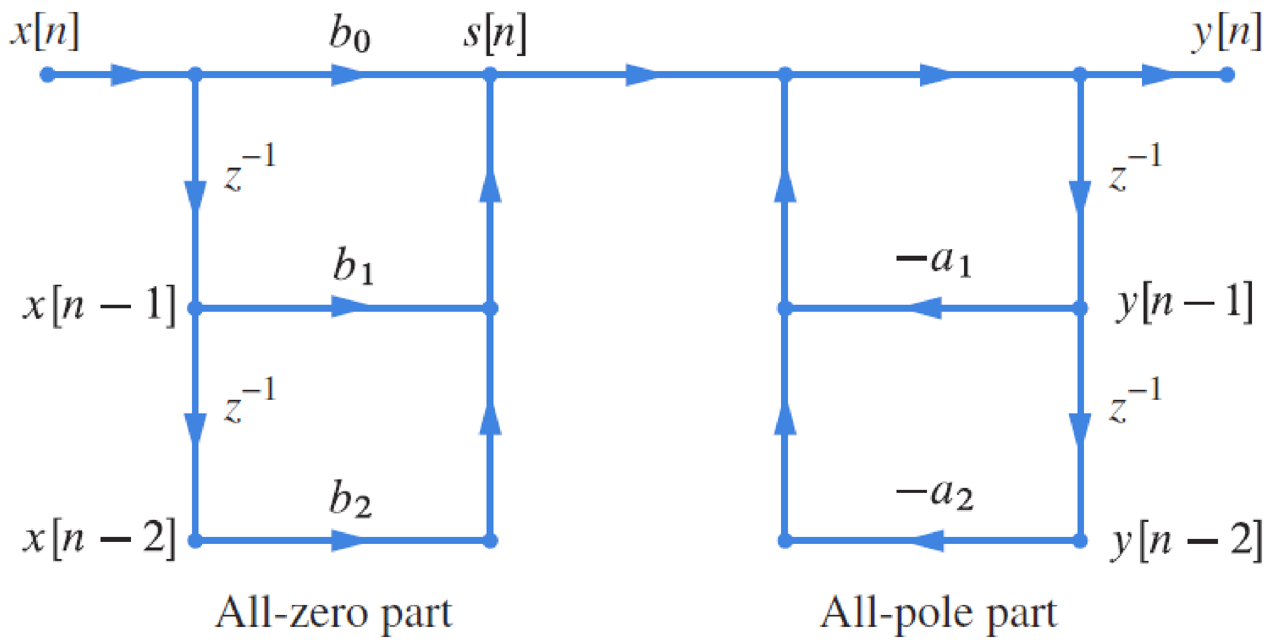
(flipped)

Mul #: N+M+1
Delay #: N+M
Path: T_m+max(M+1,N)*T_a

Direct form I (IIR)

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

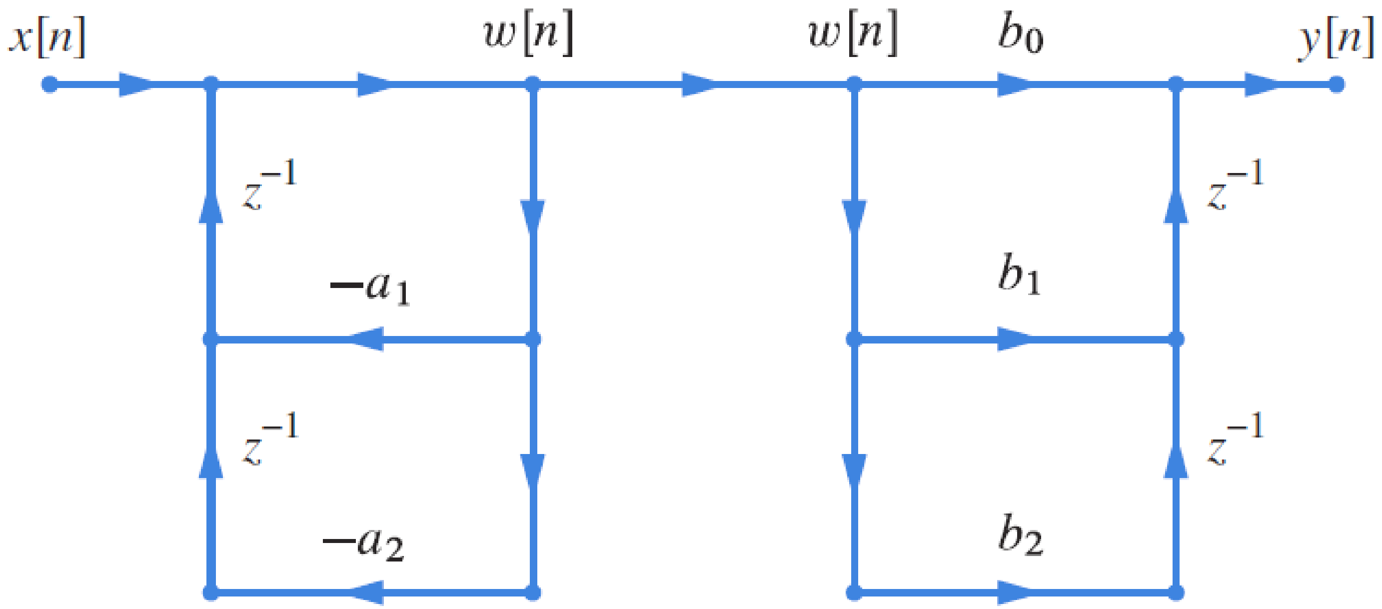


$$H_1(z) = \frac{S(z)}{X(z)} = \sum_{k=0}^{M} b_k z^{-k}$$

$$H_2(z) = \frac{Y(z)}{S(z)} = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

Mul #: N+M+1
Delay #: N+M
Path: $T_m + 2 * T_a$

Transposed direct form I



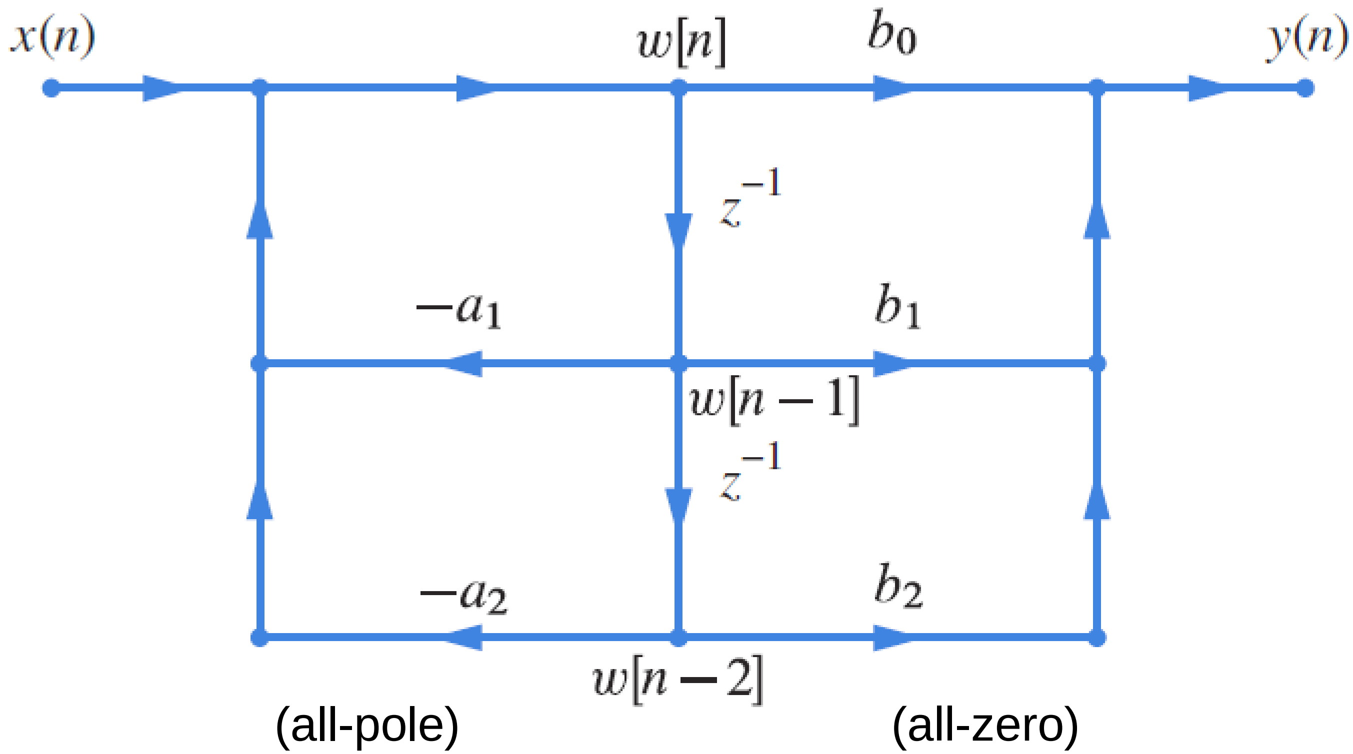
(all-pole) (all-zero)

$$w[n] = -\sum_{k=1}^{N} a_k w[n-k] + x[n]$$

$$y[n] = \sum_{k=0}^{M} b_k w[n-k]$$

Mul #: N+M+1
Delay #: max(N,M)
Path: $2*T_m+(N+1)*T_a$

Direct form II
(Canonical direct form)



$$W(z) = H_2(z)X(z) = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}} X(z)$$

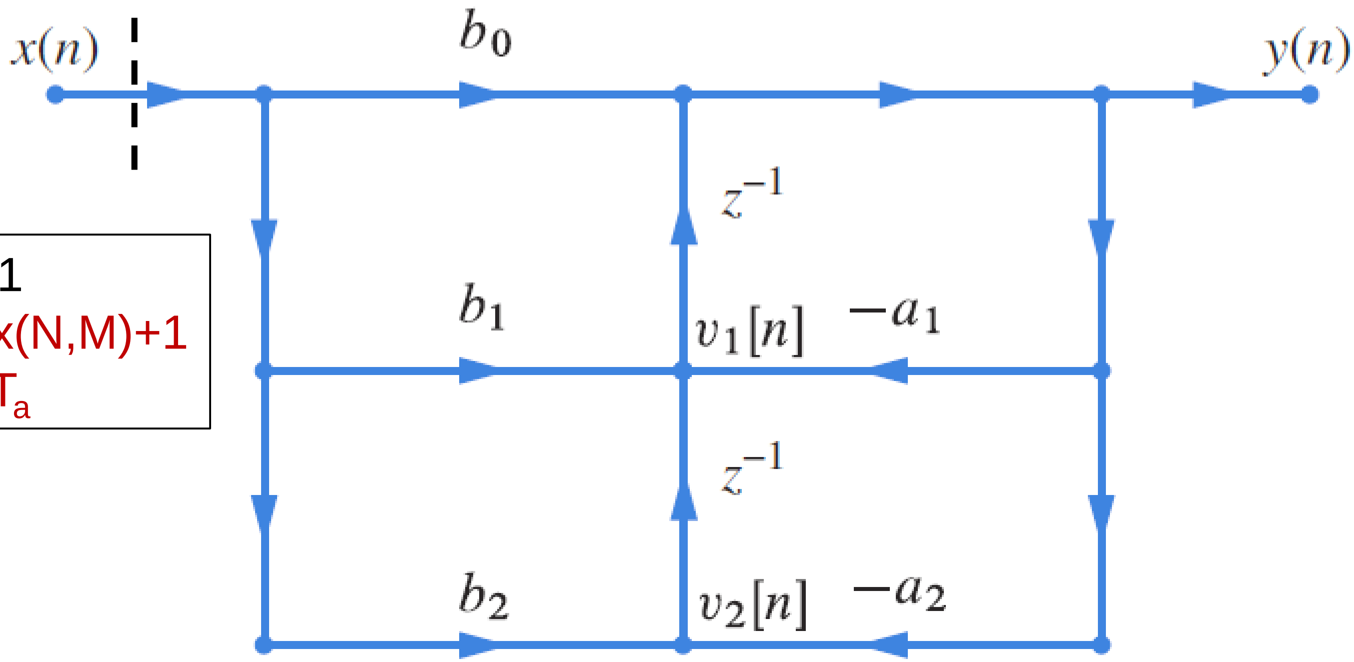
$$Y(z) = \sum_{k=0}^{M} b_k z^{-k} W(z)$$

Mul #: N+M-1
Delay #: max(N,M)
Path: 2*T_m+2*T_a

Transposed direct form II

OR

Mul #: N+M-1
Delay #: max(N,M)+1
Path: T_m+2*T_a



$$Y(z) = z^{-1}V_1(z) + b_0X(z),$$

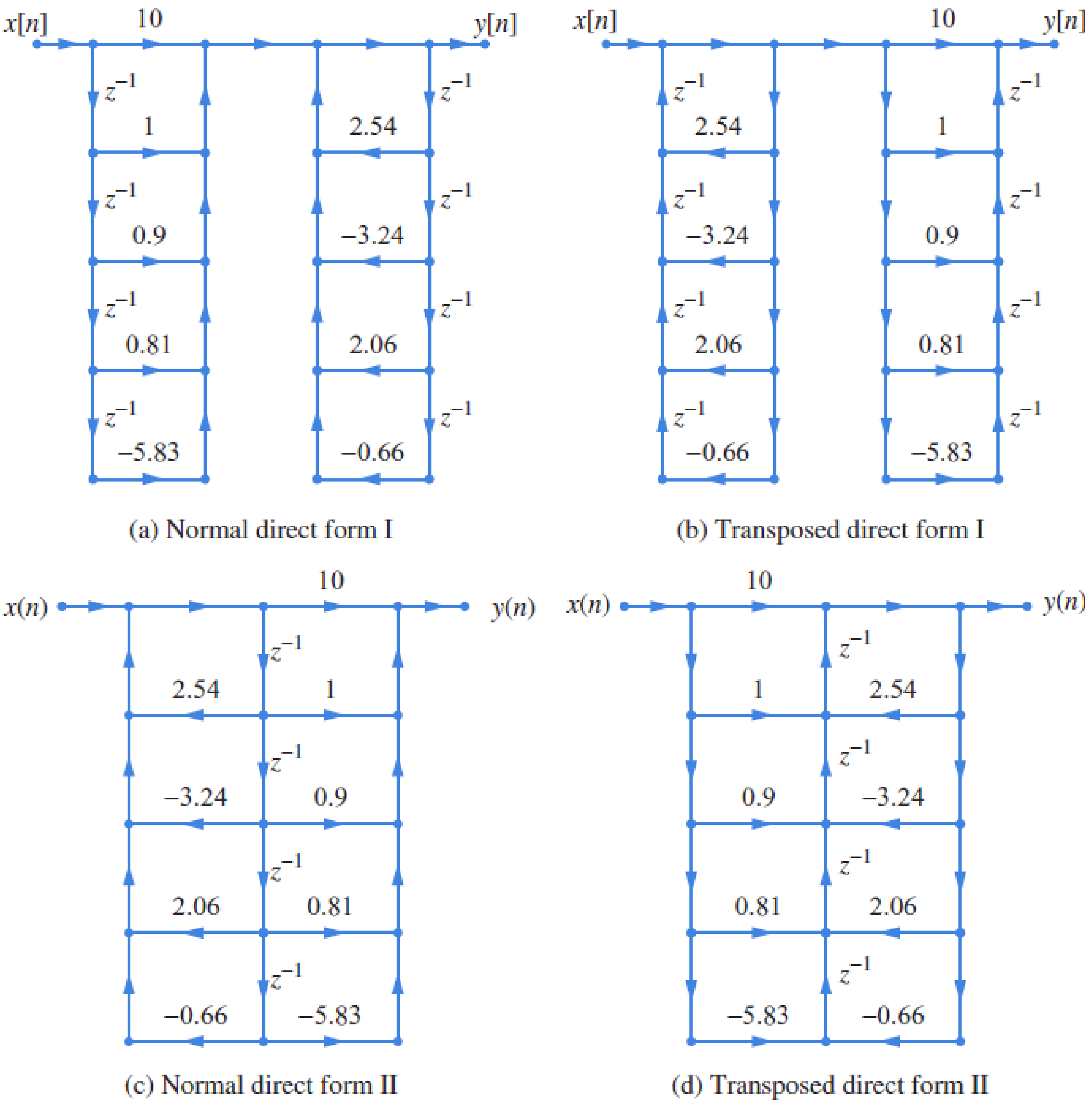
$$V_1(z) = z^{-1}V_2(z) - a_1Y(z) + b_1X(z)$$

$$V_2(z) = b_2X(z) - a_2Y(z).$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1z^{-1} + b_2z^{-2}}{1 + a_1z^{-1} + a_2z^{-2}}$$

Examples of direct forms

$$H(z) = \frac{10 + z^{-1} + 0.9z^{-2} + 0.8z^{-3} - 5.8z^{-4}}{1 - 2.54z^{-1} + 3.24z^{-2} - 2.06z^{-3} + 0.66z^{-4}}$$



Cascade form

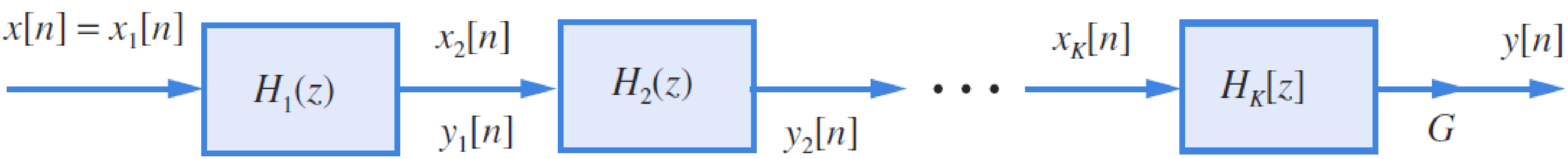
Conjugate pairs for real-coefficient systems

$$H(z) = b_0 \frac{\prod_{k=1}^{M_1}\left(1 - z_k z^{-1}\right) \prod_{k=1}^{M_2}\left(1 - z_k z^{-1}\right)\left(1 - z_k^* z^{-1}\right)}{\prod_{k=1}^{N_1}\left(1 - p_k z^{-1}\right) \prod_{k=1}^{N_2}\left(1 - p_k z^{-1}\right)\left(1 - p_k^* z^{-1}\right)}$$

(non-unique pairing)

$$H(z) \triangleq G \prod_{k=1}^{K} \frac{B_{k0} + B_{k1} z^{-1} + B_{k2} z^{-2}}{1 + A_{k1} z^{-1} + A_{k2} z^{-2}} \triangleq G \prod_{k=1}^{K} H_k(z)$$

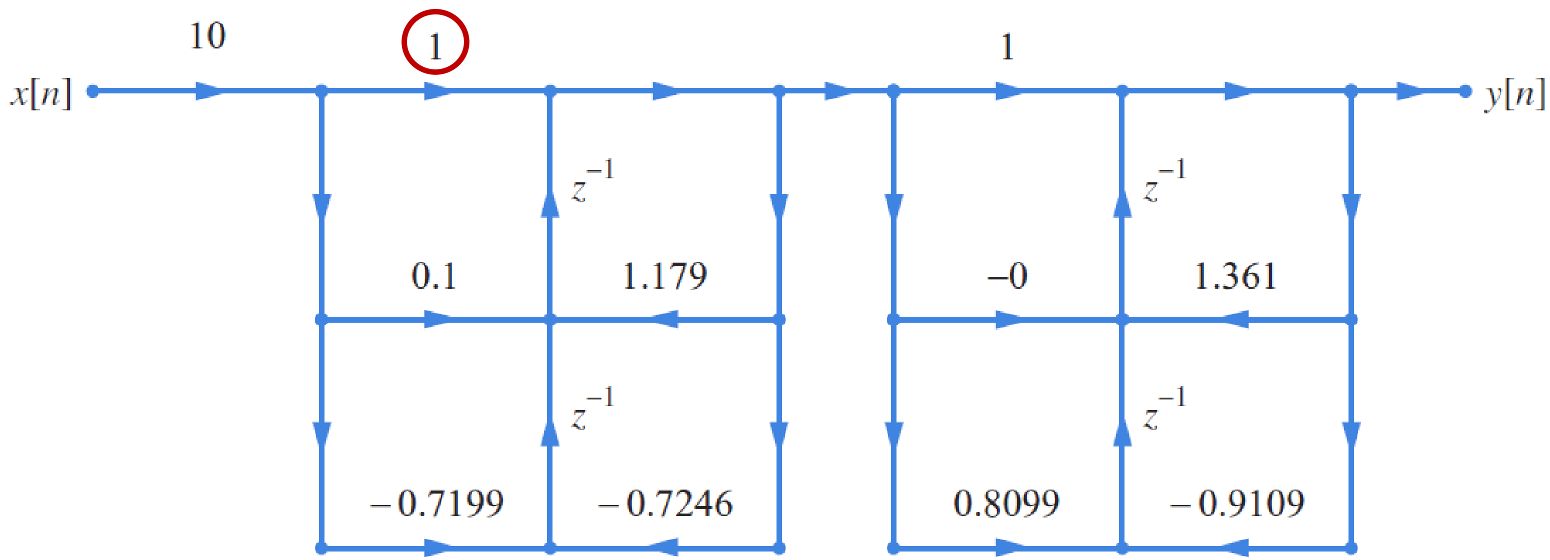
Can reuse same function (SW) or module (HW)



Example of cascade form

$$H(z) = \frac{10 + z^{-1} + 0.9z^{-2} + 0.8z^{-3} - 5.8z^{-4}}{1 - 2.54z^{-1} + 3.24z^{-2} - 2.06z^{-3} + 0.66z^{-4}}$$

$$H(z) = 10 \times \frac{1 + 0.1z^{-1} - 0.7199z^{-2}}{1 - 1.1786z^{-1} + 0.7246z^{-2}} \times \frac{1 + 0z^{-1} + 0.8099z^{-2}}{1 - 1.3614z^{-1} + 0.9109z^{-2}}$$

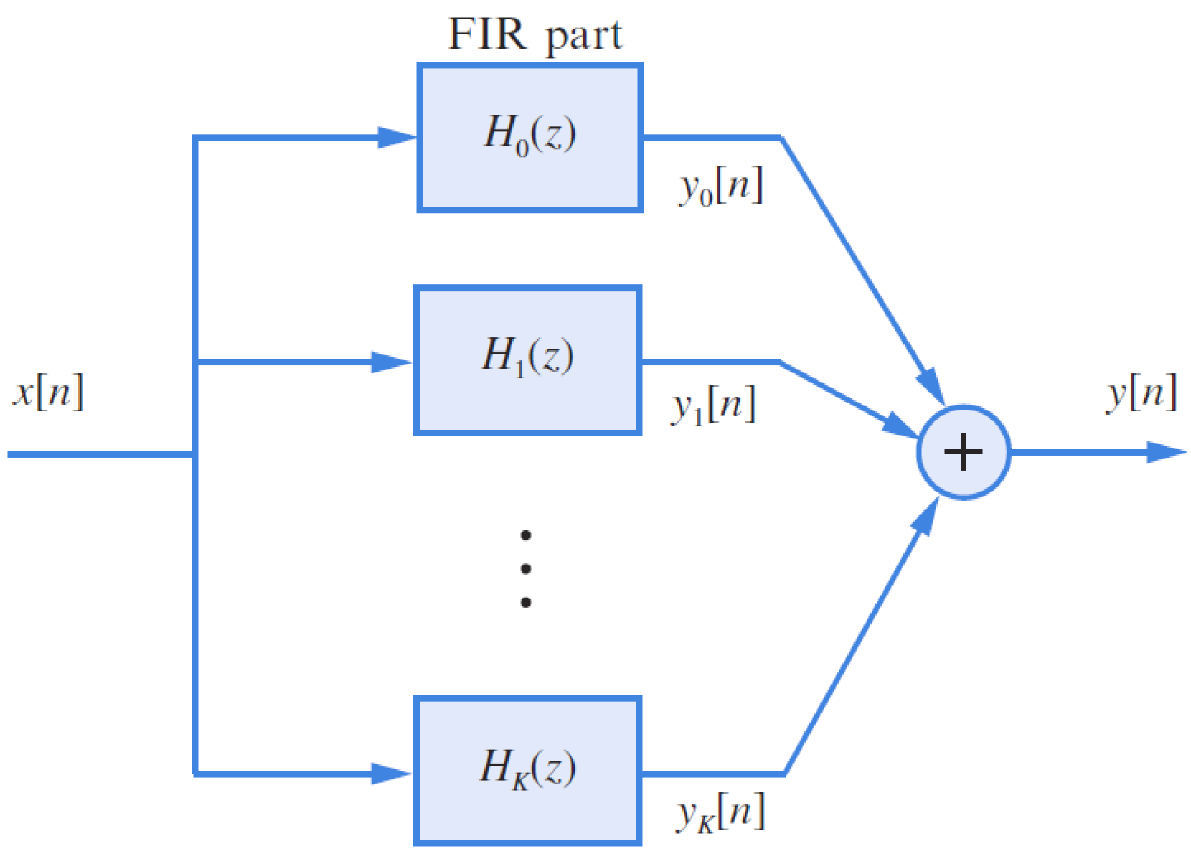


Timing path accumulated
Quantization error propagated

Parallel form

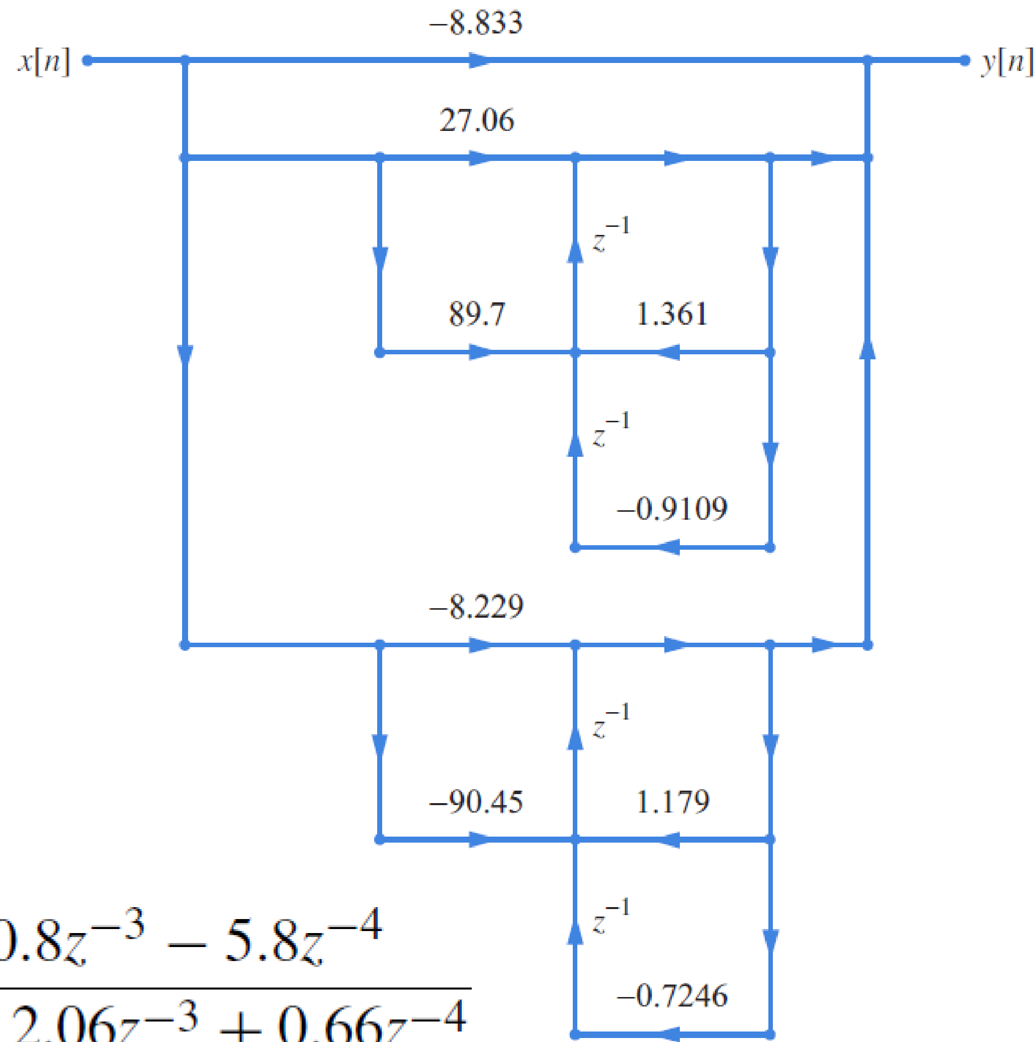
(unique expression from partial fraction expansion)

$$H(z) = \sum_{k=0}^{M-N} C_k z^{-k} + \sum_{k=1}^{K} \frac{B_{k0} + B_{k1} z^{-1}}{1 + A_{k1} z^{-1} + A_{k2} z^{-2}}$$



Timing path paralleled
Quantization error summed

Example of parallel form

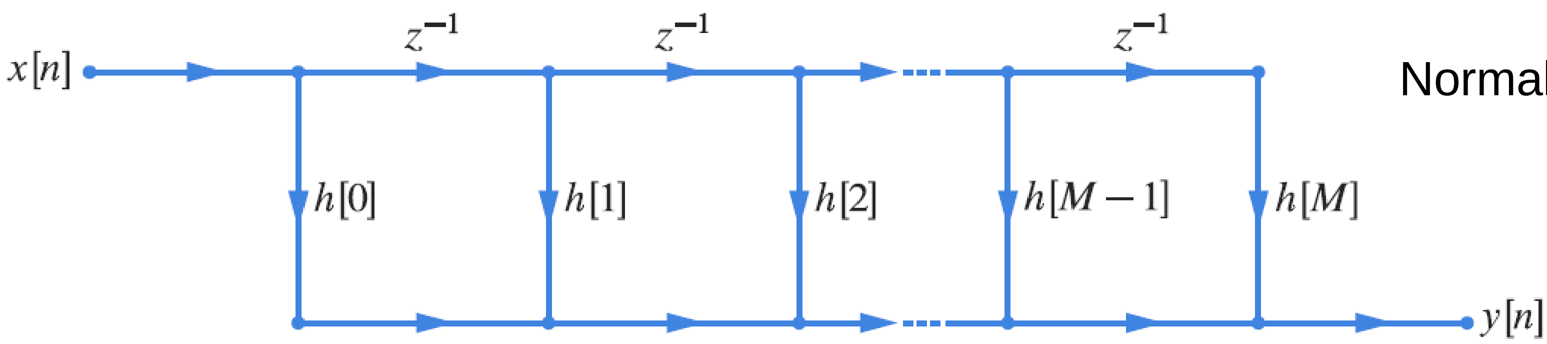


$$H(z) = \frac{10 + z^{-1} + 0.9z^{-2} + 0.8z^{-3} - 5.8z^{-4}}{1 - 2.54z^{-1} + 3.24z^{-2} - 2.06z^{-3} + 0.66z^{-4}}$$

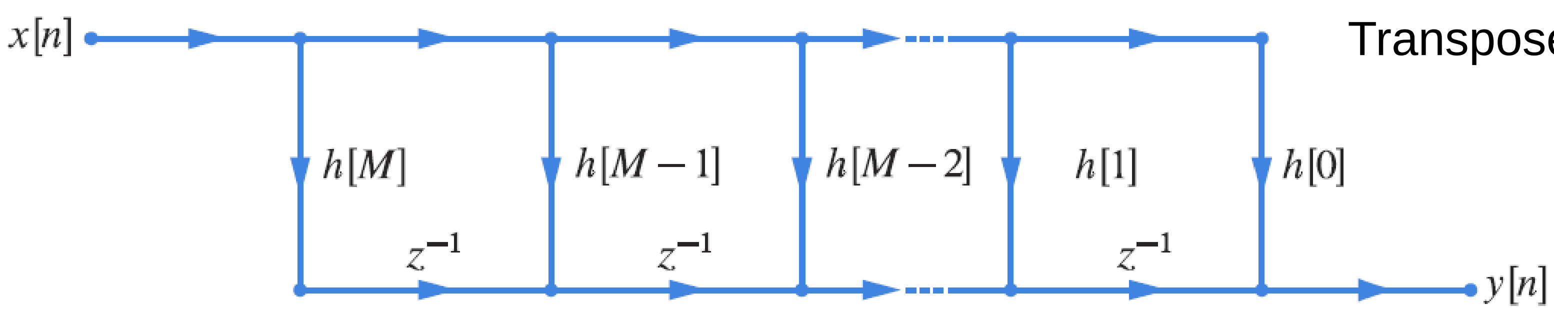
$$H(z) = -8.83 + \frac{27.06 + 89.70z^{-1}}{1 - 1.36z^{-1} + 0.91z^{-2}} + \frac{-8.23 - 90.45z^{-1}}{1 - 1.18z^{-1} + 0.72z^{-2}}$$

Direct form

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{n=0}^{M} b_n z^{-n} = \sum_{n=0}^{M} h[n] z^{-n}$$



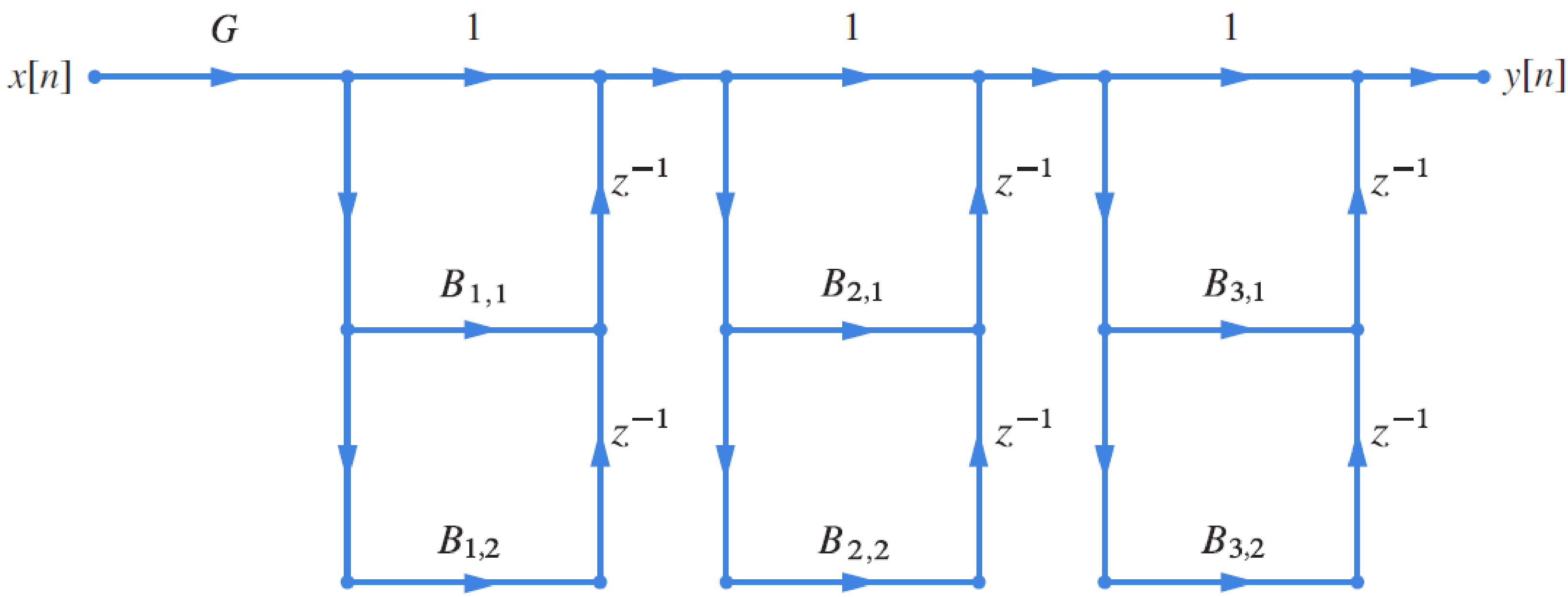
Normal form



Transposed form

Cascade form

$$H(z) = \sum_{n=0}^{M} h[n]z^{-n} \triangleq G \prod_{k=0}^{K} \left(1 + \tilde{B}_{k1} z^{-1} + \tilde{B}_{k2} z^{-2}\right)$$

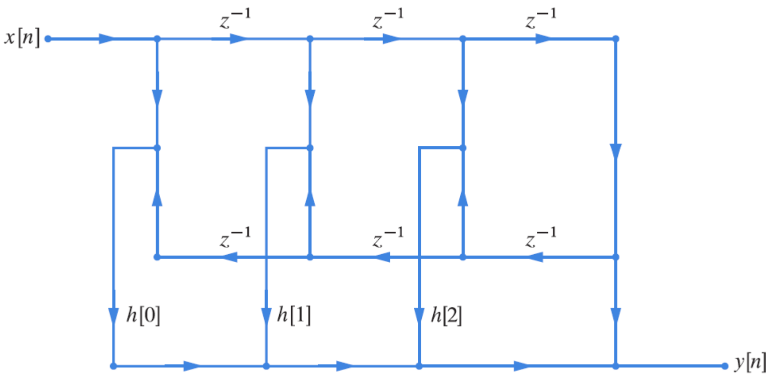


Direct form for linear-phase FIR

$$h[n] = \pm h[M-n], \quad 0 \le n \le M$$

Type I:	M even,	symmetric
Type II:	M odd,	symmetric
Type III:	M even,	anti-symmetric
Type IV:	M odd,	anti-symmetric

$$\begin{aligned} y[n] &= \sum_{k=0}^{M} h[k]x[n-k] \\ &= \sum_{k=0}^{\frac{M}{2}-1} h[k]x[n-k] + h[\tfrac{M}{2}]x[n-\tfrac{M}{2}] + \sum_{k=\frac{M}{2}+1}^{M} h[k]x[n-k] \\ &= \sum_{k=0}^{\frac{M}{2}-1} h[k]x[n-k] + h[\tfrac{M}{2}]x[n-\tfrac{M}{2}] + \sum_{k=0}^{\frac{M}{2}-1} h[M-k]x[n-M+k] \\ &= \sum_{k=0}^{\frac{M}{2}-1} h[k]\Big(x[n-k] + x[n-M+k]\Big) + h[\tfrac{M}{2}]x[n-\tfrac{M}{2}]. \end{aligned}$$

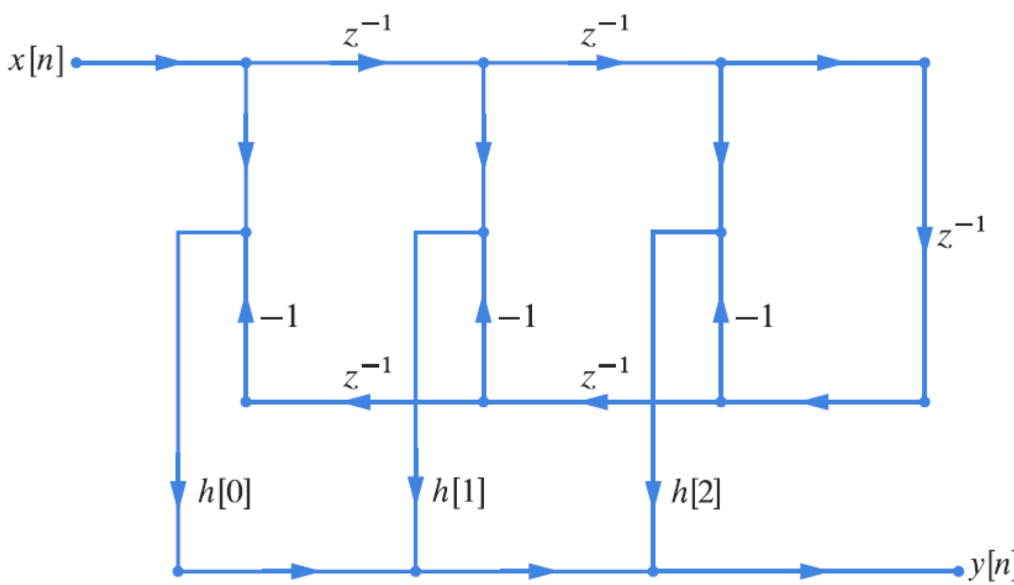


Direct form for linear-phase FIR

$$h[n] = \pm h[M-n], \quad 0 \le n \le M$$

Type I:	M even,	symmetric
Type II:	M odd,	symmetric
Type III:	M even,	anti-symmetric
Type IV:	M odd,	anti-symmetric

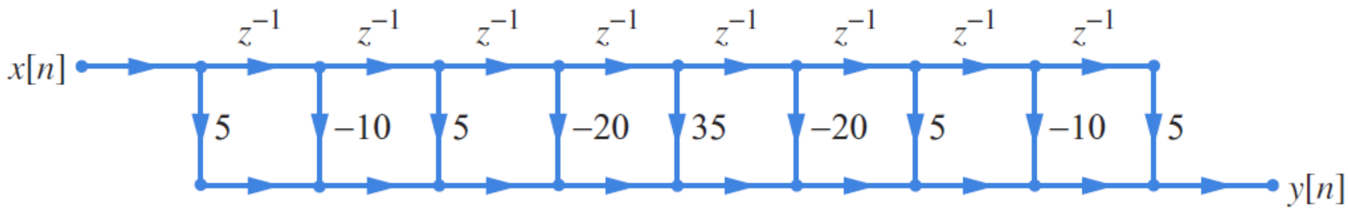
$$y[n] = \sum_{k=0}^{\frac{M-1}{2}} h[k]\Big(x[n-k] - x[n-M+k]\Big)$$



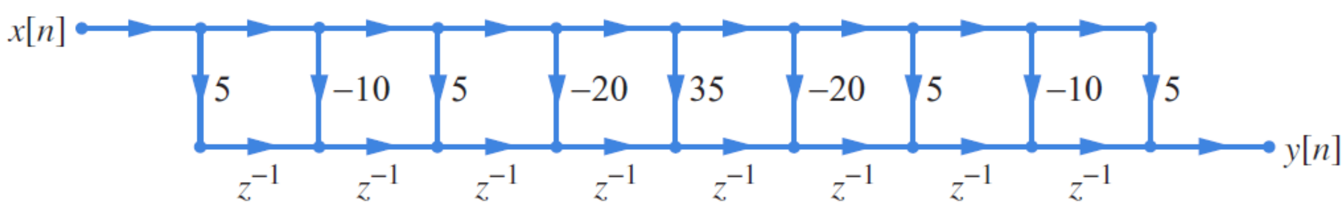
FIR example

$$H(z) = 5 - 10z^{-1} + 5z^{-2} - 20z^{-3} + 35z^{-4} - 20z^{-5} + 5z^{-6} - 10z^{-7} + 5z^{-8}$$

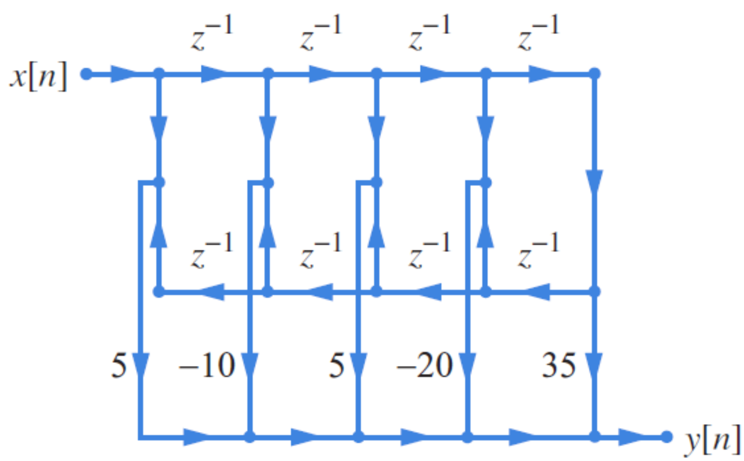
Direct form



Transposed form



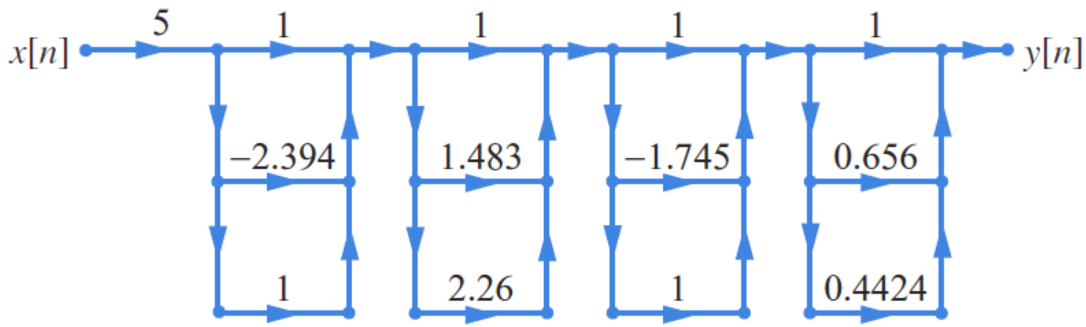
Direct-form linear-phase



FIR example

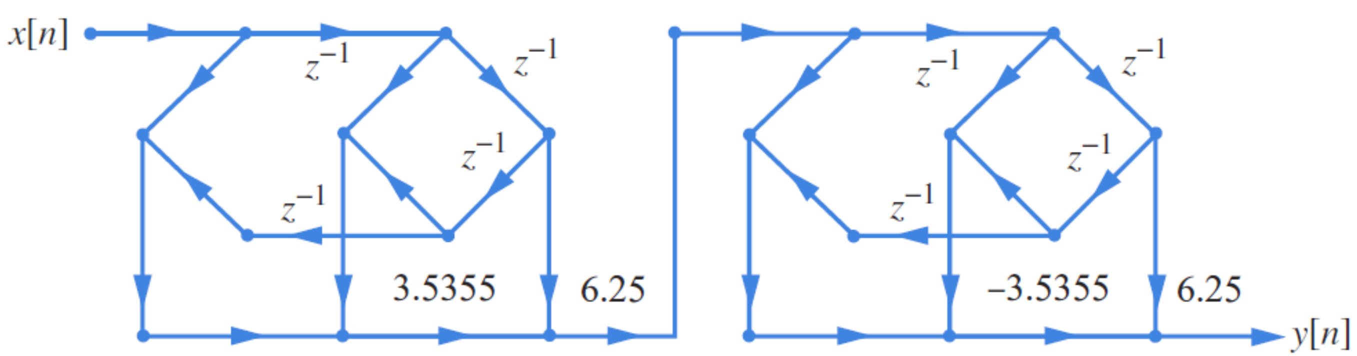
Cascade form

$$H(z) = 5\left(1 - 2.394z^{-1} + z^{-2}\right)\left(1 + 1.4829z^{-1} + 2.2604z^{-2}\right)$$
$$\times\left(1 - 1.745z^{-1} + z^{-2}\right)\left(1 + 0.656z^{-1} + 0.4424z^{-2}\right)$$



Cascade-form linear-phase

$$H(z) = 5\left(1 - 4.139z^{-1} + 6.1775z^{-2} - 4.139z^{-3} + z^{-4}\right)$$
$$\times\left(1 + 2.139z^{-1} + 3.6757z^{-2} + 2.139z^{-3} + z^{-4}\right)$$



Polynomial Lagrange interpolation (Chap7.3)

$$X(z) = \sum_{n=0}^{N-1} x[n] z^{-n}$$

Substitute IDFT for x[n]

$$X(z) = \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}(e^{j\frac{2\pi}{N}k}) e^{j\frac{2\pi}{N}kn} \right] z^{-n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}(e^{j\frac{2\pi}{N}k}) \sum_{n=0}^{N-1} \left(e^{j\frac{2\pi}{N}k} z^{-1} \right)^{n}$$

Reconstruct X(z) by DFT sampling

$$X(z) = \frac{1 - z^{-N}}{N} \sum_{k=0}^{N-1} \frac{\tilde{X}(e^{j\frac{2\pi}{N}k})}{1 - e^{j\frac{2\pi}{N}k} z^{-1}}$$

Frequency sampling form

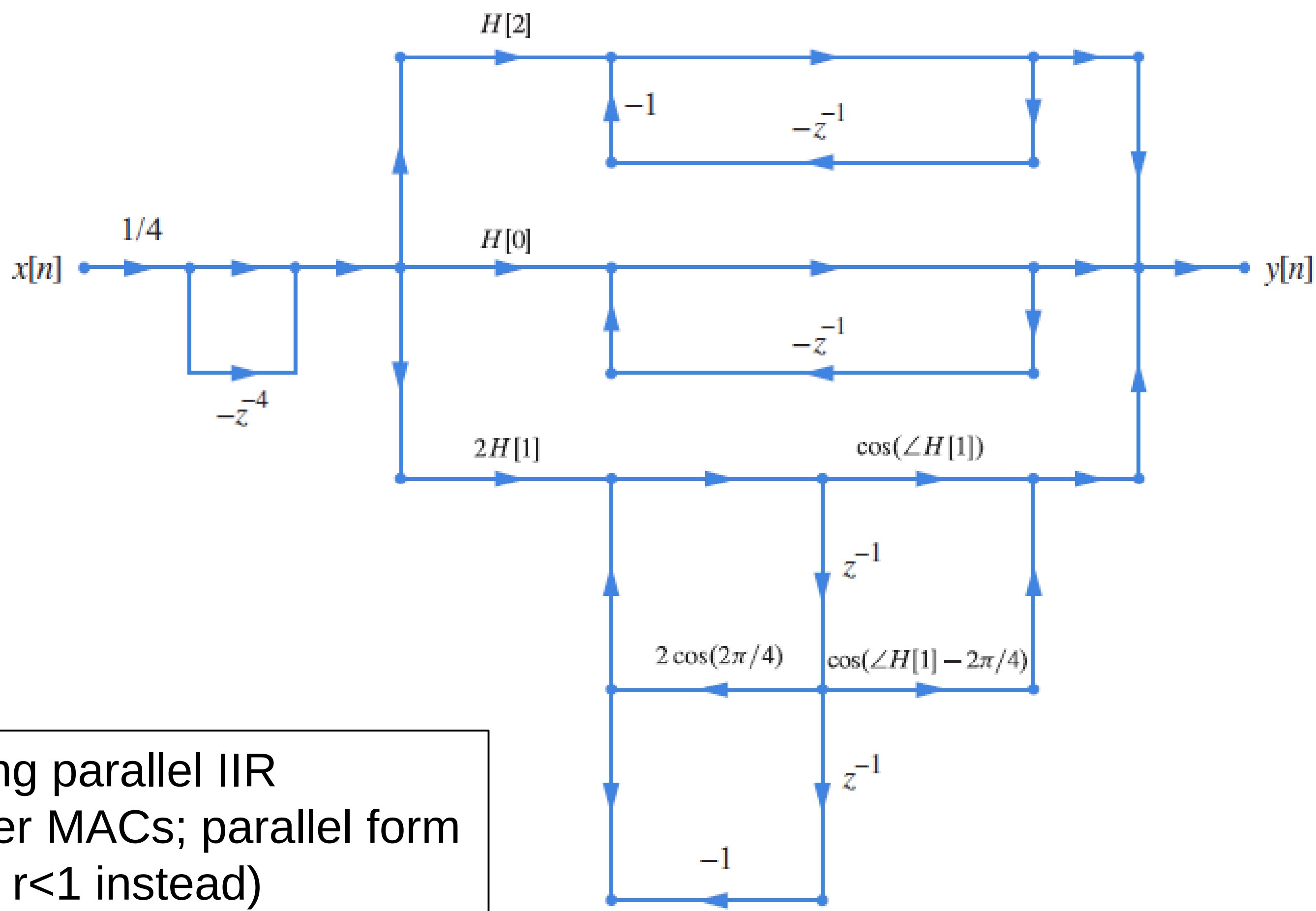
$$H(z) = \frac{1 - z^{-N}}{N} \sum_{k=0}^{N-1} \frac{H[k]}{1 - z^{-1} e^{j\frac{2\pi}{N}k}}, \quad H[k] = H(z)\big|_{z=e^{j2\pi k/N}}$$

⇩ Real-coefficient system

$$H(z) = \frac{1 - z^{-N}}{N} \left\{ \frac{H[0]}{1 - z^{-1}} + \frac{H\left[\frac{N}{2}\right]}{1 + z^{-1}} + \sum_{k=1}^{K} 2\left|H[k]\right| H_k(z) \right\}$$

$$H_k(z) = \frac{\cos\left(\angle H[k]\right) - z^{-1}\cos\left(\angle H[k] - \frac{2\pi k}{N}\right)}{1 - 2\cos\left(\frac{2\pi k}{N}\right) z^{-1} + z^{-2}}$$

Frequency sampling form

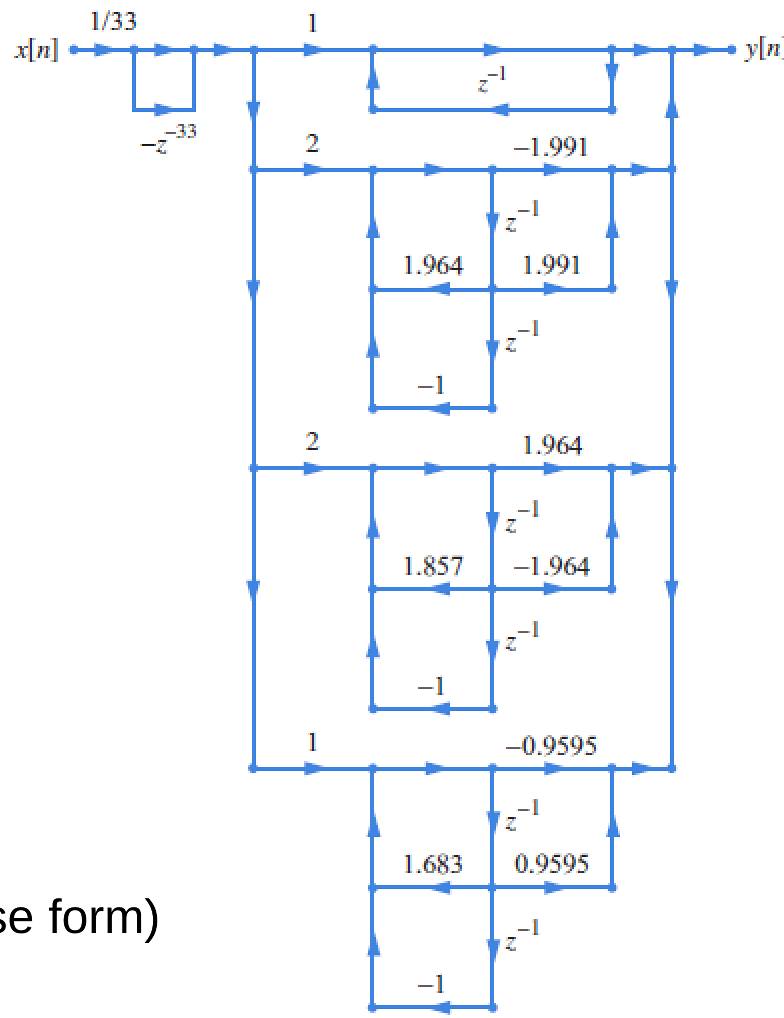


Implement FIR using parallel IIR
Pro: could use fewer MACs; parallel form
Con: unstable (use r<1 instead)

Example of frequency sampling form

$$H[k] = H(e^{j\frac{2\pi}{33}k}) = e^{-j\frac{32\pi}{33}k} \times \begin{cases} 1, & k = 0, 1, 2, 31, 3 \\ 0.5, & k = 3, 30 \\ 0. & \text{otherwise} \end{cases}$$

$$H(z) = \frac{1 - z^{-33}}{33}\left[\frac{1}{1 - z^{-1}} + \frac{-1.99 + 1.99z^{-1}}{1 - 1.964z^{-1} + z^{-2}} + \frac{1.964 - 1.964z^{-1}}{1 - 1.857z^{-1} + z^{-2}} + \frac{-1.96 + 1.96z^{-1}}{1 - 1.683z^{-1} + z^{-2}}\right.$$



Only 9 multipliers required (17 for linear-phase form)

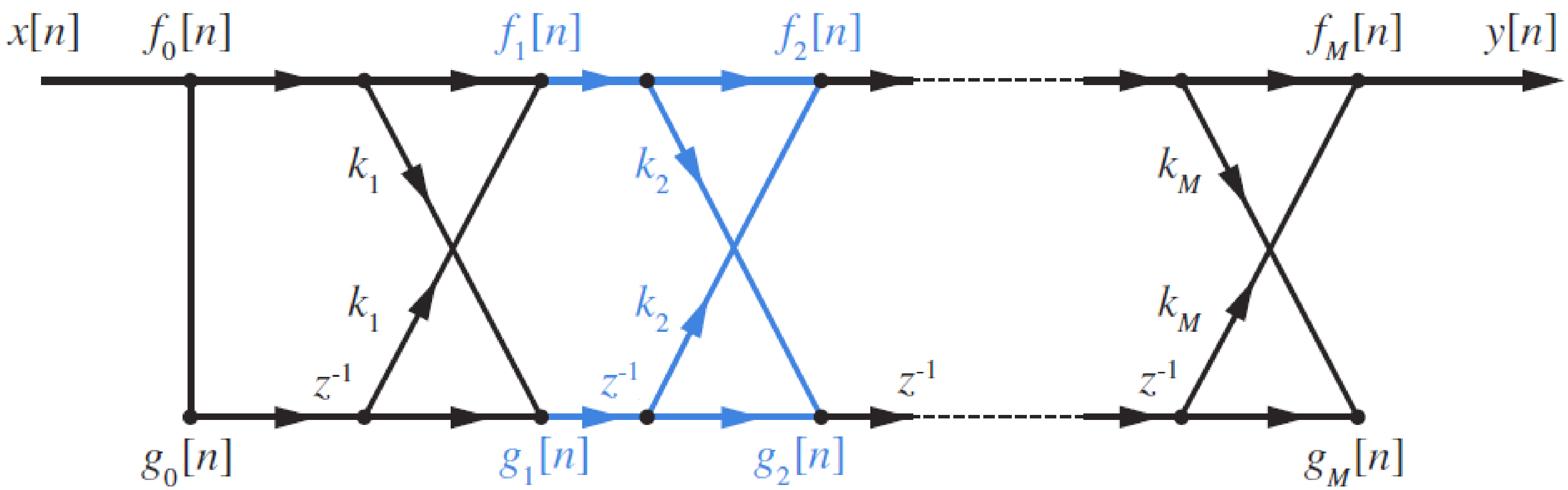
All-zero lattice structure (FIR)

$$f_m[n] = f_{m-1}[n] + k_m g_{m-1}[n-1], \quad m = 1, 2, \ldots, M$$

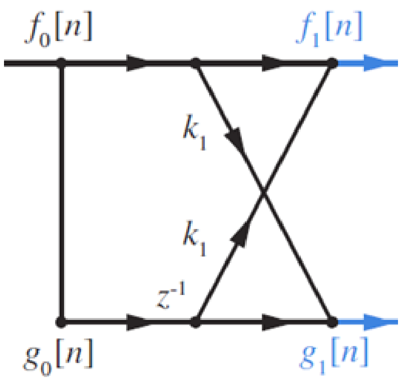
$$g_m[n] = k_m f_{m-1}[n] + g_{m-1}[n-1]. \quad m = 1, 2, \ldots, M$$

$$x[n] = f_0[n] = g_0[n]$$

$$y[n] = f_M[n].$$



Recursive solving for lattice coefficients

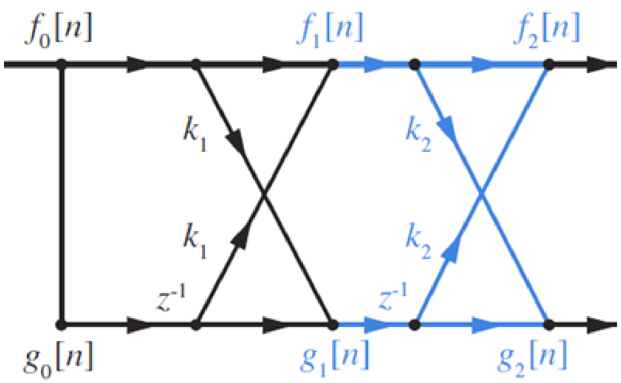


$$f_1[n] = x[n] + k_1 x[n-1]$$

$$g_1[n] = k_1 x[n] + x[n-1]$$

$$y[n] = a_0^{(1)} x[n] + a_1^{(1)} x[n-1]$$

$$a_0^{(1)} \triangleq 1, \quad a_1^{(1)} \triangleq k_1$$



$$\begin{aligned} f_2[n] &= f_1[n] + k_2 g_1[n-1] \\ &= (x[n] + k_1 x[n-1]) + k_2(k_1 x[n-1] + x[n-2]) \\ &= x[n] + (k_1 + k_1 k_2) x[n-1] + k_2 x[n-2], \end{aligned}$$

$$y[n] \triangleq f_2[n] = a_0^{(2)} x[n] + a_1^{(2)} x[n-1] + a_2^{(2)} x[n-2]$$

$$g_2[n] = a_2^{(2)} x[n] + a_1^{(2)} x[n-1] + a_0^{(2)} x[n-2]$$

$$a_0^{(2)} \triangleq 1, \quad a_1^{(2)} \triangleq k_1(1+k_2), \quad a_2^{(2)} \triangleq k_2$$

Recursive solving for lattice coefficients

Build recursive formulation on $A_m(z)$ and $B_m(z)$:

$$f_m[n] = \sum_{i=0}^{m} a_i^{(m)} x[n-i], \quad m = 1, 2, \ldots, M$$

$$g_m[n] = \sum_{i=0}^{m} a_{m-i}^{(m)} x[n-i]. \quad m = 1, 2, \ldots, M$$

$$A_m(z) \triangleq \frac{F_m(z)}{F_0(z)} = \sum_{i=0}^{m} a_i^{(m)} z^{-i}, \quad a_0^{(0)} = 1$$

$$B_m(z) \triangleq \frac{G_m(z)}{G_0(z)} = \sum_{i=0}^{m} a_{m-i}^{(m)} z^{-i} \triangleq \sum_{i=0}^{m} b_i^{(m)} z^{-i}$$

$$B_m(z) = z^{-m} A_m(1/z)$$

Recursive formulation:

$$A_m(z) = A_{m-1}(z) + k_m z^{-1} B_{m-1}(z)$$

$$B_m(z) = k_m A_{m-1}(z) + z^{-1} B_{m-1}(z)$$

$$k_m = a_m^{(m)}$$

Recursive solving for lattice coefficients

Initial condition $$H(z) = \sum_{k=0}^{M} h[k]\, z^{-k} \qquad a_k^{(M)} = h[k]/h[0] \qquad k_M = a_M^{(M)}$$

⇩

$$A_{m-1}(z) = \frac{1}{1-k_m^2}\left[A_m(z) - k_m B_m(z)\right]$$

⇩

k_{m-1}

⇩

A_{m-2}

⋮

Until solving k_1