Chap7
The Discrete Fourier Transform

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Chap 7 Discrete Fourier transform

- 7.1 Computational Fourier analysis
- 7.2 The discrete Fourier transform (DFT)
- 7.3 Sampling the DTFT
- 7.4 Properties of the DFT
- 7.5 Linear convolution using the DFT
- 7.6 Fourier analysis of signals using the DFT

Operations for Fourier transforms

		Direct transform (spectral analysis)	Inverse transform (signal reconstruction)	Exact computation
DFT	DTFS	$\tilde{c}_k = \frac{1}{N}\sum_{n=0}^{N-1}\tilde{x}[n]e^{-j\frac{2\pi}{N}kn}$ finite summation	$\tilde{x}[n] = \sum_{k=0}^{N-1}\tilde{c}_k e^{j\frac{2\pi}{N}kn}$ finite summation	yes
	DTFT	$\tilde{X}(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$ infinite summation	$x[n] = \frac{1}{2\pi}\int_0^{2\pi}\tilde{X}(e^{j\Omega})e^{j\Omega}d\omega$ integration	no
	CTFS	$c_k = \frac{1}{T_0}\int_0^{T_0}\tilde{x}_c(t)e^{-jk\Omega_0 t}dt$ integration	$\tilde{x}_c(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\Omega_0 t}$ infinite summation	no
	CTFT	$X_c(j\Omega) = \int_{-\infty}^{\infty} x_c(t)e^{-j\Omega t}dt$ integration	$x_c(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_c(j\Omega)e^{j\Omega t}d\Omega$ integration	no

Computing CTFT, CTFS, and DTFT

Computing CTFT
(via DTFT)

$$X_c(j\Omega) = \int_{-\infty}^{\infty} x_c(t)e^{-j\Omega t}dt \approx \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\Omega nT}(T) \triangleq \hat{X}_c(j\Omega)$$

$$X_c(j\Omega) = \begin{cases} T\tilde{X}(e^{j\Omega T}), & |\Omega| < \pi/T \\ 0. & \text{otherwise} \end{cases} \qquad \tilde{X}(e^{j\omega}) \text{ is the DTFT}$$

Approximate CTFT by DTFT but need to consider the effect of periodic spectrum and aliasing distortion (due to undersampling).

Computing CTFS
(via DTFS)

$$c_k \approx \frac{1}{T_0}\sum_{n=0}^{N-1} \tilde{x}_c(nT)e^{-jk\Omega_0 nT}(T) = \frac{1}{N}\sum_{n=0}^{N-1} \tilde{x}[n]e^{-j\frac{2\pi}{N}kn} = \tilde{c}_k$$

Approximate CTFS by DTFS but need to consider the effect of aliasing distortion (due to undersampling).

Computing DTFT
(via DTFS)

$$\tilde{X}(e^{j\omega}) \approx \sum_{n=0}^{N-1} x[n]e^{-j\omega n} \triangleq \tilde{X}_N(e^{j\omega}) \qquad x_N[n] \triangleq x[n]p_N[n] \qquad \tilde{c}_k = \frac{1}{N}X[k]$$

Approximate DTFT by DTFS but need to consider the effect of finite-segment windowing.

Discrete Fourier Transform (DFT)

Analysis equation **Synthesis equation**

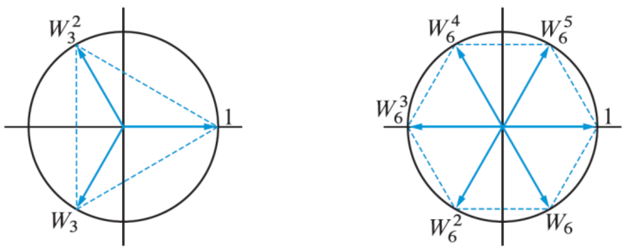
$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \quad \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} \quad x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}$$

Twiddle factor

$$W_N \triangleq e^{-j\frac{2\pi}{N}}$$

Roots of unity

$$\left(W_N^{-k}\right)^N = \left(e^{j\frac{2\pi}{N}k}\right)^N = e^{j2\pi k} = 1$$



$N = 3$ $N = 6$

Orthogonality

$$\frac{1}{N} \sum_{n=0}^{N-1} W_N^{(k-m)n} = \frac{1}{N} \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-m)n} = \begin{cases} 1, & k - m = rN \\ 0. & \text{otherwise} \end{cases}$$

Matrix formulation of DFT

DFT

$$\begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & W_N & \cdots & W_N^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

$$\boldsymbol{X}_N = \boldsymbol{W}_N \boldsymbol{x}_N. \quad \text{(DFT)}$$

IDFT

$$\boldsymbol{x}_N = \boldsymbol{W}_N^{-1} \boldsymbol{X}_N$$

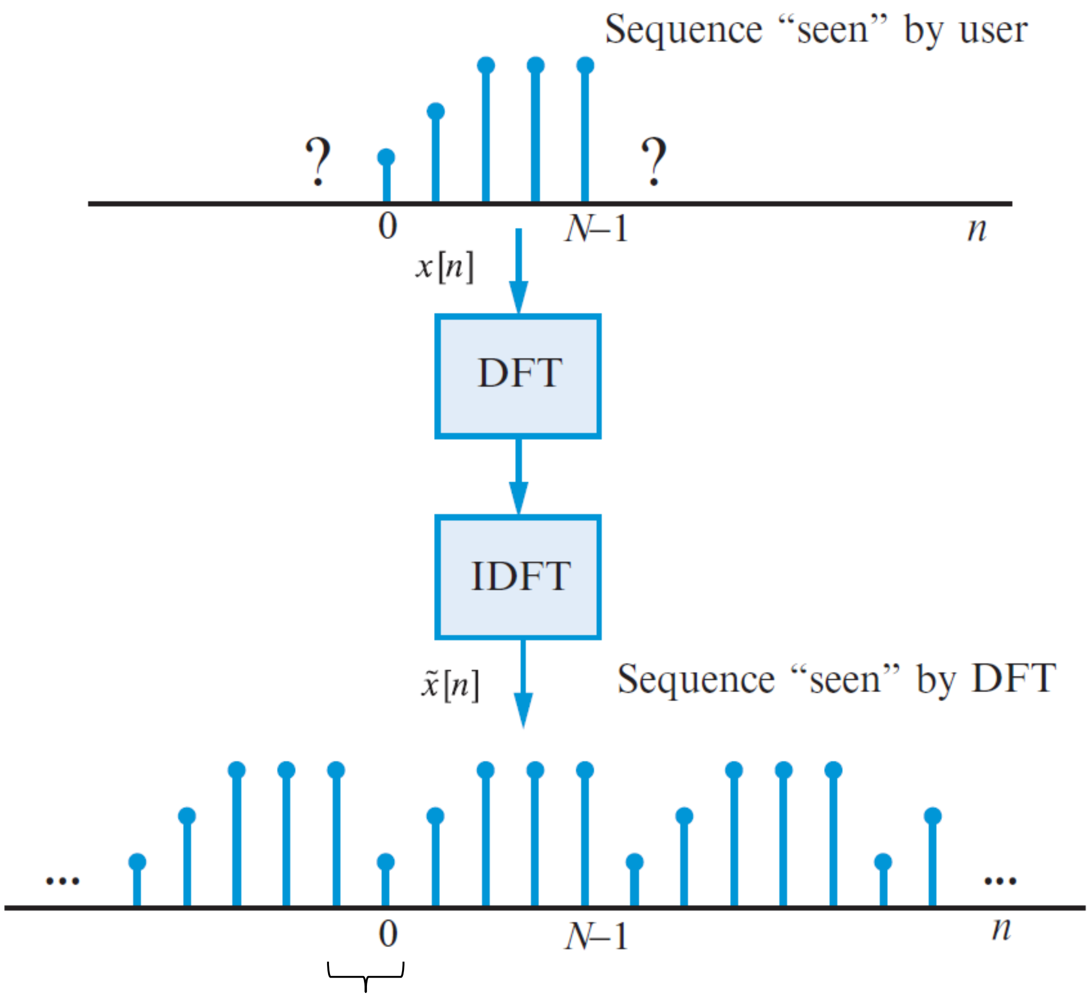
$$\boldsymbol{W}_N^{-1} = \frac{1}{N} \boldsymbol{W}_N^H = \frac{1}{N} \boldsymbol{W}_N^*$$

Conjugate transpose

Complexity

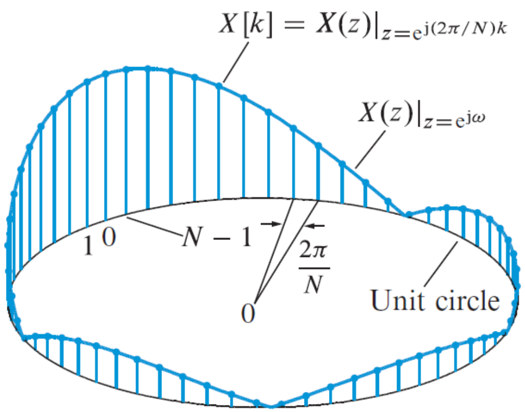
Matrix-by-vector multiplication requires $O(N^2)$ operations;
Fast Fourier transform (FFT), in Chap 8, needs only O(NlogN).

Periodicity of DFT



Could introduce high-intensity high-frequency components which do not belong to the signal itself => any alternative?

Sampling DTFT in frequency domain



Aperiodic signal

$$x[n] = \frac{1}{2\pi}\int_0^{2\pi} \tilde{X}(e^{j\omega})e^{-j\omega n}d\omega$$

Sampling DTFT as DFT

$$X[k] \triangleq \tilde{X}\left(e^{j\frac{2\pi}{N}k}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\frac{2\pi}{N}kn}$$

$$X[k] = \sum_{n=0}^{N-1}\left(\sum_{\ell=-\infty}^{\infty} x[n-\ell N]\right)e^{-j\frac{2\pi}{N}kn}$$

IDFT

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}\left(e^{j\frac{2\pi}{N}k}\right)e^{j\frac{2\pi}{N}kn} = \frac{1}{N}\sum_{k=0}^{N-1}X[k]W_N^{-kn}$$

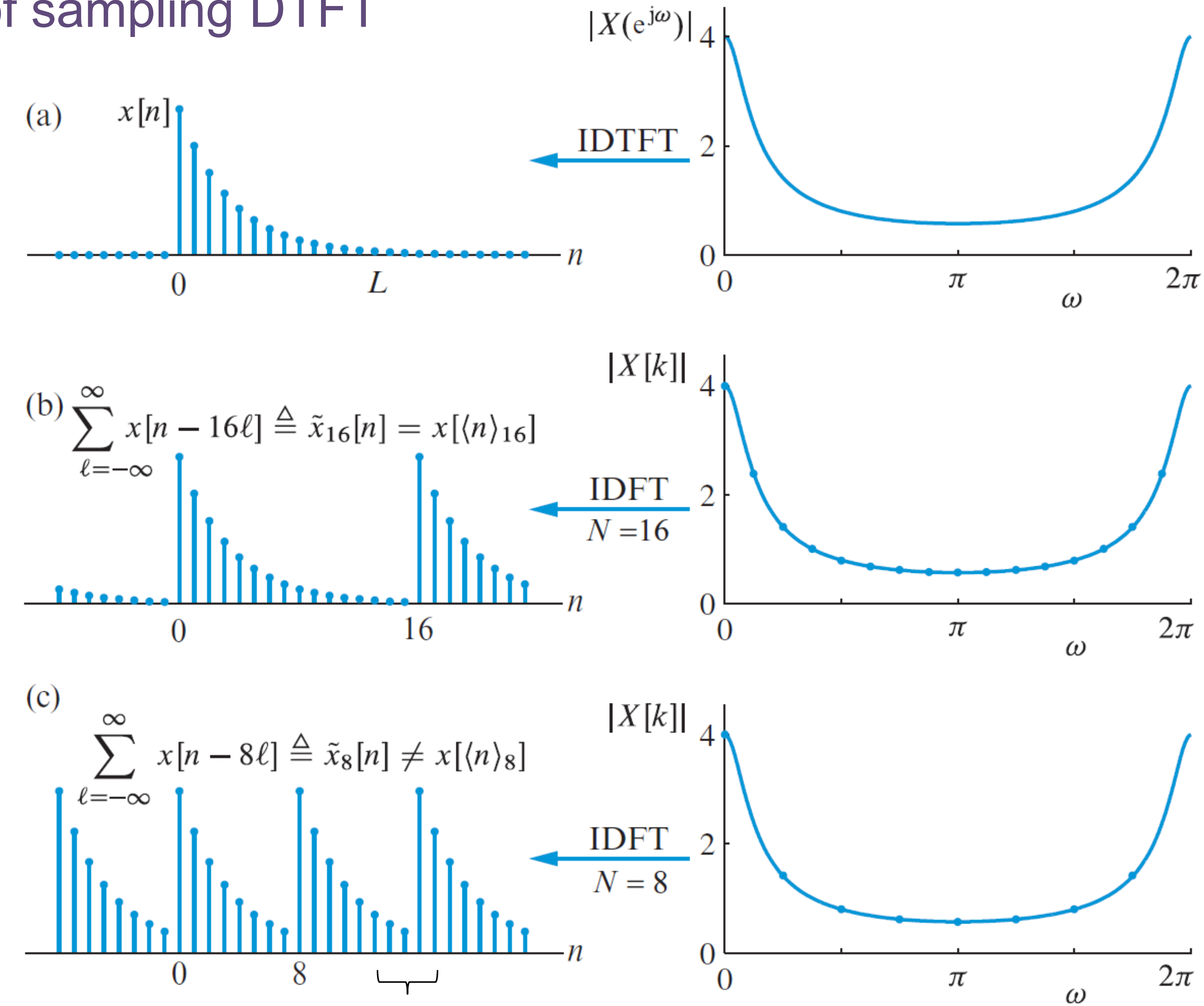
Periodic extension

$$\tilde{x}[n] \triangleq \sum_{\ell=-\infty}^{\infty} x[n-\ell N]$$

(may introduce time-domain aliasing)

IDTFT approximation $x[n] = \tilde{x}[n]p_N[n]$

Example of sampling DTFT



time-domain aliasing due to insufficient sampling

Example of sampling and reconstruction of DTFT

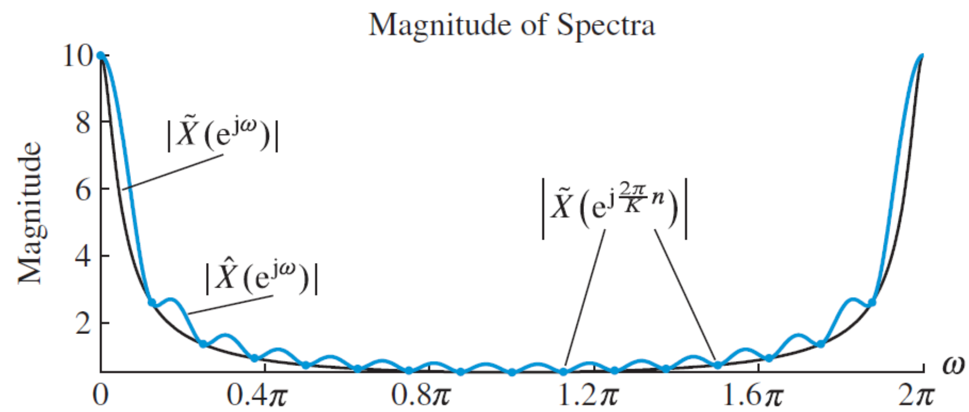
Causal exponential sequence $x[n] = a^n u[n], \quad 0 < a < 1 \overset{\text{DTFT}}{\longleftrightarrow} \tilde{X}(e^{j\omega}) = \dfrac{1}{1 - ae^{-j\omega}}$

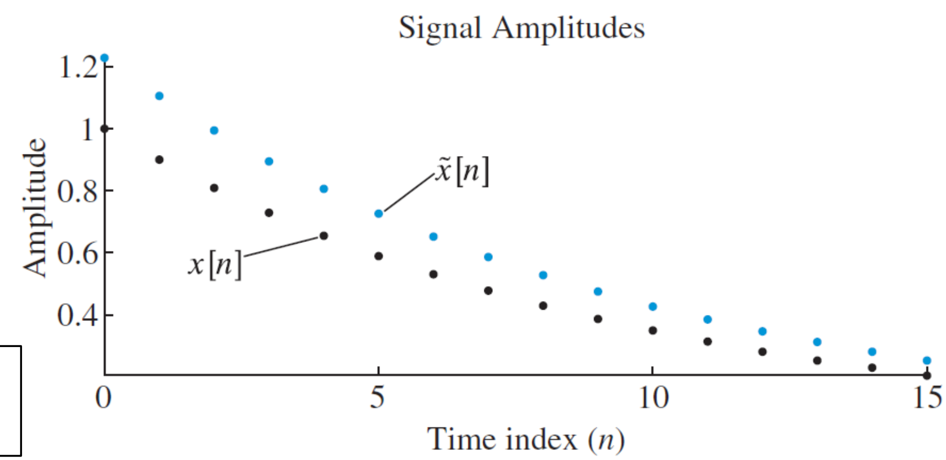
Sampling of DTFT $\tilde{X}(e^{j\omega})$ is sampled at frequencies $\omega_k = (2\pi/N)k,\ 0 \le k \le N-1$

Reconstruction of DTFT

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}\left(e^{j\frac{2\pi}{N}k}\right) e^{j\frac{2\pi}{N}kn}$$

$$\tilde{X}_N\left(e^{j\frac{2\pi}{K}k}\right) = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{K}kn}$$





If time-domain aliasing exists, DTFT is unable to be reconstructed from its sampled DFT.

Ideal DTFT reconstruction for timelimited signals

DTFT of N-point sequence

$$\tilde{X}(e^{j\omega}) = \sum_{n=0}^{N-1} x[n]e^{-j\omega n}$$

Signal reconstruction from DTFT samples

$$x[n] = \left[\frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}(e^{j\frac{2\pi}{N}k})e^{j\frac{2\pi}{N}kn}\right]p_N[n]$$

Ideal interpolation for DTFT reconstruction

$$\tilde{X}(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}(e^{j\frac{2\pi}{N}k})\tilde{P}_N\left[e^{j(\omega-\frac{2\pi}{N}k)}\right]$$

$$\tilde{P}_N(e^{j\omega}) = \frac{\sin(\omega N/2)}{N\sin(\omega/2)}e^{-j\omega(N-1)/2}$$

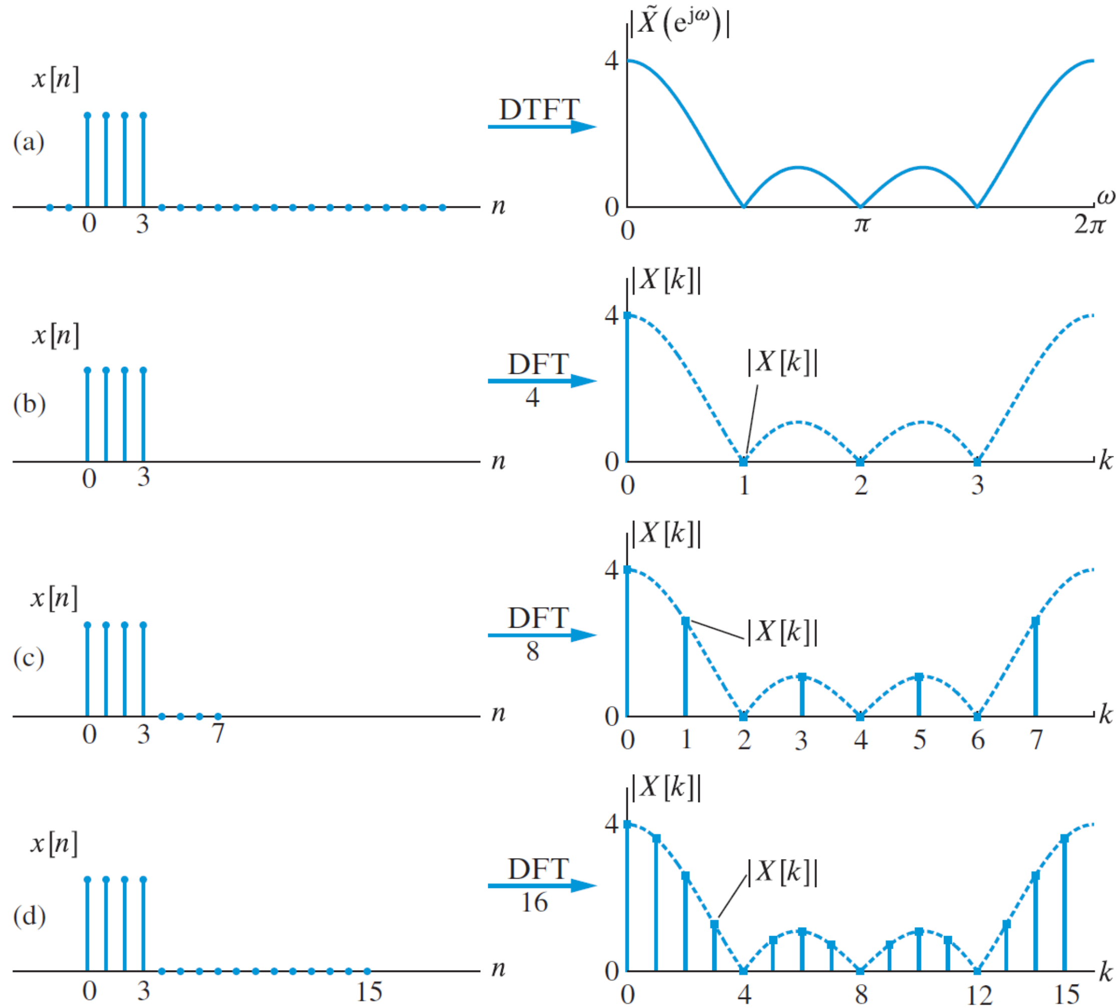
Practical DTFT reconstruction by zero padding

Zero-padded sequence

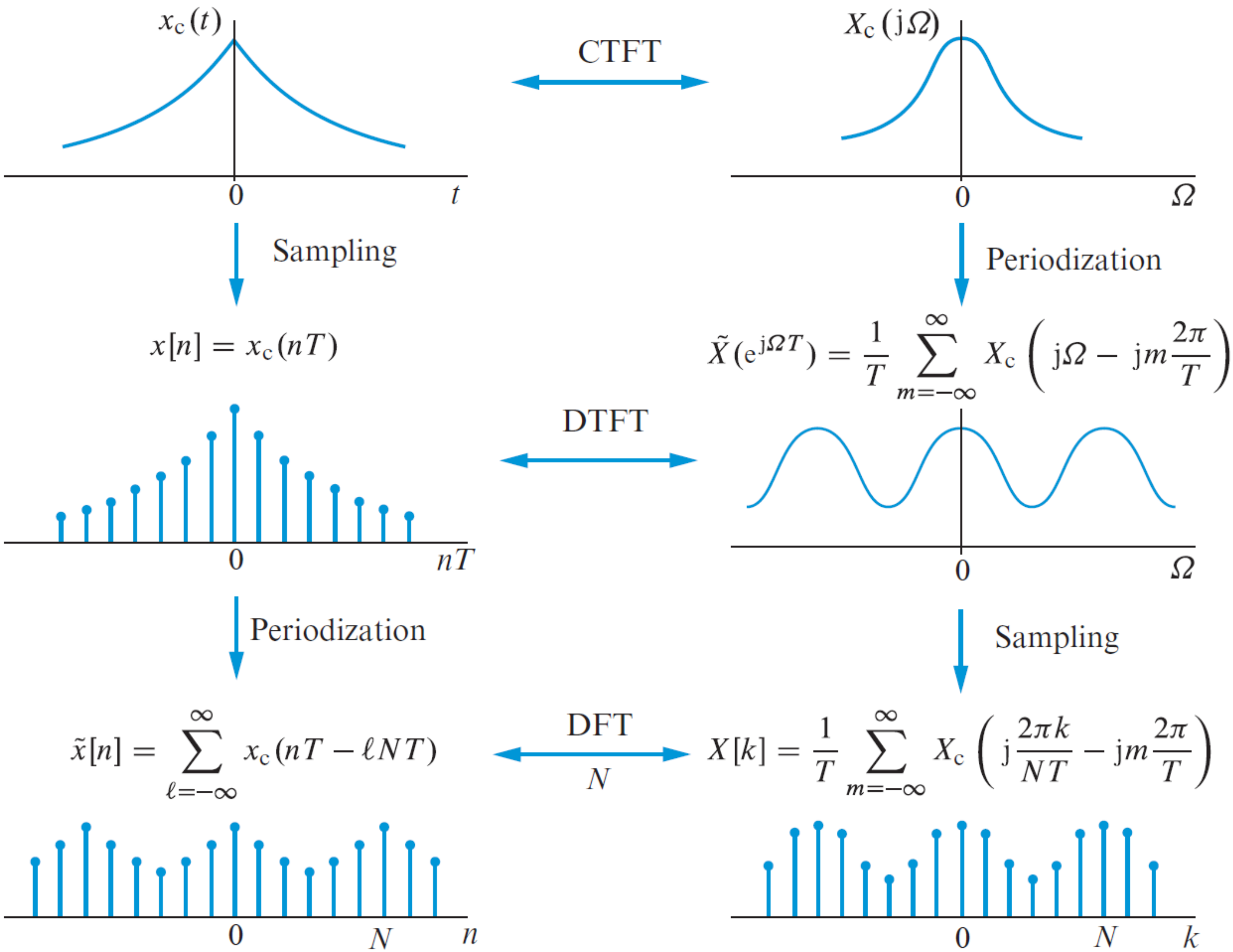
$$x_{\text{zp}}[n] \triangleq \begin{cases} x[n], & 0 \le n \le N-1 \\ 0, & N \le n \le K-1 \end{cases}$$

K-point IDFT

$$\tilde{X}(e^{j\frac{2\pi}{K}k}) = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{K}kn} = \sum_{n=0}^{K-1} x_{\text{zp}}[n] W_K^{kn} = X_{\text{zp}}$$



Relationship between CTFT, DTFT, and DFT



Periodic and circular properties of DFT

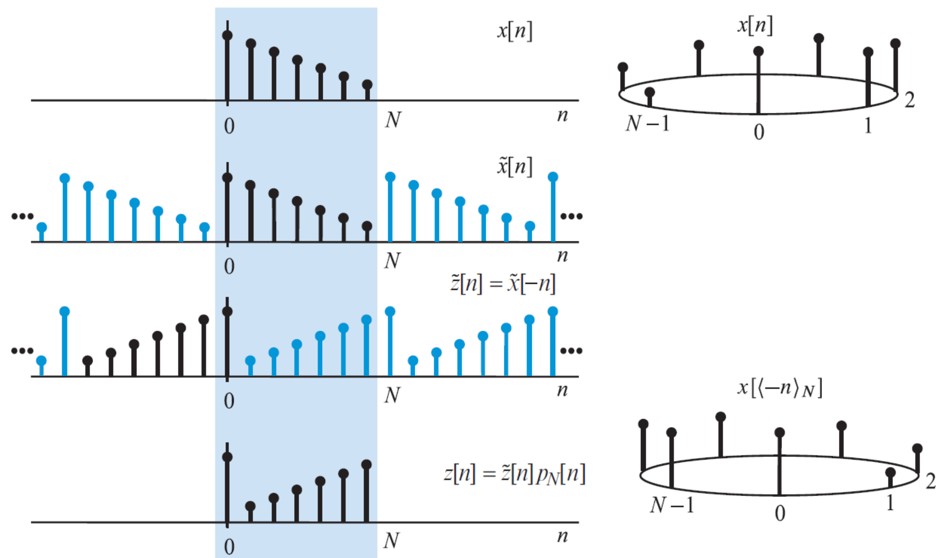
Modulo-N operation $n = \ell N + r,\ 0 \le r \le N-1 \Rightarrow \langle n \rangle_N \triangleq n \text{ modulo } N = r$

Periodic extension $\tilde{x}[n] = x[\langle n \rangle_N], \quad \text{for all } n \qquad \tilde{X}[k] = X[\langle k \rangle_N]. \quad \text{for all } k$

Circular folding

$$z[n] = x[\langle -n \rangle_N] \triangleq \begin{cases} x[0], & n = 0 \\ x[N-n]. & 1 \le n \le N-1 \end{cases}$$

$$x[\langle -n \rangle_N] \xleftrightarrow[N]{\text{DFT}} X[\langle -k \rangle_N]$$



Symmetry properties of DFT for real-valued sequence

$$x[n] = x_R[n] + jx_I[n], \quad 0 \le n \le N-1$$

$$X[k] = X_R[k] + jX_I[k]. \quad 0 \le k \le N-1$$

$$x_I[n] = 0$$

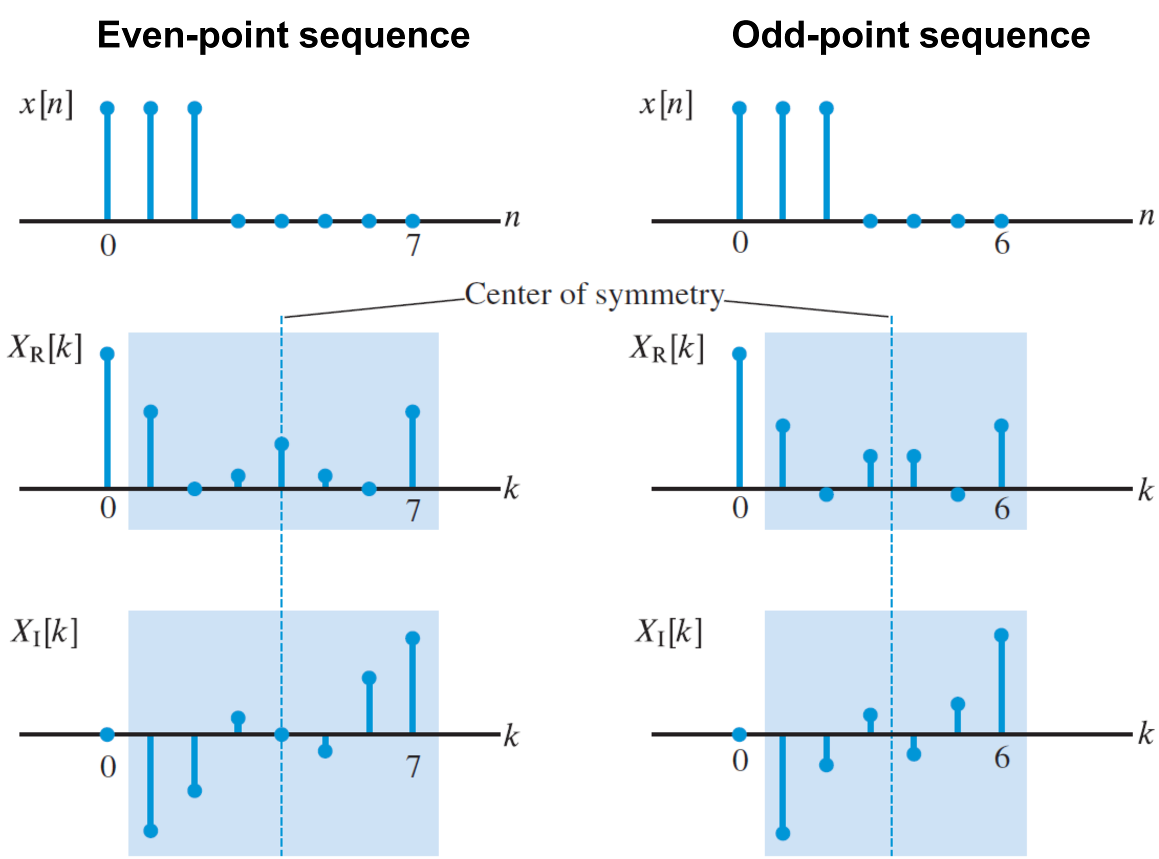
$$X_R[k] = \sum_{n=0}^{N-1} x_R[n]\cos\left(\frac{2\pi}{N}kn\right) = \sum_{n=0}^{N-1} x[n]\cos\left(\frac{2\pi}{N}kn\right)$$

$$X_I[k] = \sum_{n=0}^{N-1} x_R[n]\sin\left(\frac{2\pi}{N}kn\right) = \sum_{n=0}^{N-1} x[n]\sin\left(\frac{2\pi}{N}kn\right)$$

$$X_R[k] = X_R[\langle -k\rangle_N]$$

$$X^*[k] = X[\langle -k\rangle_N]$$

$$X_I[k] = -X_I[\langle -k\rangle_N]$$



Circular-even/-odd decomposition

A real-valued sequence can be decomposed into two sequences: one is circular-even symmetric and another circular-odd:

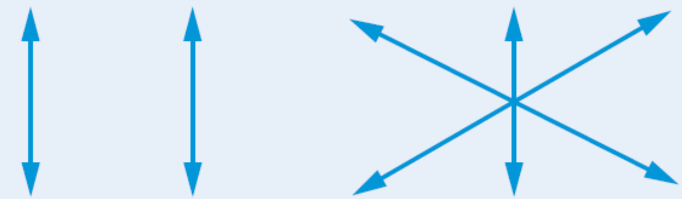
$$x[n] = x^{\text{ce}}[n] + x^{\text{co}}[n]$$

$$x^{\text{ce}}[n] \triangleq \frac{x[n] + x[\langle -n \rangle_N]}{2} = x^{\text{ce}}[\langle -n \rangle_N]$$

$$x^{\text{co}}[n] \triangleq \frac{x[n] - x[\langle -n \rangle_N]}{2} = -x^{\text{co}}[\langle -n \rangle_N]$$

A complex-valued sequence can be decomposed into four circular symmetric sequence:

$$x[n] = x_R^{\text{ce}}[n] + x_R^{\text{co}}[n] + \text{j}x_I^{\text{ce}}[n] + \text{j}x_I^{\text{co}}[n]$$



$$X[k] = X_R^{\text{ce}}[k] + X_R^{\text{co}}[k] + \text{j}X_I^{\text{ce}}[k] + \text{j}X_I^{\text{co}}[k]$$

N-point sequence	*N*-point DFT
Real	real part is even - imaginary part is odd
Imaginary	real part is odd - imaginary part is even
Real and even	real and even
Real and odd	imaginary and odd
Imaginary and even	imaginary and even
Imaginary and odd	real and odd

Symmetry properties of DFT

N-point Sequence	*N*-point DFT
Complex signals	
$x^*[n]$	$X^*[\langle -k\rangle_N]$
$x^*[\langle -n\rangle_N]$	$X^*[k]$
$x_R[n]$	$X^{cce}[k] = \frac{1}{2}(X[k] + X^*[\langle -k\rangle_N])$
$jx_I[n]$	$X^{cco}[k] = \frac{1}{2}(X[k] - X^*[\langle -k\rangle_N])$
$x^{cce}[n] = \frac{1}{2}(x[n] + x^*[\langle -n\rangle_N])$	$X_R[k]$
$x^{cco}[n] = \frac{1}{2}(x[n] - x^*[\langle -n\rangle_N])$	$jX_I[k]$
Real signals	
Any real $x[n]$	$\begin{cases} X[k] = \tilde{X}^*[\langle -k\rangle_N] \\ X_R[k] = X_R[\langle -k\rangle_N] \\ X_I[k] = -X_I[\langle -k\rangle_N] \\ \lvert X[k]\rvert = \lvert X[\langle -k\rangle_N]\rvert \\ \angle X[k] = -\angle X[\langle -k\rangle_N] \end{cases}$

Fast computation for DFT of two real-valued sequences

$$x[n] = x_1[n] + jx_2[n]$$

$$X_1[k] = X^{\text{cce}}[k] \quad \text{and} \quad jX_2[k] = X^{\text{cco}}[k]$$

$$X^{\text{cce}}[k] = \frac{1}{2}(X[k] + X^*[\langle -k \rangle_N])$$

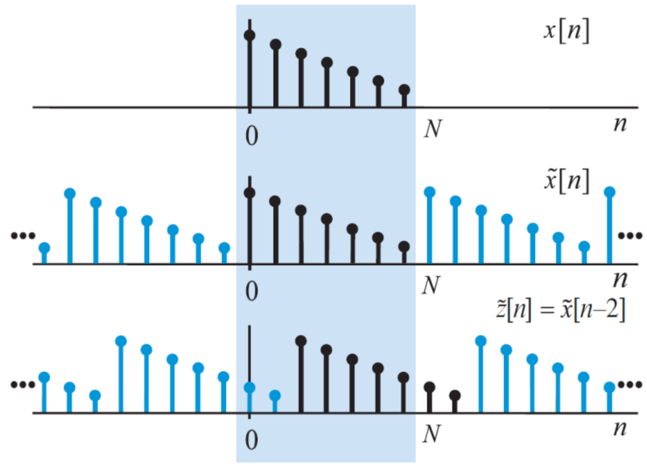
$$X^{\text{cco}}[k] = \frac{1}{2}(X[k] - X^*[\langle -k \rangle_N])$$

One, instead of two, complex-valued DFT computation for two real-valued sequences.

Circular operations

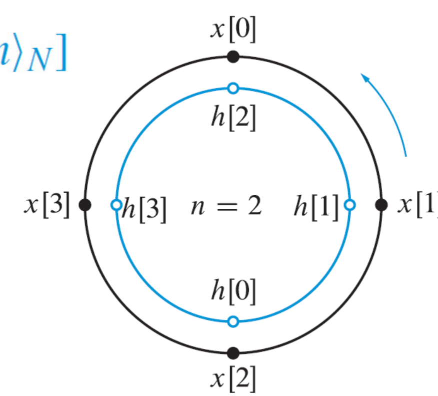
Circular shift $z[n] = x[\langle n-m\rangle_N]$

$$x[\langle n-m\rangle_N] \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} W_N^{km}X[k]$$



Circular convolution $y[n] \triangleq h[n] \circledN x[n]$ $\quad y[n] = \sum_{m=0}^{N-1} h[m]x[\langle n-m\rangle_N]$

$$y[n] = h[n] \circledN x[n] \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} Y[k] = H[k]X[k]$$



Circular correlation

$$r_{xy}[\ell] \triangleq \sum_{n=0}^{N-1} x[n]y[\langle n-\ell\rangle_N]$$

$$r_{xy}[\ell] = x[n] \circledN y[\langle -n\rangle_N] \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} R_{xy}[k] = X[k]Y[\langle -k\rangle_N]$$

DFT of upsampled and downsampled sequences

Time-domain upsampling
(leads to DFT-domain periodic extension)

N-point

$$\underset{\text{LN-point}}{x^{(L)}[n]} \triangleq \begin{cases} x[n/L], & n = 0, L, \ldots, (N-1)L \\ 0. & \text{otherwise.} \end{cases}$$

$$x^{(L)}[n] \overset{\text{DFT}}{\underset{NL}{\longleftrightarrow}} \tilde{X}[k] = X[\langle k \rangle_N]$$

DFT-domain upsampling
(leads to time-domain periodic extension)

$$\frac{1}{L} x[\langle n \rangle_N] = \frac{1}{L} \tilde{x}[n] \overset{\text{DFT}}{\underset{NL}{\longleftrightarrow}} X^{(L)}[k]$$

Time-domain downsampling
(leads to DFT-domain overlapping/aliasing)

$$x_{(M)}[n] = x[nM]. \quad 0 \le n \le \frac{N}{M} - 1$$

$$x_{(M)}[n] \overset{\text{DFT}}{\underset{N/M}{\longleftrightarrow}} \frac{1}{M} \sum_{m=0}^{M-1} X\left[k + m\frac{N}{M}\right]$$

DFT-domain downsampling
(leads to time-domain overlapping/aliasing)

$$\frac{1}{M} \sum_{m=0}^{M-1} x\left[n + m\frac{N}{M}\right] \overset{\text{DFT}}{\underset{N/M}{\longleftrightarrow}} X_{(M)}[k]$$

Summary of DFT properties

	Property	N-point sequence	N-point DFT
		$x[n], h[n], v[n]$	$X[k], H[k], V[k]$
		$x_1[n], x_2[n]$	$X_1[k], X_2[k]$
1.	Linearity	$a_1x_1[n] + a_2x_2[n]$	$a_1X_1[k] + a_2X_2[k]$
2.	Time shifting	$x[\langle n-m\rangle_N]$	$W_N^{km}X[k]$
3.	Frequency shifting	$W_N^{-mn}x[n]$	$X[\langle k-m\rangle_N]$
4.	Modulation	$x[n]\cos(2\pi/N)k_0n$	$\frac{1}{2}X[\langle k+k_0\rangle_N] + \frac{1}{2}X[\langle k-k_0\rangle_N]$
5.	Folding	$x[\langle -n\rangle_N]$	$X[\langle -k\rangle_N]$
6.	Conjugation	$x^*[n]$	$X^*[\langle -k\rangle_N]$
7.	Duality	$X[n]$	$Nx[\langle -k\rangle]_N]$
8.	Convolution	$h[n] \circledN x[n]$	$H[k]X[k]$
9.	Correlation	$x[n] \circledN y[\langle -n\rangle_N]$	$X[k]Y[\langle -k\rangle_N]$
10.	Windowing	$v[n]x[n]$	$\frac{1}{N}V[k] \circledN X[k]$
11.	Parseval's theorem	$\sum_{n=0}^{N-1} x[n]y^*[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]Y^*[k]$	
12.	Parseval's relation	$\sum_{n=0}^{N-1} \|x[n]\|^2 = \frac{1}{N}\sum_{k=0}^{N-1} \|X[k]\|^2$	

Linear convolution using DFT

Linear convolution

$x[n],\ 0 \le n \le L-1$ L-point input signal

$h[n],\ 0 \le n \le M-1$ M-point impulse response

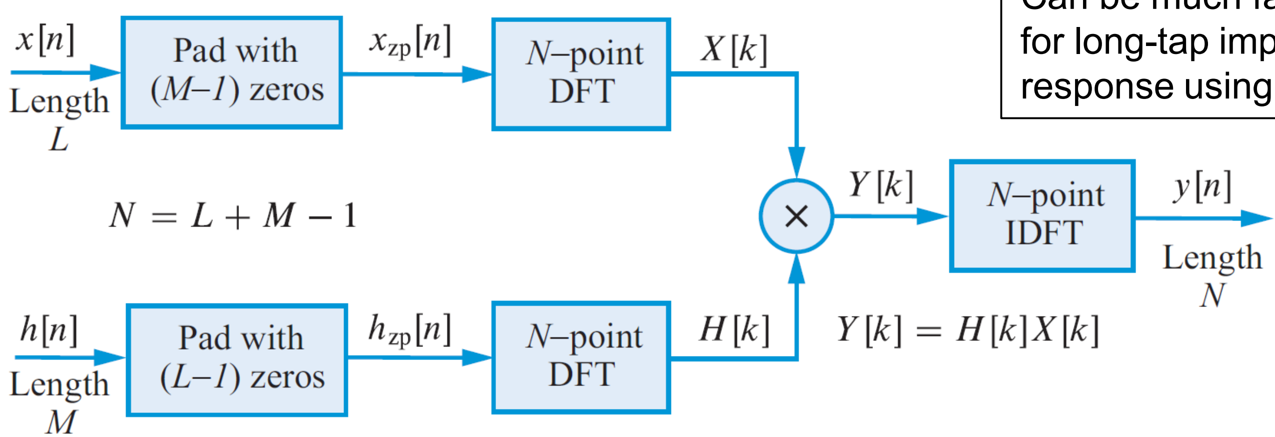
$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]. \quad 0 \le n \le L+M-2$$

(L+M-1)-point output signal

Zero padding +
Circular convolution

$$y_{zp}[n] = h_{zp}[n] \, \textcircled{N} \, x_{zp}[n] \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} Y[k] = H[k]X[k]$$

Can be much faster for long-tap impulse response using FFT



Matrix interpretation

Linear convolution

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \\ y[5] \\ y[6] \end{bmatrix} = \begin{bmatrix} x[0] & 0 & 0 \\ x[1] & x[0] & 0 \\ x[2] & x[1] & x[0] \\ x[3] & x[2] & x[1] \\ x[4] & x[3] & x[2] \\ 0 & x[4] & x[3] \\ 0 & 0 & x[4] \end{bmatrix} \begin{bmatrix} h[0] \\ h[1] \\ h[2] \end{bmatrix}$$

Zero padding +
Circular convolution

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \\ y[5] \\ y[6] \end{bmatrix} = \begin{bmatrix} x[0] & 0 & 0 & x[4] & x[3] & x[2] & x[1] \\ x[1] & x[0] & 0 & 0 & x[4] & x[3] & x[2] \\ x[2] & x[1] & x[0] & 0 & 0 & x[4] & x[3] \\ x[3] & x[2] & x[1] & x[0] & 0 & 0 & x[4] \\ x[4] & x[3] & x[2] & x[1] & x[0] & 0 & 0 \\ 0 & x[4] & x[3] & x[2] & x[1] & x[0] & 0 \\ 0 & 0 & x[4] & x[3] & x[2] & x[1] & x[0] \end{bmatrix} \begin{bmatrix} h[0] \\ h[1] \\ h[2] \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Overlap-add method for indefinite-length input signals

Partition input signals into non-overlapped blocks to have overlapped output blocks. Add the overlapped part.

$$
\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \\ y[5] \\ y[6] \\ y[7] \\ y[8] \\ y[9] \end{bmatrix}
=
\begin{bmatrix}
h[0] & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
h[1] & h[0] & 0 & 0 & 0 & 0 & 0 & 0 \\
h[2] & h[1] & h[0] & 0 & 0 & 0 & 0 & 0 \\
0 & h[2] & h[1] & h[0] & 0 & 0 & 0 & 0 \\
0 & 0 & h[2] & h[1] & h[0] & 0 & 0 & 0 \\
0 & 0 & 0 & h[2] & h[1] & h[0] & 0 & 0 \\
0 & 0 & 0 & 0 & h[2] & h[1] & h[0] & 0 \\
0 & 0 & 0 & 0 & 0 & h[2] & h[1] & h[0] \\
0 & 0 & 0 & 0 & 0 & 0 & h[2] & h[1] \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & h[2]
\end{bmatrix}
\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \\ x[6] \\ x[7] \end{bmatrix}
$$

Overlap M-1 points between neighboring blocks

Overlap-save method for indefinite-length input signals

Partition input signals into Q-point overlapped blocks to generate (Q-M+1)-point output directly without additional additions.

$$
\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \\ y[5] \\ y[6] \\ y[7] \\ y[8] \\ y[9] \end{bmatrix}
=
\begin{bmatrix}
h[0] & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
h[1] & h[0] & 0 & 0 & 0 & 0 & 0 & 0 \\
h[2] & h[1] & h[0] & 0 & 0 & 0 & 0 & 0 \\
0 & h[2] & h[1] & h[0] & 0 & 0 & 0 & 0 \\
0 & 0 & h[2] & h[1] & h[0] & 0 & 0 & 0 \\
0 & 0 & 0 & h[2] & h[1] & h[0] & 0 & 0 \\
0 & 0 & 0 & 0 & h[2] & h[1] & h[0] & 0 \\
0 & 0 & 0 & 0 & 0 & h[2] & h[1] & h[0] \\
0 & 0 & 0 & 0 & 0 & 0 & h[2] & h[1] \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & h[2]
\end{bmatrix}
\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \\ x[6] \\ x[7] \end{bmatrix}
$$

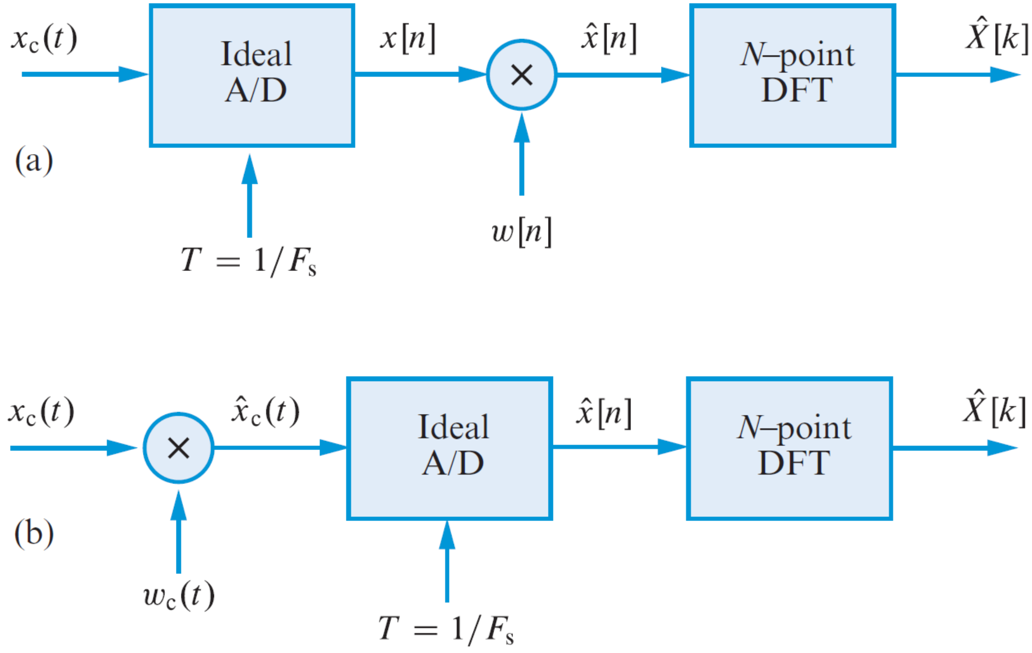
Can be implemented by QxQ circular convolution (without zero padding) since the first M-1 outputs will be discarded.

$$
\begin{bmatrix} y_{cir}[3] \\ y_{cir}[4] \\ y[5] \\ y[6] \\ y[7] \end{bmatrix}
=
\begin{bmatrix}
h[0] & 0 & 0 & h[2] & h[1] \\
h[1] & h[0] & 0 & 0 & h[2] \\
h[2] & h[1] & h[0] & 0 & 0 \\
0 & h[2] & h[1] & h[0] & 0 \\
0 & 0 & h[2] & h[1] & h[0]
\end{bmatrix}
\begin{bmatrix} x[3] \\ x[4] \\ x[5] \\ x[6] \\ x[7] \end{bmatrix}
$$

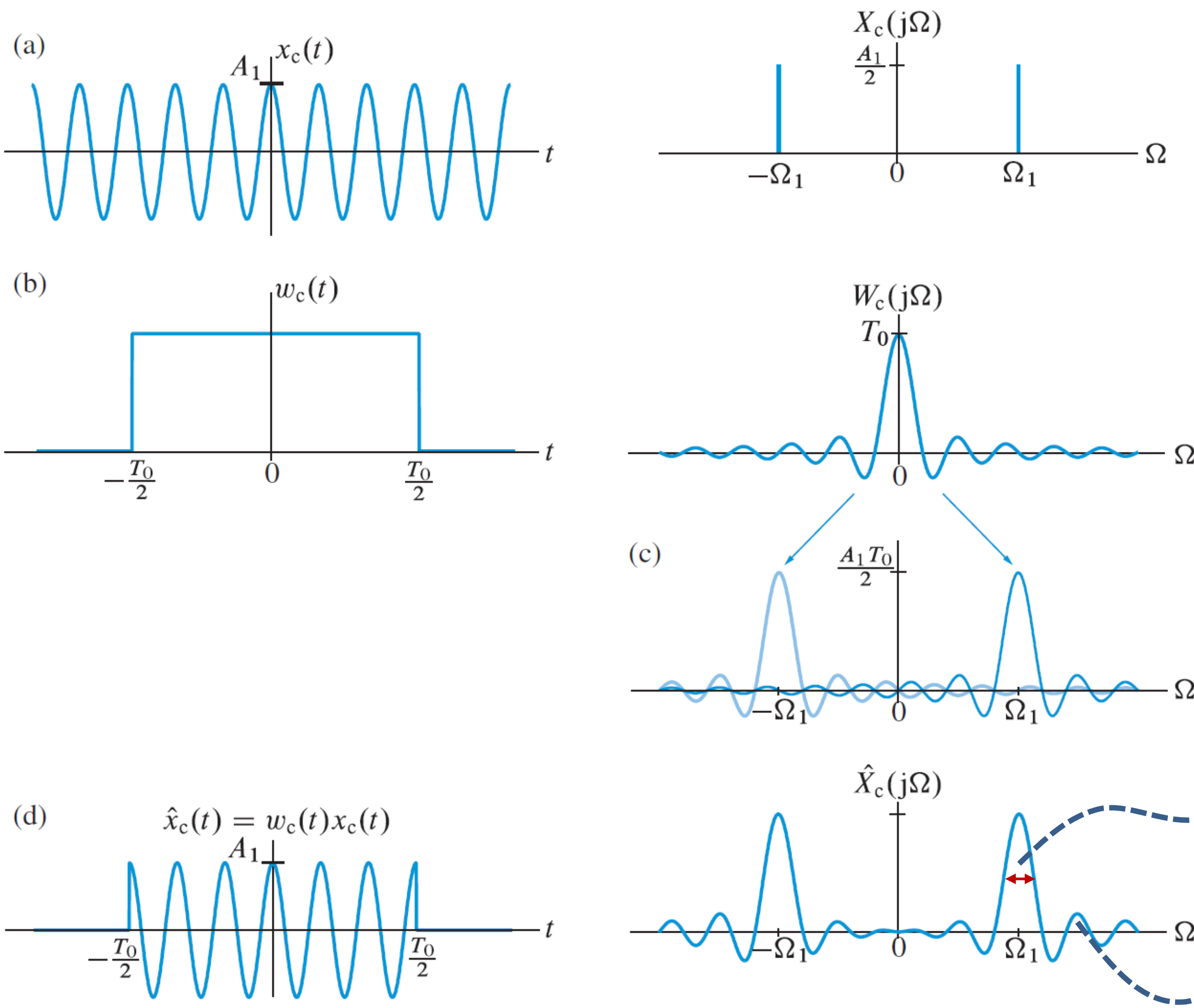
Fourier analysis of signals using DFT

Three steps to apply DFT for Fourier analysis:

1. Sample continuous-time signals (periodic sampling)
2. Select a finite-length segment (time windowing)
3. Compute the spectrum at a finite number of frequencies (frequency sampling)



Time-windowing on sinusoidal signals



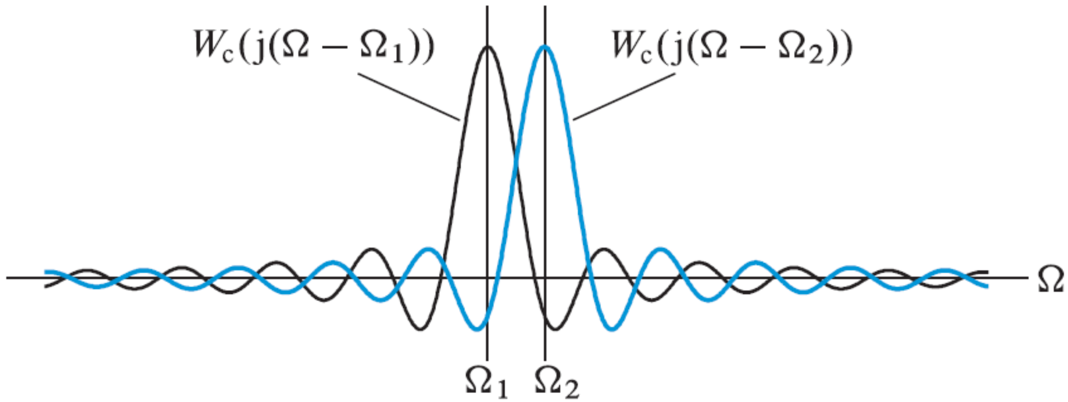
$$x_c(t) = \frac{1}{2}A_1 e^{-j\phi_1} e^{-j\Omega_1 t} + \frac{1}{2}A_1 e^{j\phi_1} e^{j\Omega_1 t}$$

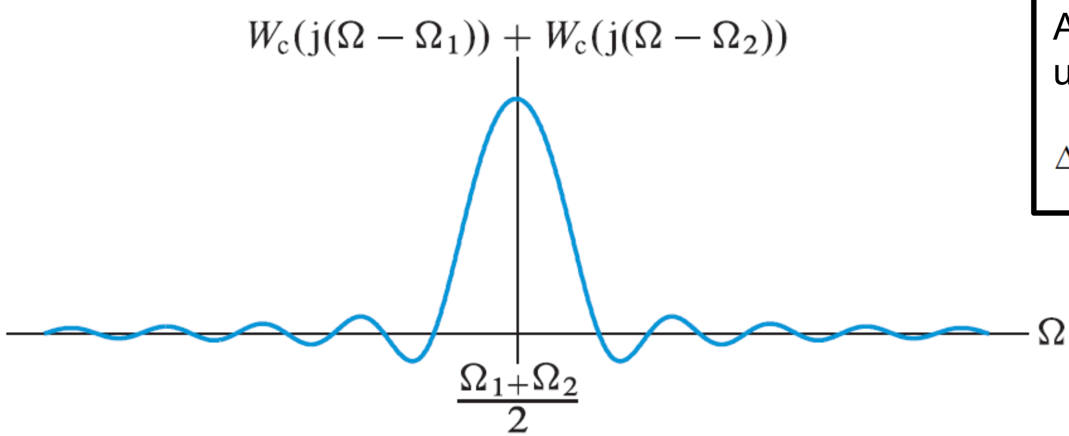
$$w_c(t) \triangleq w_R(t) = \begin{cases} 1, & 0 \le t \le T_0 \\ 0, & \text{otherwise} \end{cases}$$

Smearing
(spectral spreading)

Leakage

Loss of spectral resolution due to peak merging

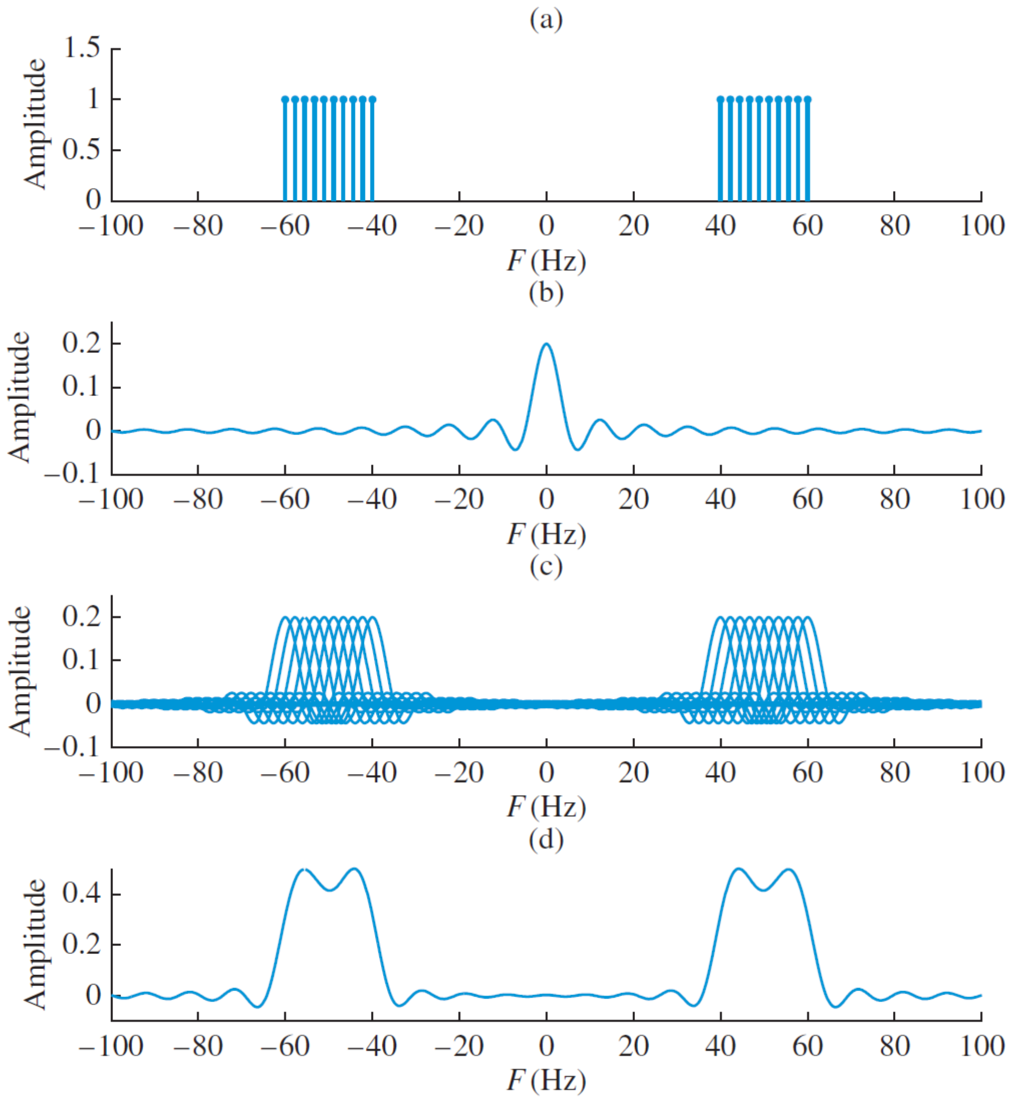




A sufficiently long window should be used to avoid peak merging.

$$\Delta\Omega = \Omega_2 - \Omega_1 \geq \frac{2\pi}{T_0} \quad \text{or} \quad T_0 \geq \frac{1}{F_2 - F_1}$$

Windowing on an ideal bandpass signal



Spectrum of infinite-length signal

Spectrum of rectangular window

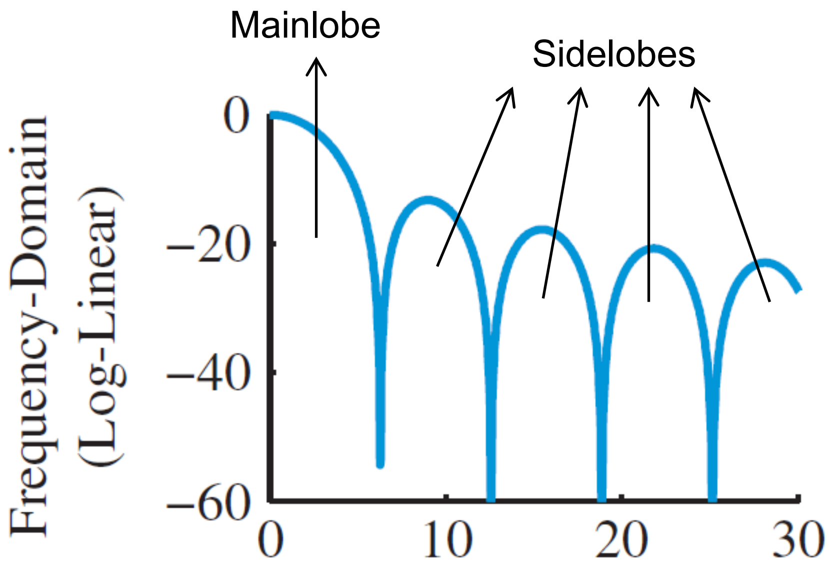
Shifted copies of window spectrum

Spectrum of windowed signal
[as “weighted average” of (a)]

Spectral distortions due to time-windowing

Smearing The mainlobe smears the original spectrum and causes loss of resolution.

Leakage The sidelobes transfer power into bands that contain little or even no power.



Magnitude response of a rectangular window

Good windows and uncertainty principle

Good window

1. Narrow mainlobe bandwidth (requires long window based on uncertainty principle)
2. Small sidelobe magnitude (reduces effective window duration)

Uncertainty principle

$$\sigma_t^2 \triangleq \int_{-\infty}^{\infty} t^2 |x_c(t)|^2 \mathrm{d}t$$

Duration

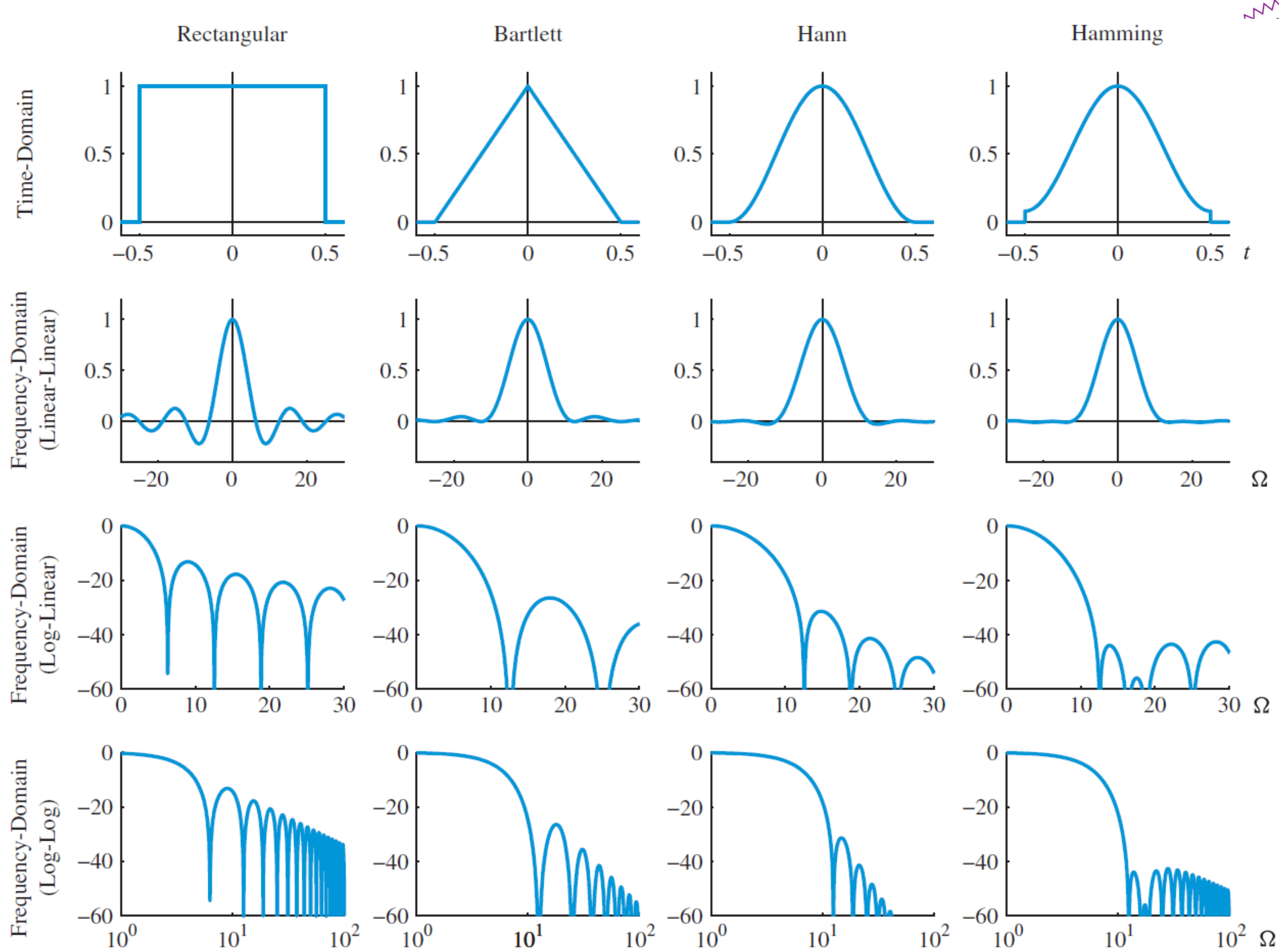
$$\sigma_\Omega^2 \triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} \Omega^2 |X_c(\mathrm{j}\Omega)|^2 \mathrm{d}\Omega$$

Bandwidth

$$\boxed{\sigma_t \sigma_\Omega \geq \frac{1}{2}}$$

Duration and bandwidth cannot be arbitrarily small simultaneously.

Window choices



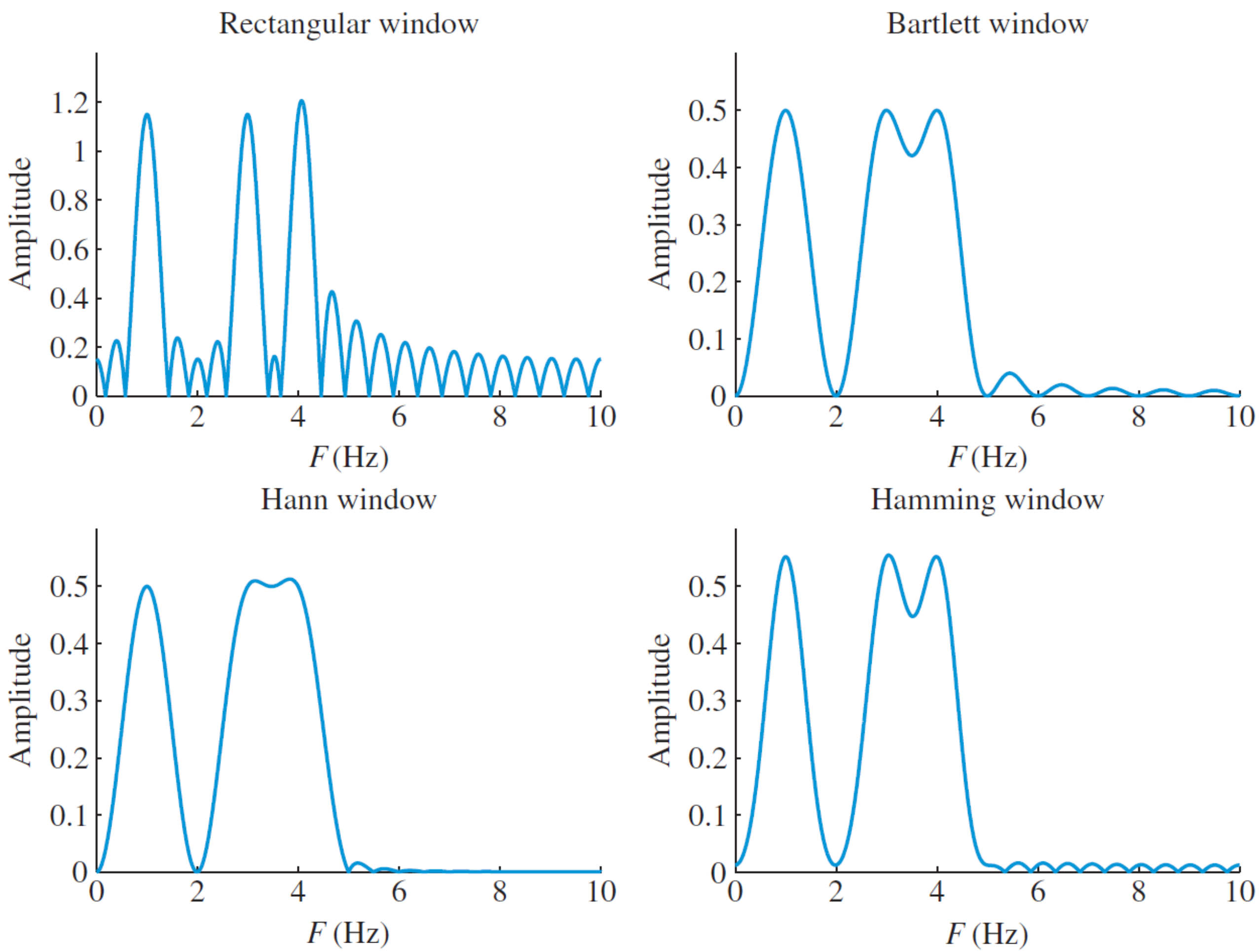
$$w_{\mathrm{Han}}(t) = \left[0.50 + 0.50\cos\left(\frac{2\pi t}{T_0}\right)\right] w_R(t)$$

$$w_{\mathrm{Ham}}(t) = \left[0.54 + 0.46\cos\left(\frac{2\pi t}{T_0}\right)\right] w_R(t)$$

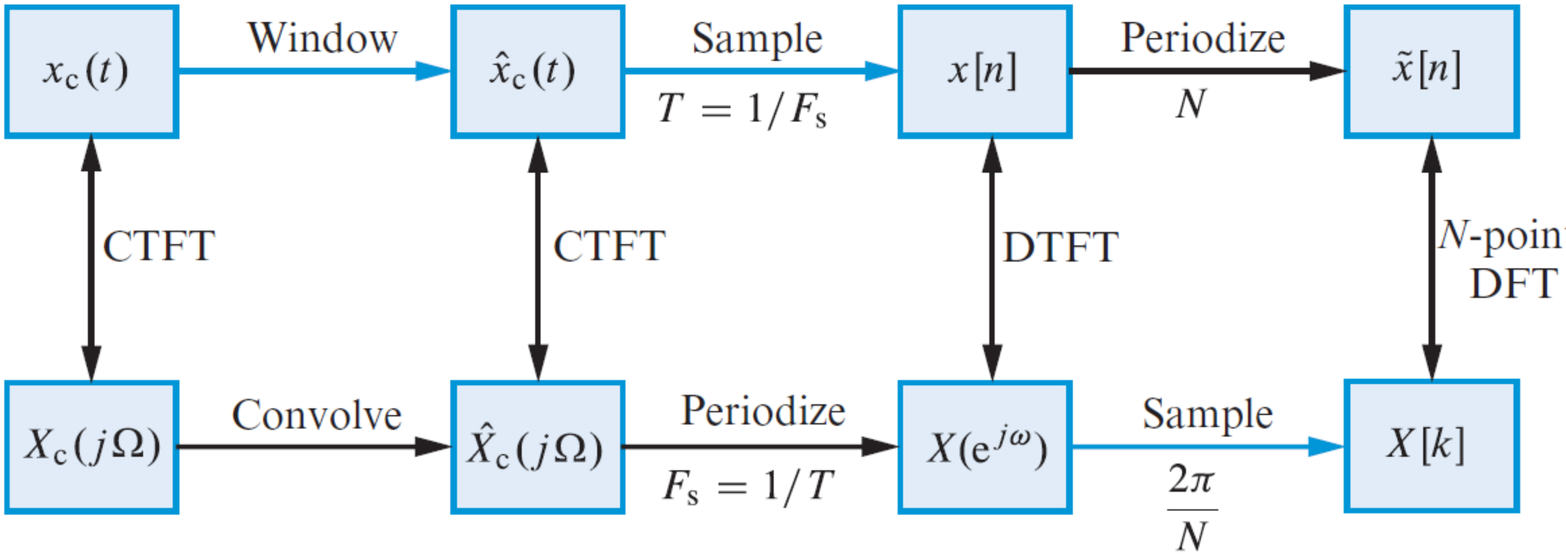
Window	Mainlobe width	Rolloff rate (dB/octave)	Peak sidelobe level (dB)
Rectangular	$4\pi/T_0$	-6	-13.3
Bartlett	$8\pi/T_0$	-12	-26.5
Hann	$8\pi/T_0$	-18	-31.5
Hamming	$8\pi/T_0$	-6	-42.9

Example of different windows

The original signal has three sinusoids at frequencies 1, 3, and 4 Hz.



Frequency-domain sampling

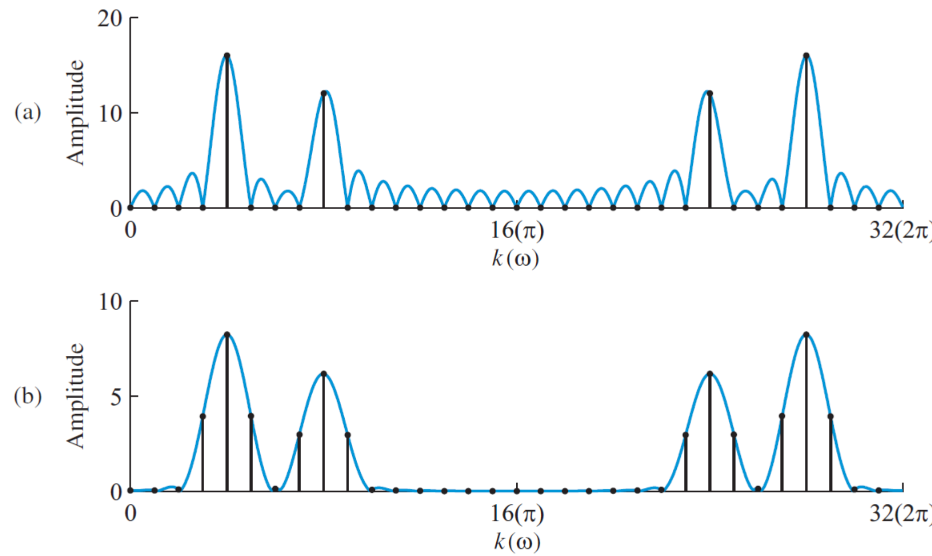


Windowing should be applied before zero-padding

Zero padding to enhance visual representation

Example of spectrum analysis using DFT

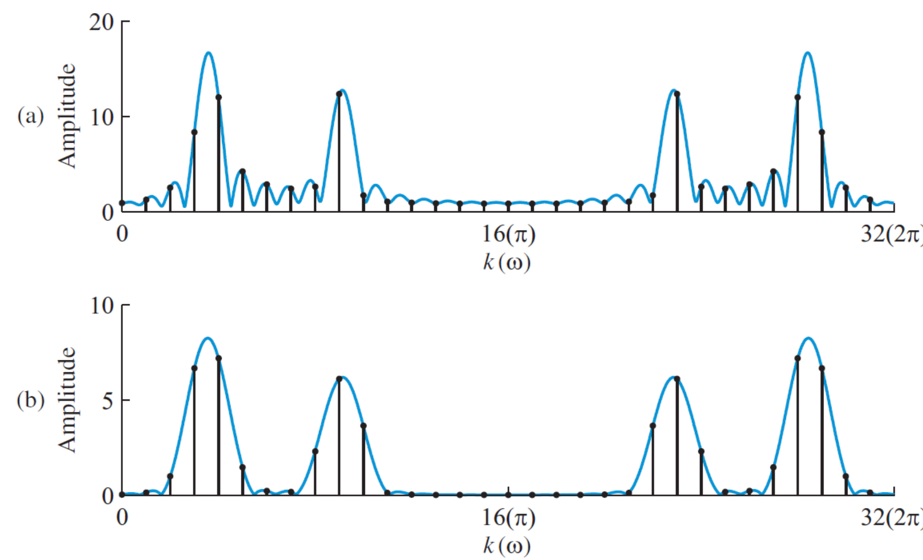
$$x[n] = \begin{cases} \cos\left(\frac{2\pi}{8}n\right) + \frac{3}{4}\cos\left(\frac{2\pi}{4}n\right), & 0 \le n \le 31 \\ 0. & \text{otherwise} \end{cases}$$



Rectangular window

Hann window

$$x[n] = \begin{cases} \cos\left(\frac{2\pi}{9}n\right) + \frac{3}{4}\cos\left(\frac{4\pi}{7}n\right), & 0 \le n \le 31 \\ 0. & \text{otherwise} \end{cases}$$



Rectangular window

Hann window

Spectrogram

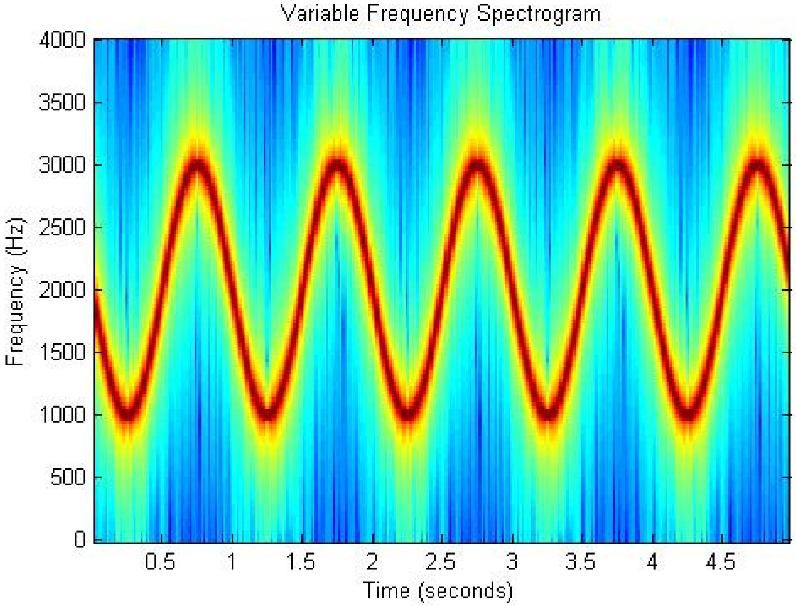
Short-time DFT
(time-dependent)

$$X[k,n] \triangleq \sum_{m=0}^{L-1} w[m]x[n+m]e^{-j(2\pi k/N)m}$$

L is the length of the window $w[n]$

$X[k,n],\ 0 \le k \le N-1$ is the N-point DFT

Example of an FM signal



Example of linear FM (chirp) signal (1/2)

Linear FM signal

$$x_c(t) = \sin[\pi(F_1/\tau)t^2]. \quad 0 \le t \le \tau$$

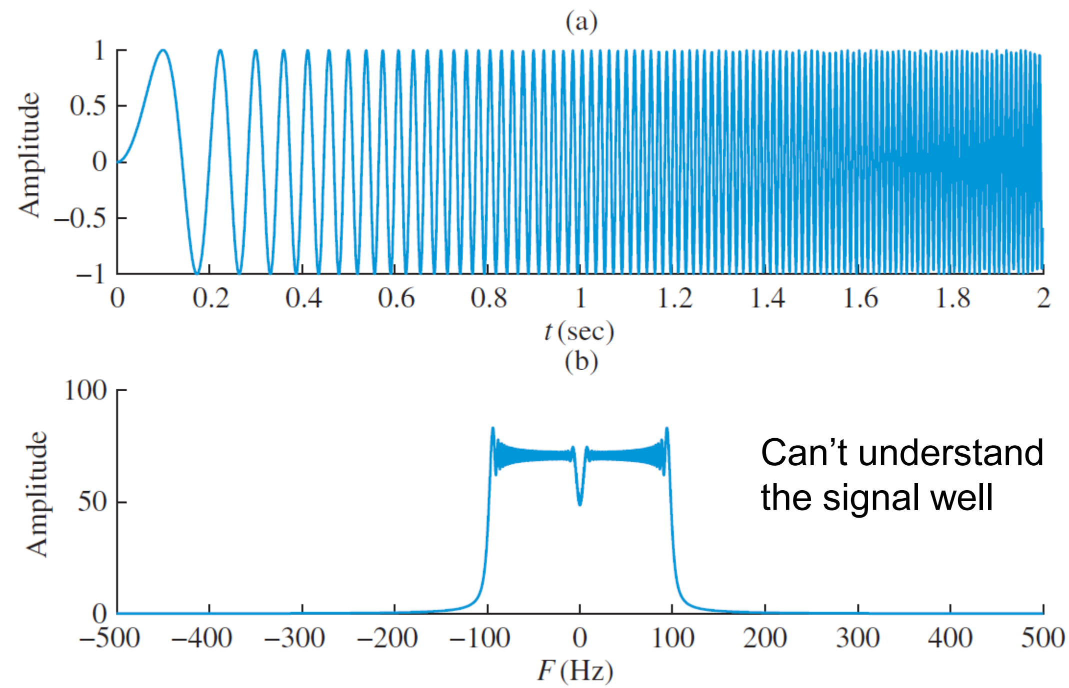
Instantaneous frequency

$$F_i(t) = \frac{1}{2\pi}\frac{d}{dt}\left(\pi t^2 F_1/\tau\right) = F_1\frac{t}{\tau}. \quad 0 \le t \le \tau$$

the frequency of $x_c(t)$ increases linearly from $F = 0$ to $F = F_1$

DTFT of 2s segment

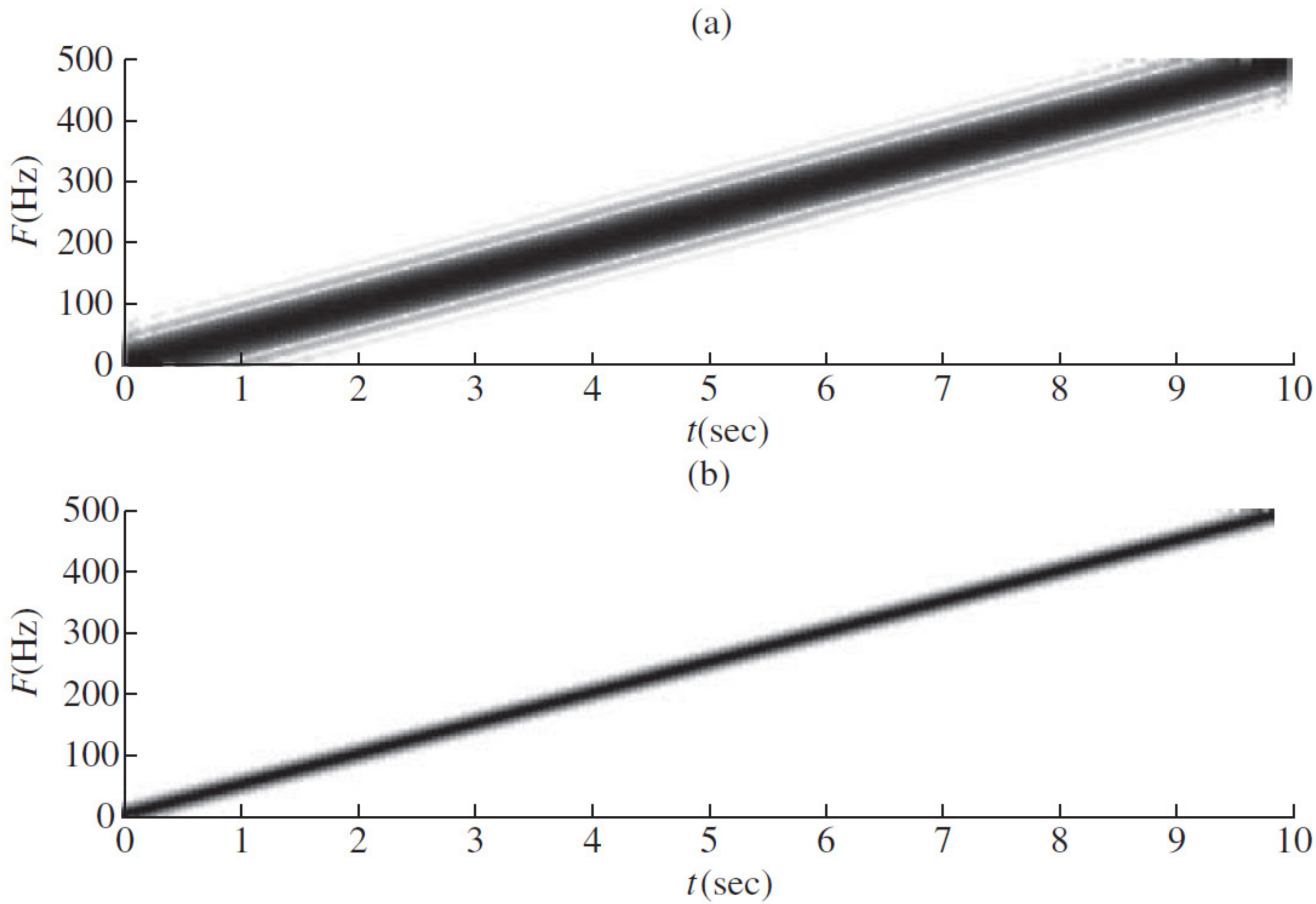
$F_1 = 500$ Hz, $\tau = 10$ s, and $F_s = 1000$ Hz



Can't understand the signal well

Example of linear FM (chirp) signal (2/2)

Spectrogram



Hann window length

$L = 50$

$L = 200$