Chap6
Sampling of continuous-time signals

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Chap 6 Sampling of continuous-time signals

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- 6.4 Discrete-time processing of continuous-time signals
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- 6.6 Sampling of bandpass signals
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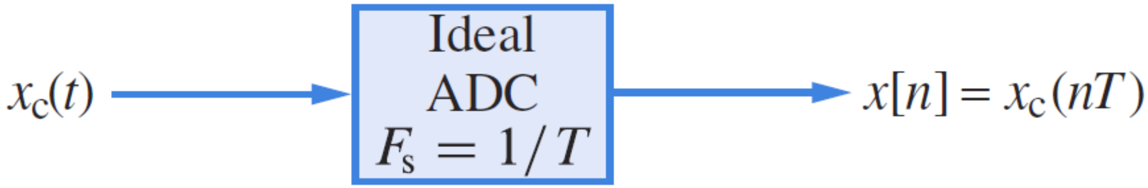
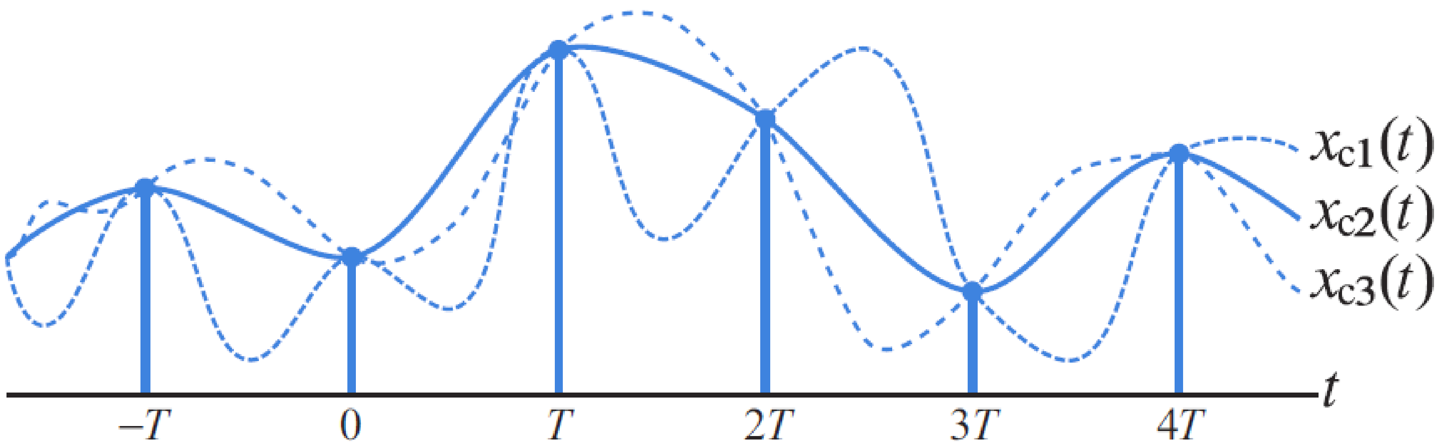
Periodic sampling of continuous-time signals



Many continuous-time signals could lead to the same sampled discrete-time signal
=> Non-invertible?



Frequency-domain relationship

$$X_c(j\Omega) = \int_{-\infty}^{\infty} x_c(t) e^{-j\Omega t} dt,$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n},$$

$$x_c(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_c(j\Omega)\, e^{j\Omega t} d\Omega$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega,$$

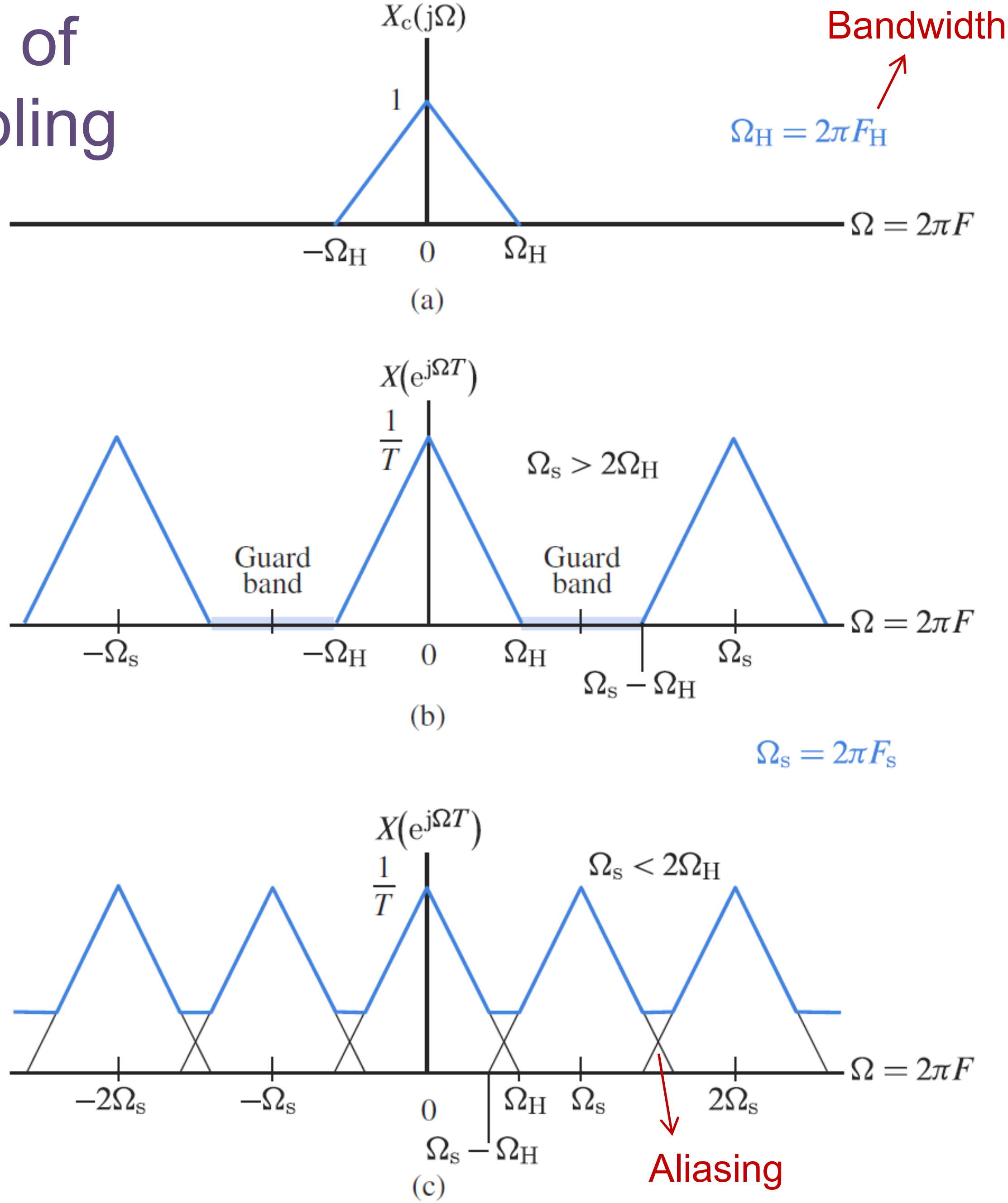
$$x[n] = x_c(nT)$$

$$\omega = \Omega T = 2\pi F T = 2\pi \frac{F}{F_s} = 2\pi f$$

$$X(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega}{T} - j\frac{2\pi}{T}k\right)$$

$$X(e^{j\Omega T}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(j\Omega - j\frac{2\pi}{T}k\right)$$

Interpretation of uniform sampling



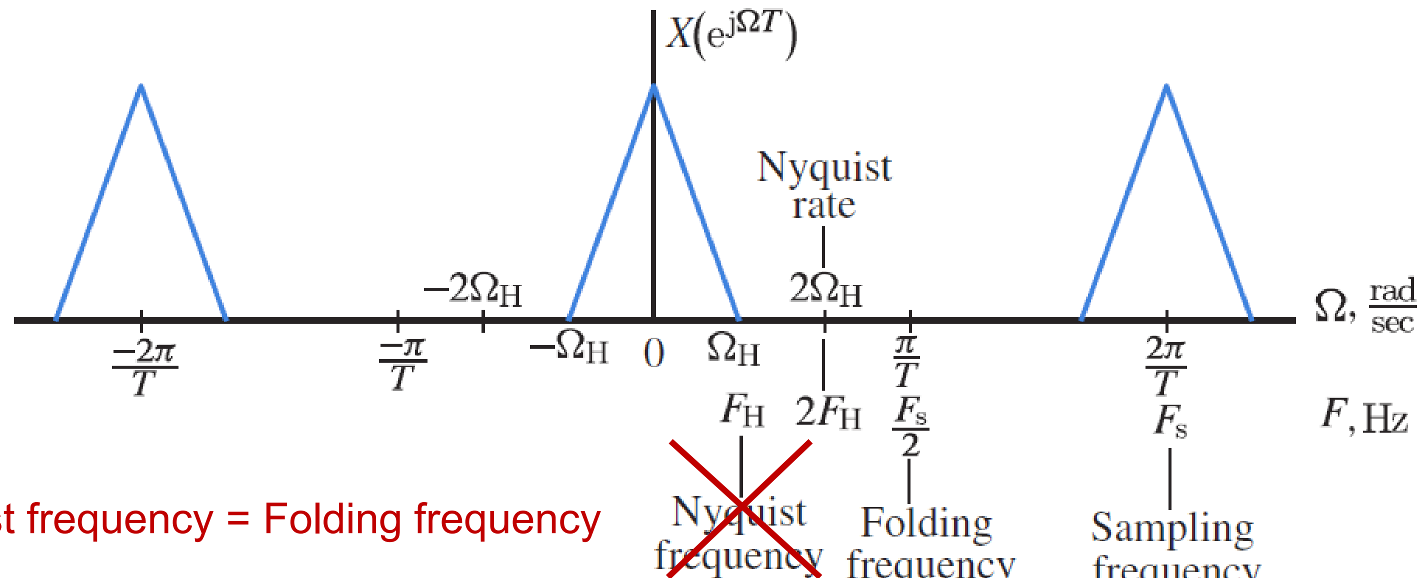
Sampling theorem: Let $x_c(t)$ be a continuous-time bandlimited signal with Fourier transform

$$X_c(j\Omega) = 0 \quad \text{for} |\Omega| > \Omega_H. \tag{6.18}$$

Then $x_c(t)$ can be uniquely determined by its samples $x[n] = x_c(nT)$, where $n = 0, \pm 1, \pm 2, \ldots$, if the sampling frequency Ω_s satisfies the condition

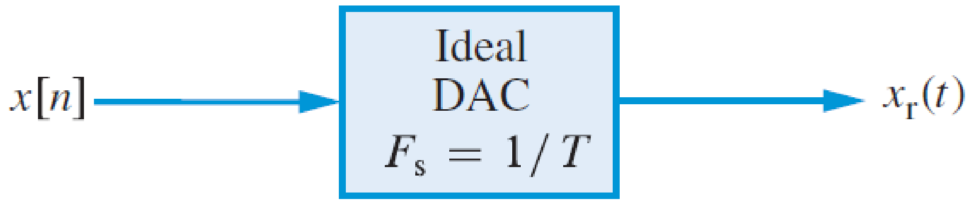
$$\Omega_s = \frac{2\pi}{T} \geq 2\Omega_H. \tag{6.19}$$

Sampling frequency ≥ Nyquist rate

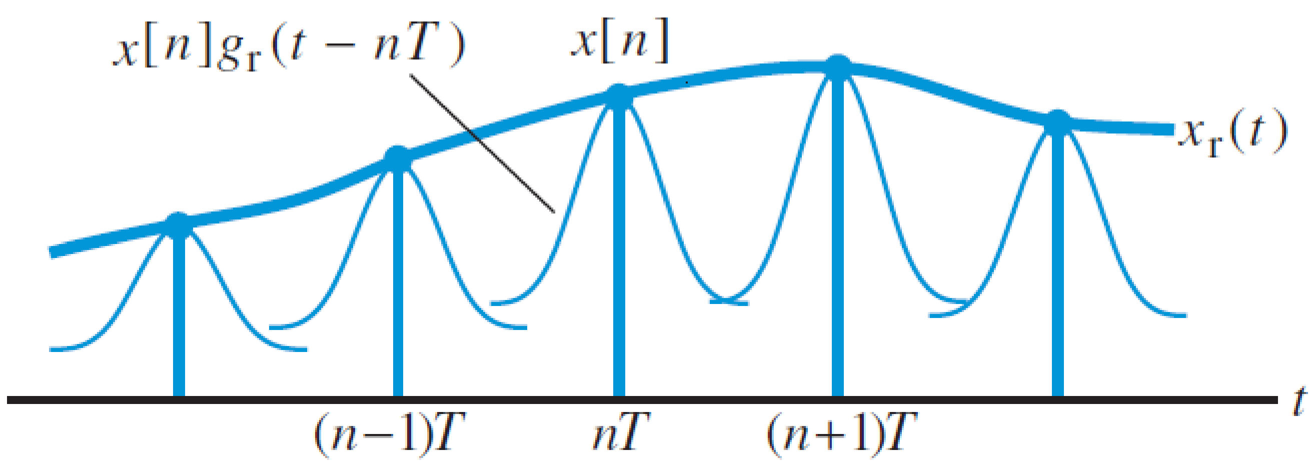


Nyquist frequency = Folding frequency

Reconstruction from discrete-time signals

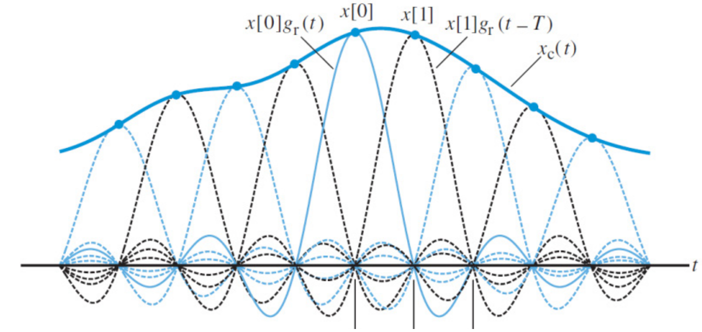


We can fully recover a bandlimited signal based on sampling theorem.
=> How to do it? Ideal vs. practical?



Ideal bandlimited interpolator

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] g_r(t - nT)$$



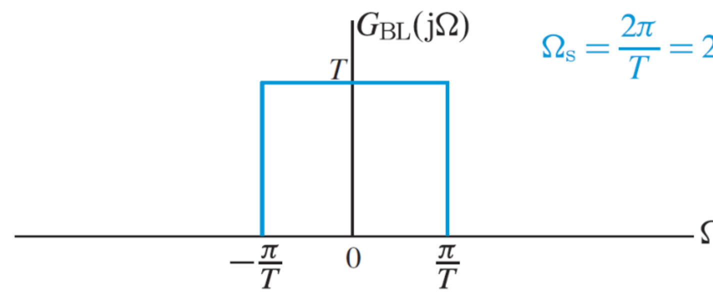
$\Downarrow$

$$X_r(j\Omega) = G_r(j\Omega)\, X(e^{j\Omega T}) \quad \Leftarrow \quad X_r(j\Omega) = \sum_{n=-\infty}^{\infty} x[n] G_r(j\Omega)\, e^{-j\Omega n T}$$

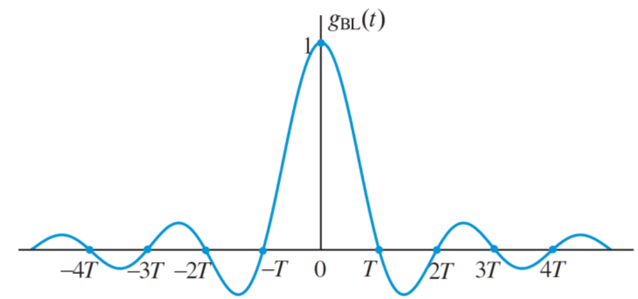
$\Downarrow$

$$G_r(j\Omega) \triangleq G_{BL}(j\Omega) = \begin{cases} T, & |\Omega| \le \Omega_s/2 \\ 0, & |\Omega| > \Omega_s/2 \end{cases}$$

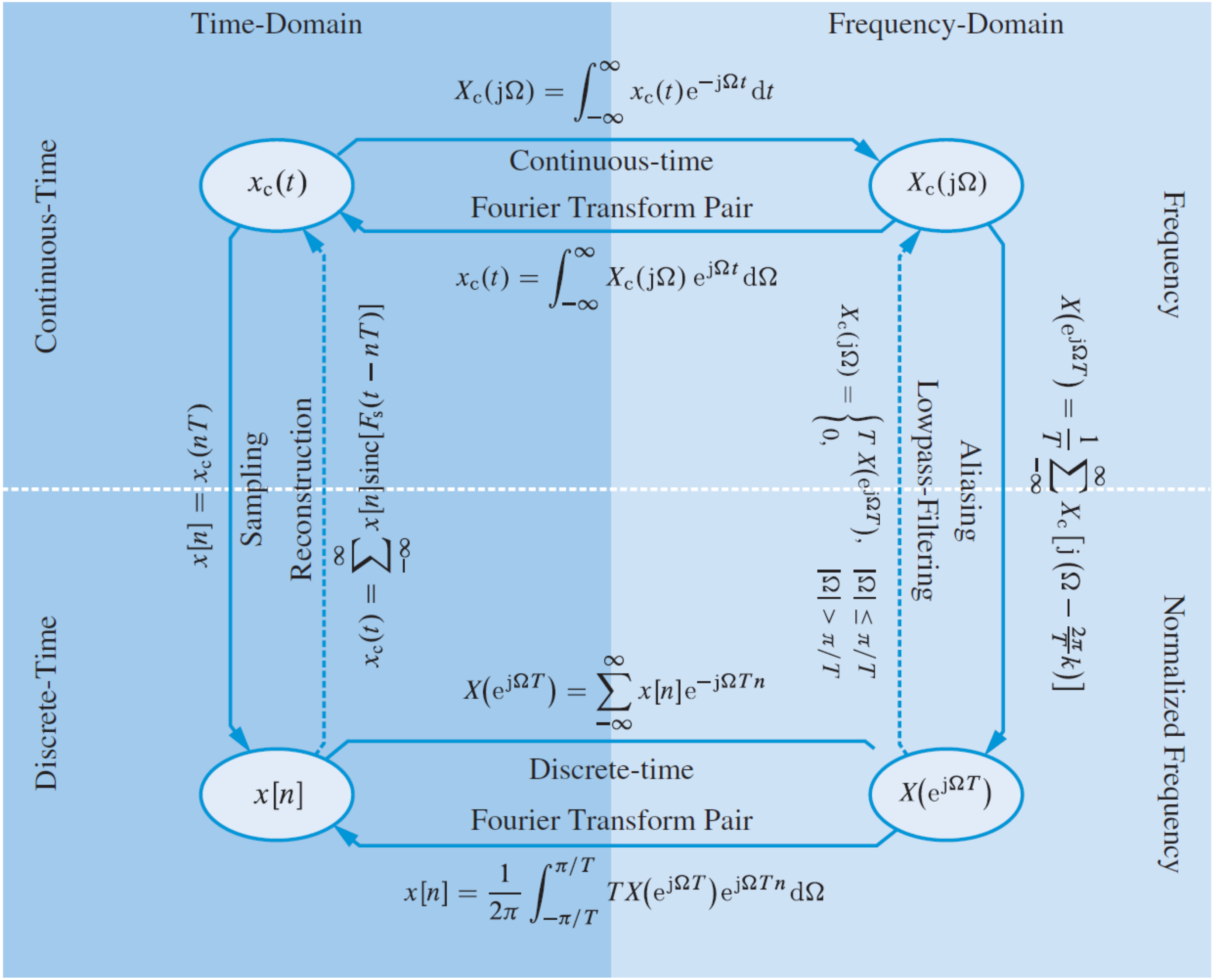
$$\Omega_s = \frac{2\pi}{T} = 2\pi F_s$$



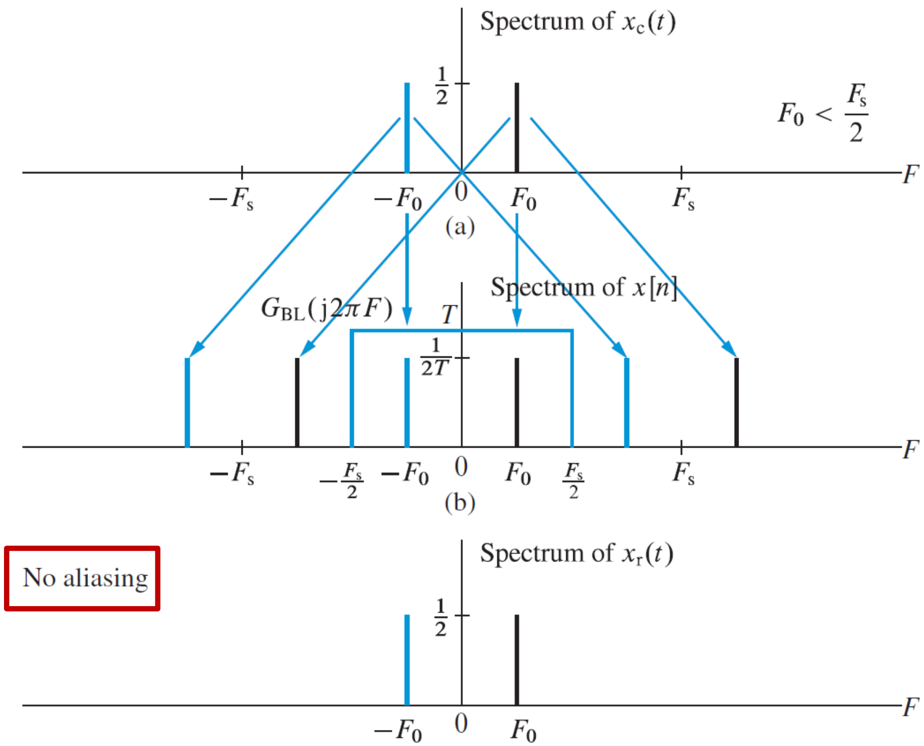
$$g_r(t) \triangleq g_{BL}(t) = \frac{\sin(\pi t/T)}{\pi t/T}$$

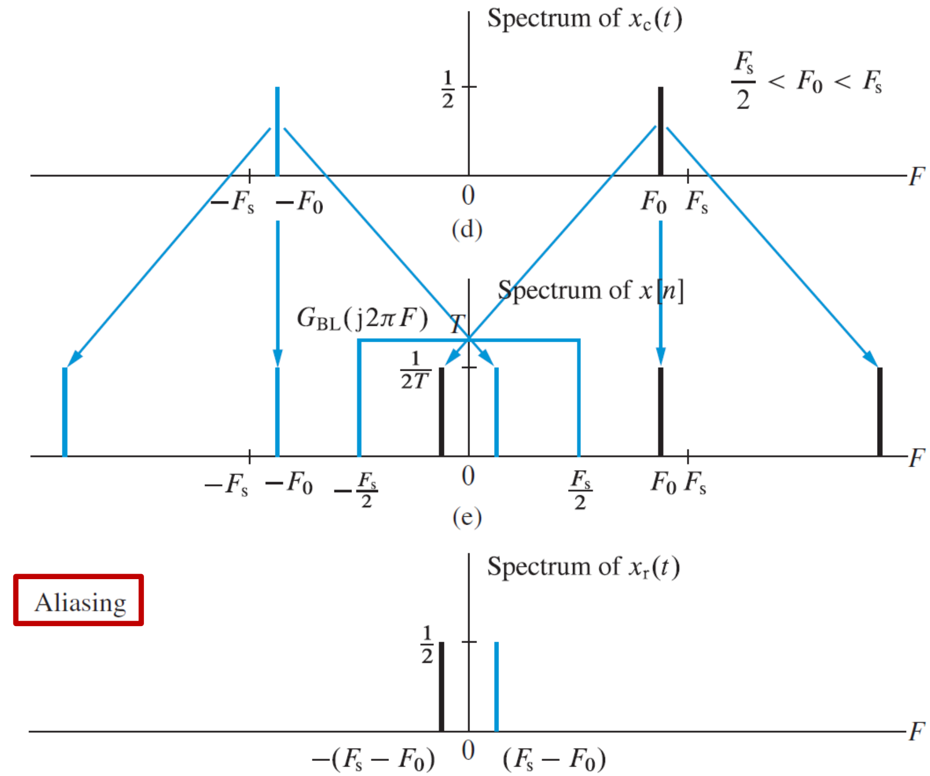


Periodic sampling vs. Bandlimited reconstruction

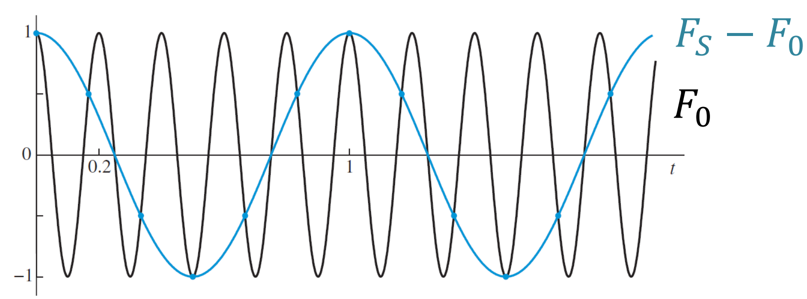


Aliasing example due to undersampling (1/3)





$$x_c(t) = \cos(2\pi F_0 t) = \frac{1}{2}e^{j2\pi F_0 t} + \frac{1}{2}e^{-j2\pi F_0 t}$$



Aliasing example due to undersampling (2/3)

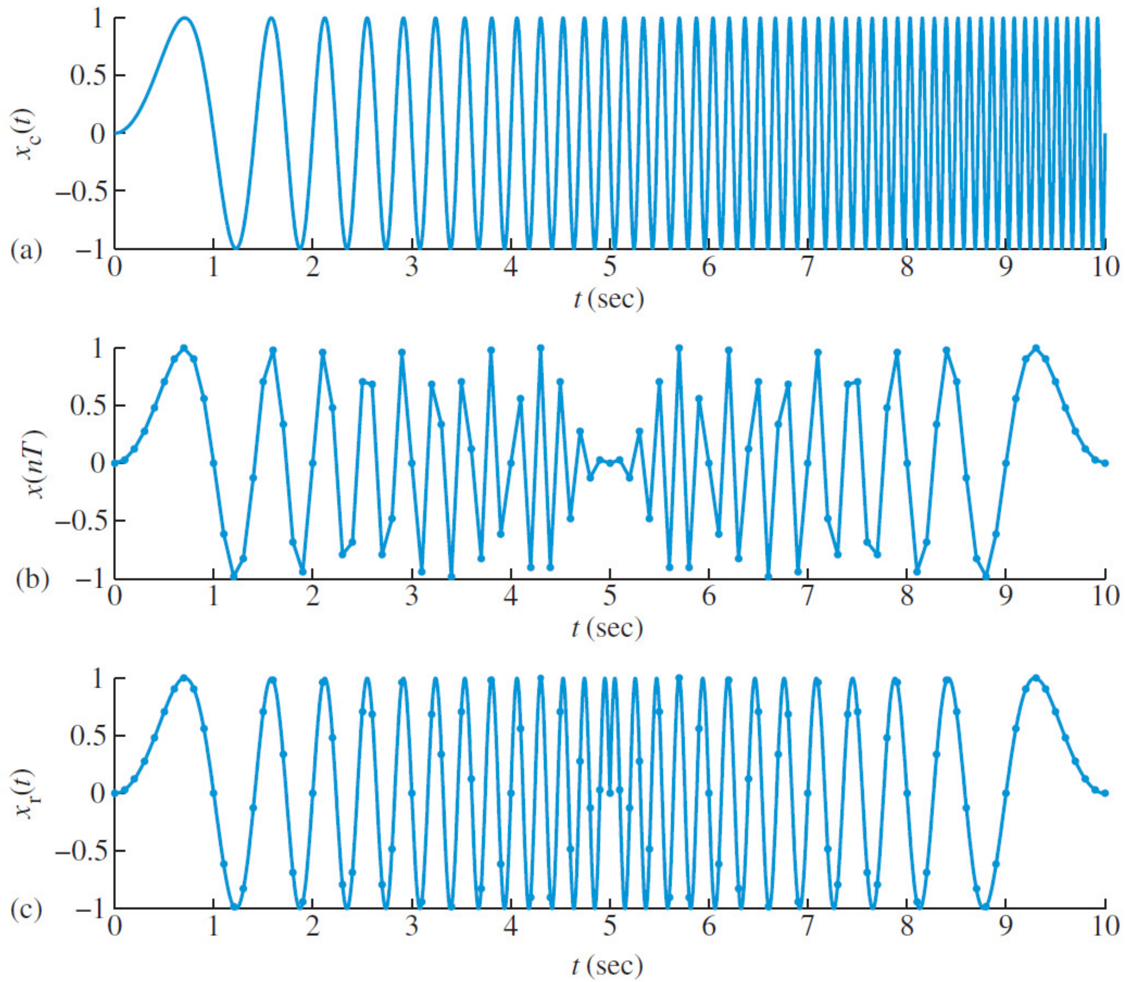


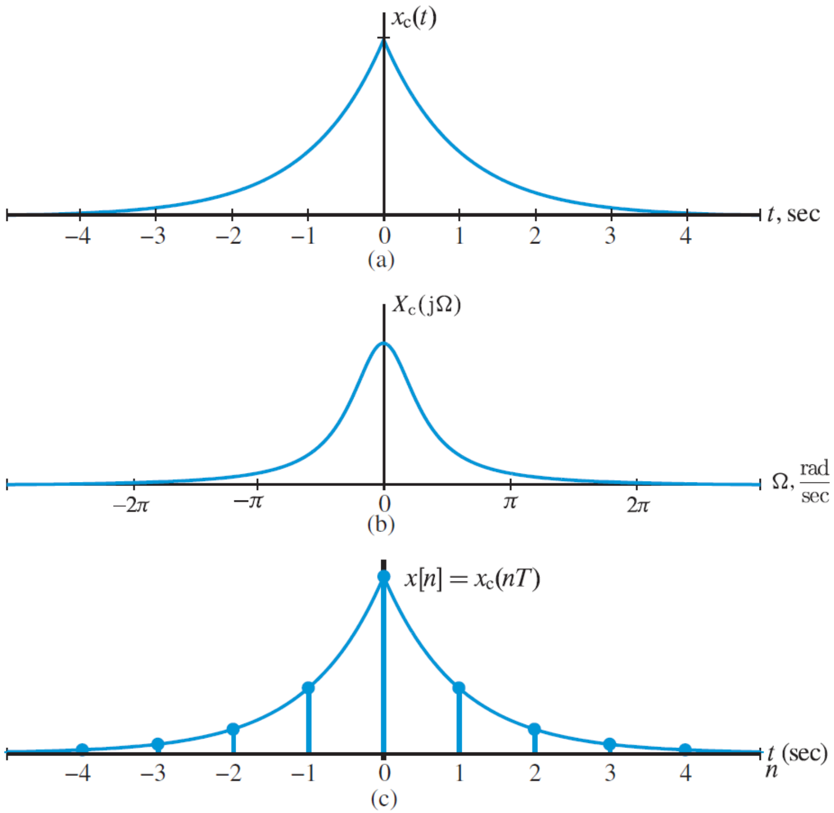
Figure 6.15 Sampling a continuous-time linear FM signal: (a) signal, (b) samples connected by line segments, and (c) output of ideal DAC.

Aliasing example due to undersampling (3/3)

$$x_c(t) = e^{-A|t|} \overset{\text{CTFT}}{\longleftrightarrow} X_c(j\Omega) = \frac{2A}{A^2 + \Omega^2}. \quad A > 0$$

$$x[n] = x_c(nT) = e^{-A|n|T} = (e^{-AT})^{|n|} = a^{|n|}, \quad a \triangleq e^{-AT}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \frac{1 - a^2}{1 - 2a\cos(\omega) + a^2}, \quad \omega = \Omega/F_s$$



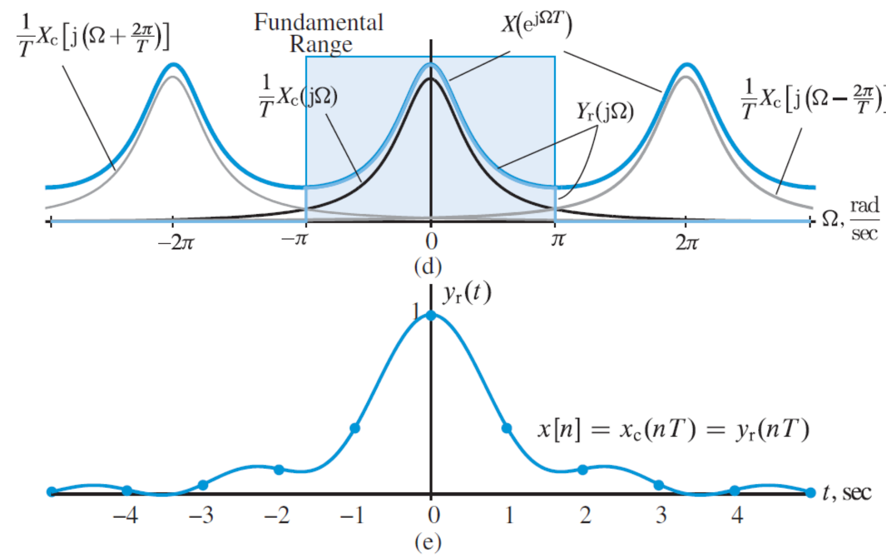
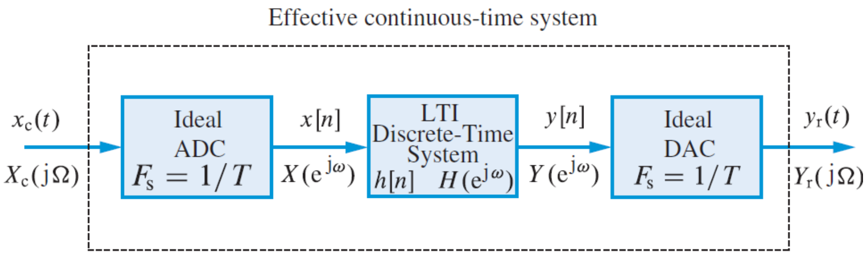


Figure 6.16 Aliasing effects in sampling and reconstruction of a continuous-time nonbandlimited signal: (a) continuous-time signal $x_c(t)$, (b) spectrum of $x_c(t)$, (c) discrete-time signal $x[n]$ sampled at $T = 1$ s, (d) spectrum of $x[n]$, and (e) bandlimited reconstruction $y_r(t)$. In this case, aliasing distortion is unavoidable.

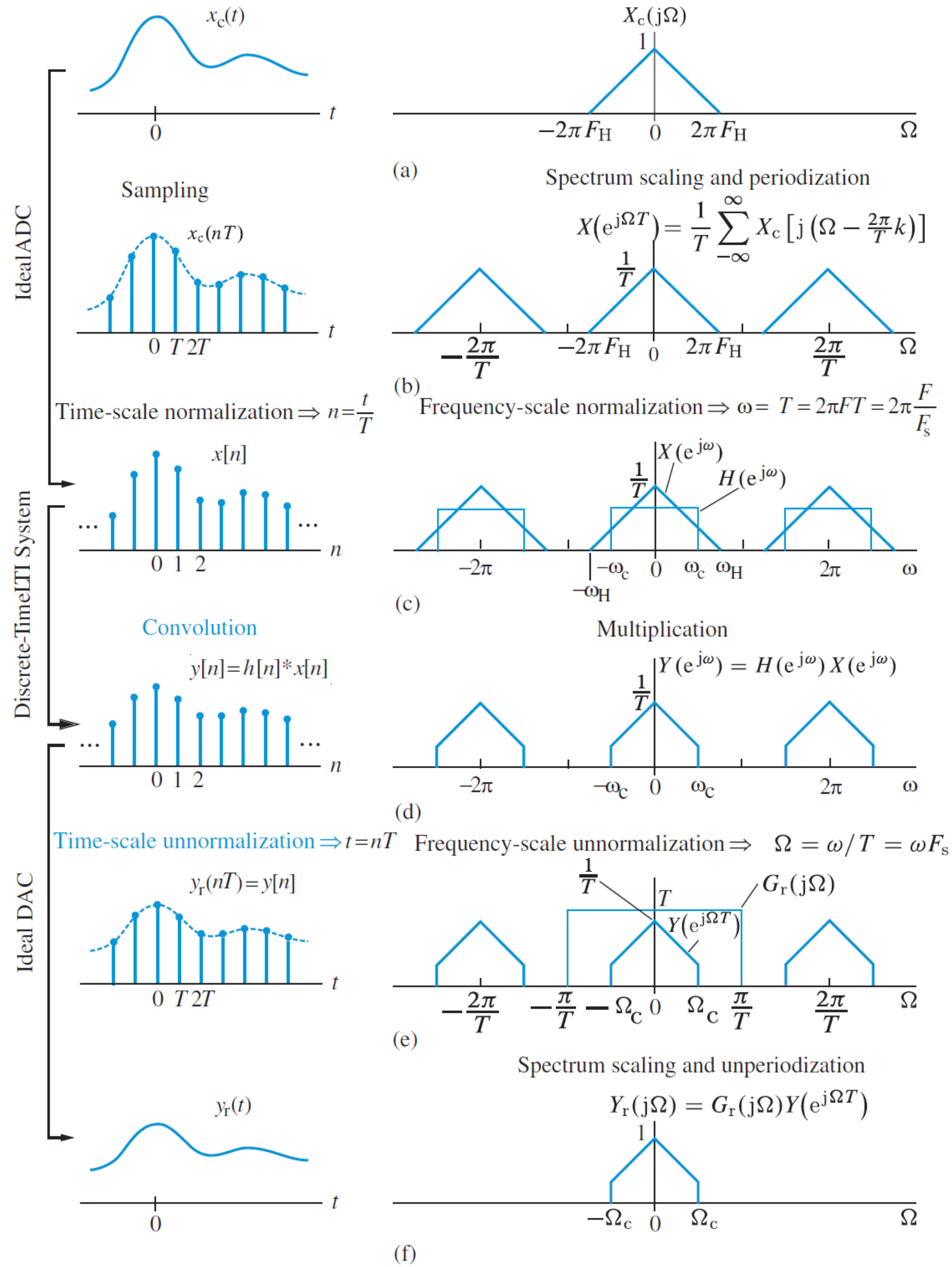
Discrete-time filtering of continuous-time signals



$$Y_r(j\Omega) = H_{eff}(j\Omega)X_c(j\Omega)$$

$$H_{eff}(j\Omega) = \begin{cases} H(e^{j\Omega T}), & |\Omega| \le \pi/T \\ 0. & |\Omega| > \pi/T \end{cases}$$

Constraints:
- Ideal ADC and DAC
- Bandlimited processing



Example of ideal bandlimited differentiator

Bandlimited differentiator

$$H_c(j\Omega) = \begin{cases} j\Omega, & |\Omega| \le \Omega_H \\ 0. & \text{otherwise} \end{cases}$$

$$\Downarrow \quad \omega = \Omega T \text{ and } \Omega_H T = \pi$$

Discrete filter

$$H(e^{j\omega}) = \frac{1}{T} H_c(j\omega/T) = \frac{j\omega}{T^2}. \quad |\omega| \le \pi$$

$$\Downarrow$$

$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{j\omega}{T^2} \right) e^{j\omega n} d\omega = \begin{cases} 0, & n = 0 \\ \frac{\cos(\pi n)}{nT^2}. & n \ne 0 \end{cases}$$

Example of second-order system (1/2)

Continuous system

$$\frac{d^2 y_c(t)}{dt^2} + 2\zeta\Omega_n \frac{dy_c(t)}{dt} + \Omega_n^2 y_c(t) = \Omega_n^2 x_c(t)$$

$$H_c(s) = \frac{Y_c(s)}{X_c(s)} = \frac{\Omega_n^2}{s^2 + 2\zeta\Omega_n s + \Omega_n^2}$$

Non-bandlimited

$$h_c(t) = \frac{\Omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\Omega_n t} \sin\left[\left(\Omega_n\sqrt{1-\zeta^2}\right)t\right]u(t)$$

⇩ Discrete sampling (impulse-invariance transformation)

$$h[n] = h_c(nT) = \frac{\Omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\Omega_n nT} \sin\left[\left(\Omega_n\sqrt{1-\zeta^2}\right)nT\right]u(n)$$

Discrete filter

$$= \frac{\Omega_n}{\sqrt{1-\zeta^2}} \left(e^{-\zeta\Omega_n T}\right)^n \sin\left[\left(\Omega_n T\sqrt{1-\zeta^2}\right)n\right]u(n).$$

Example of second-order system (2/2)

Z transform

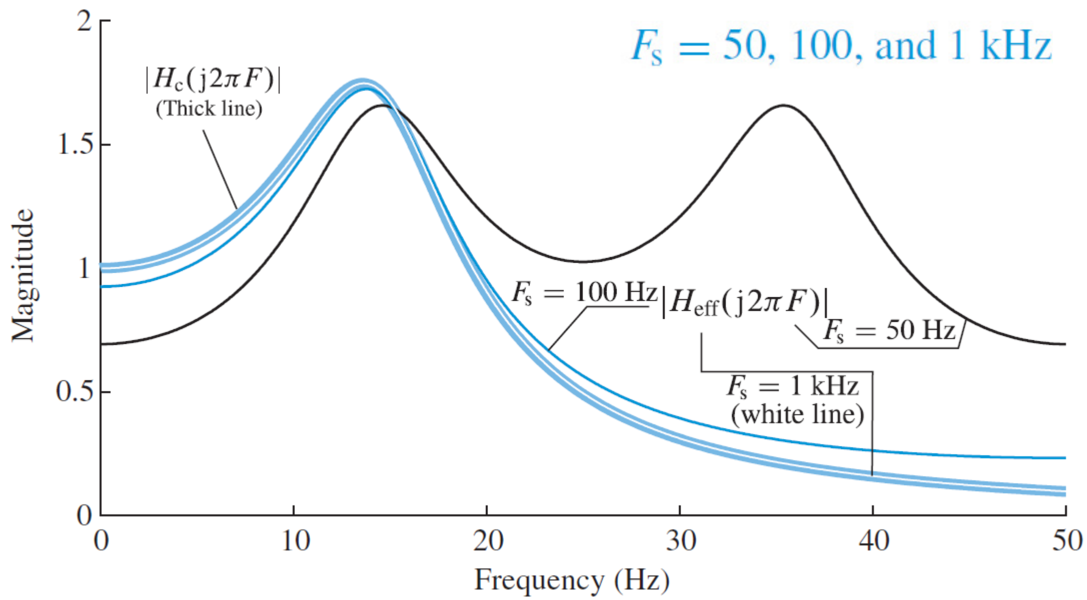
$$H(z)=\frac{\Omega_n}{\sqrt{1-\zeta^2}}\frac{e^{-\zeta\Omega_n T}\sin\left(\Omega_n T\sqrt{1-\zeta^2}\right)z^{-1}}{1-2e^{-\zeta\Omega_n T}\cos\left(\Omega_n T\sqrt{1-\zeta^2}\right)z^{-1}+e^{-2\zeta\Omega_n T}z^{-2}}$$

Difference equation

$$y[n]=\frac{\Omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\Omega_n T}\sin\left(\Omega_n T\sqrt{1-\zeta^2}\right)x[n-1]$$
$$+2e^{-\zeta\Omega_n T}\cos\left(\Omega_n T\sqrt{1-\zeta^2}\right)y[n-1]-e^{-2\zeta\Omega_n T}y[n-2]$$

Example

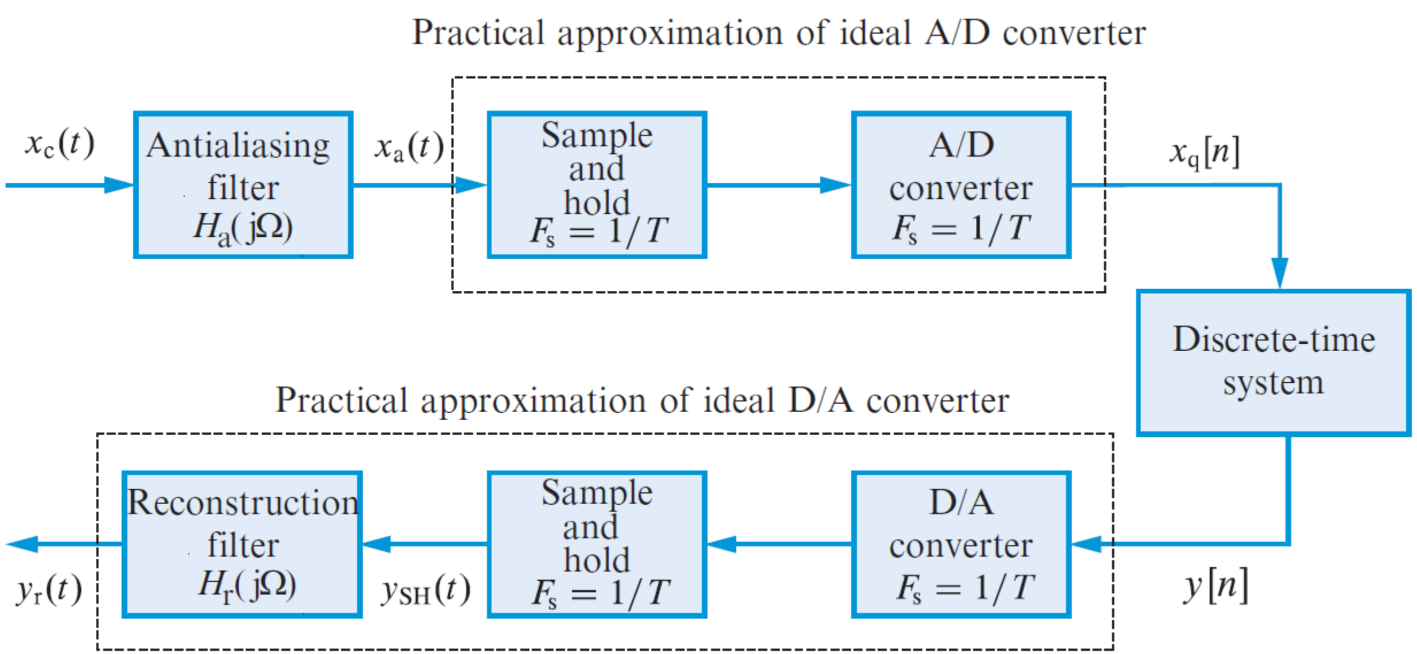
$$\zeta=0.3,\ \Omega_n=30\pi$$



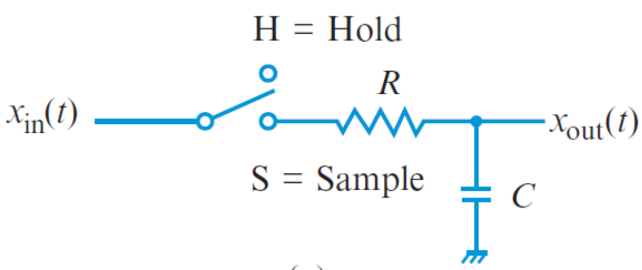
Practical discrete-time processing of continuous signals

Main differences from ideal processing:

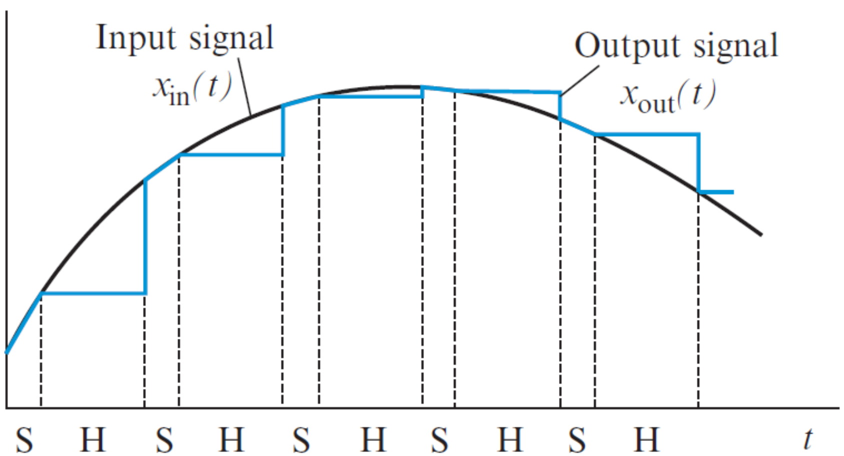
- Practical signals are time-limited, thus not bandlimited => Need analog antialiasing pre-filtering
- Impulse sampling is not practical for ADC => Use sample-and-hold circuits instead
- Discrete signal values are quantized => Need to consider quantization noise
- Ideal interpolator (sinc) is not practical for DAC => Use S/H reconstruction



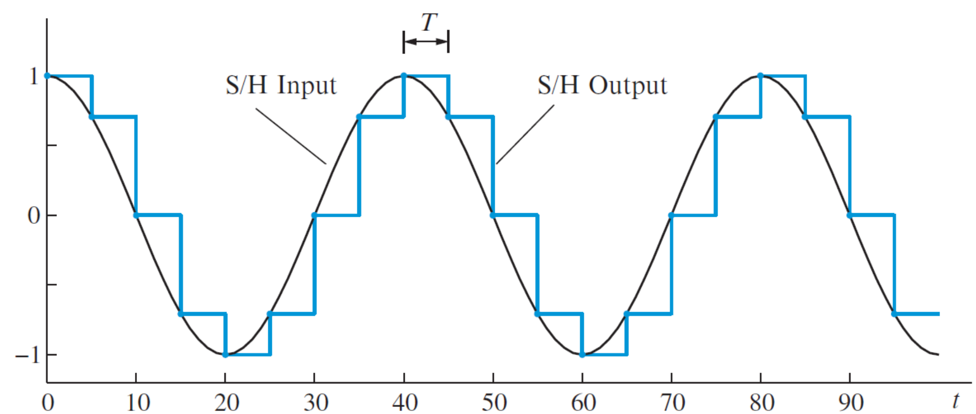
Sample-and-hold circuits for ADC



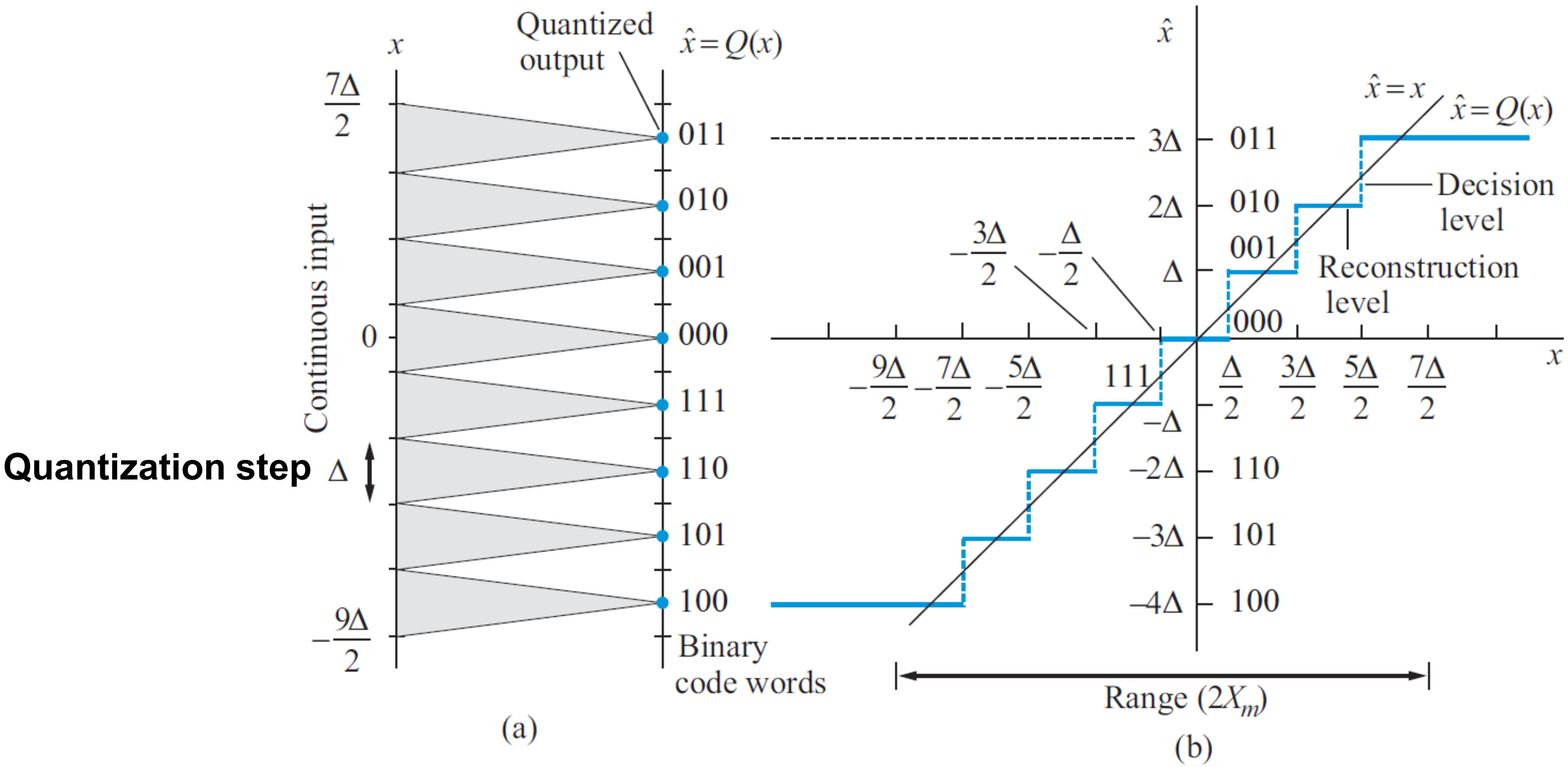
(a)

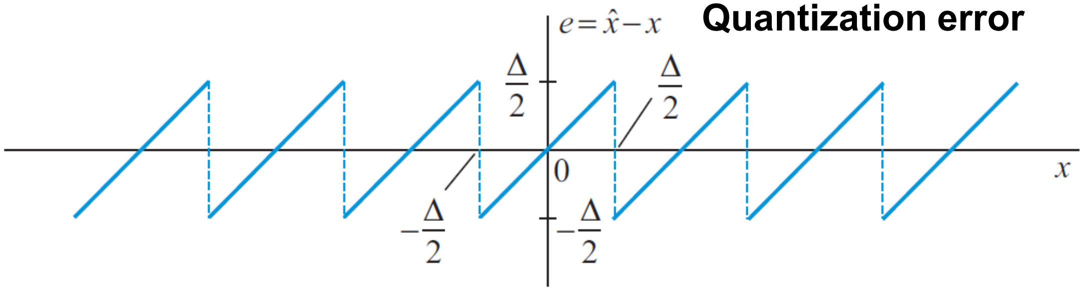


(b)



Quantization of ADC converter





Quantization Noise

Quantization error power
(assume error is uniformly distributed)

$$P_Q = \frac{1}{2\tau}\int_{-\tau}^{\tau} e_c^2(t)\,dt = \frac{\Delta^2}{12}$$

Signal power
(assume sinusoidal signals)

$$P_S = \frac{1}{T_p}\int_0^{T_p} X_m^2 \sin^2\left(\frac{2\pi}{T_p}\right) dt = \frac{X_m^2}{2}$$

$$x_c(t) = X_m \sin\left(\frac{2\pi}{T_p} t\right)$$

Signal-to-quantization-noise-ratio

$$\mathrm{SQNR} \triangleq \frac{P_S}{P_Q} = \frac{3}{2} \times 2^{2B}$$

$$\mathrm{SQNR(dB)} = 10\log_{10} \mathrm{SQNR} = 6.02B + 1.76$$

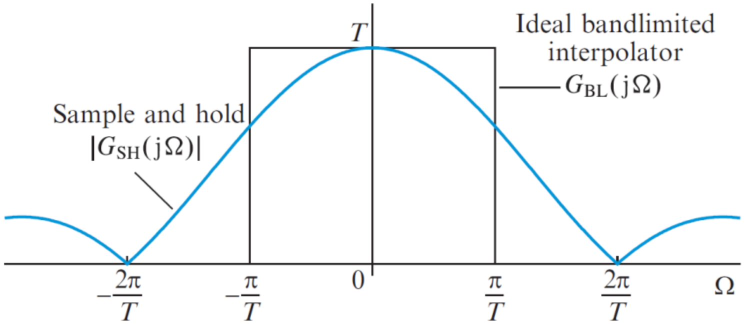
(one additional bit adds 6dB)

S/H reconstruction

S/H amplifier

$$x_{SH}(t) = \sum_{n=-\infty}^{\infty} x_q[n] g_{SH}(t - nT)$$

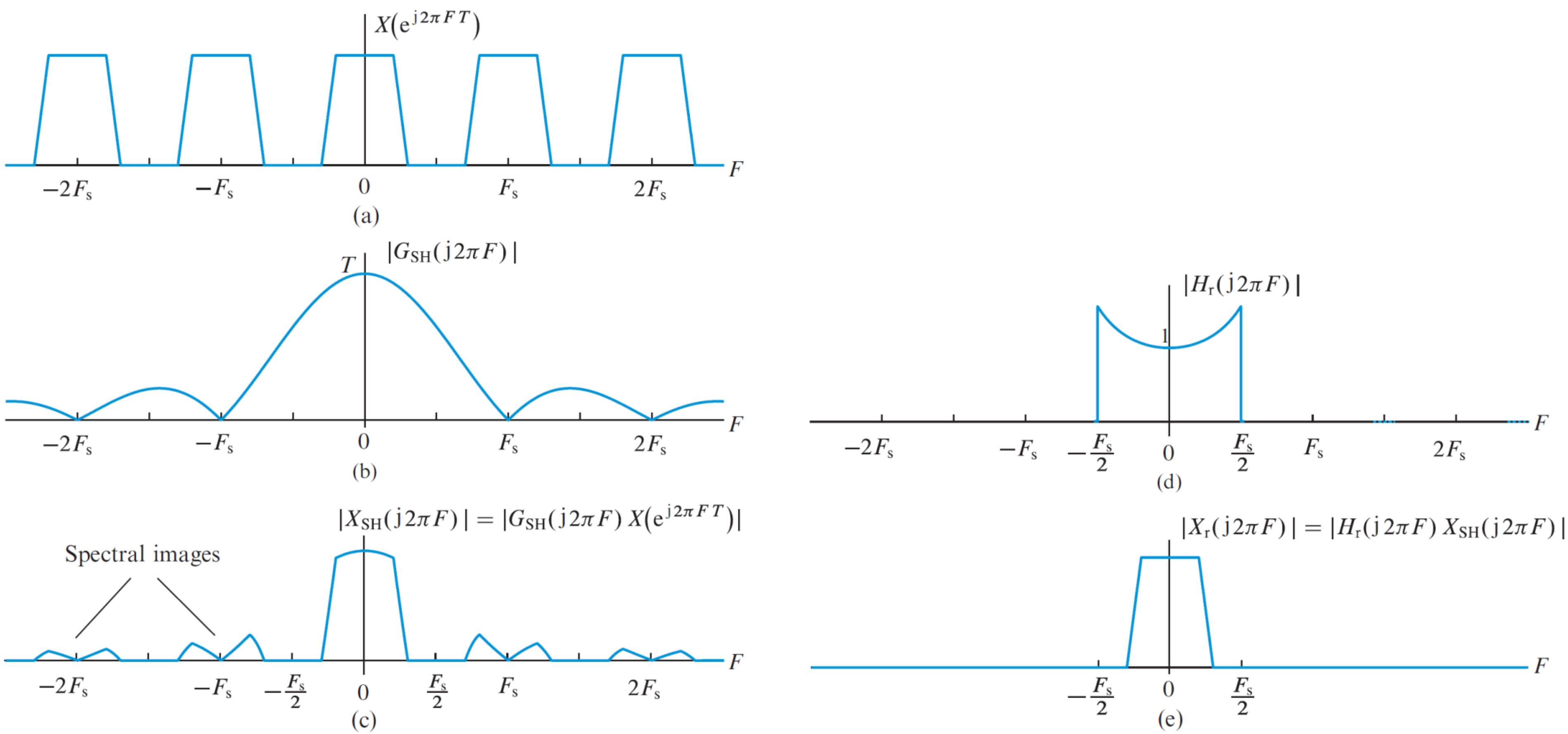
$$g_{SH}(t) = \begin{cases} 1, & 0 \le t \le T \\ 0. & \text{otherwise} \end{cases} \overset{\text{CTFT}}{\longleftrightarrow} G_{SH}(j\Omega) = \frac{2\sin(\Omega T/2)}{\Omega} e^{-j\Omega T/2}$$



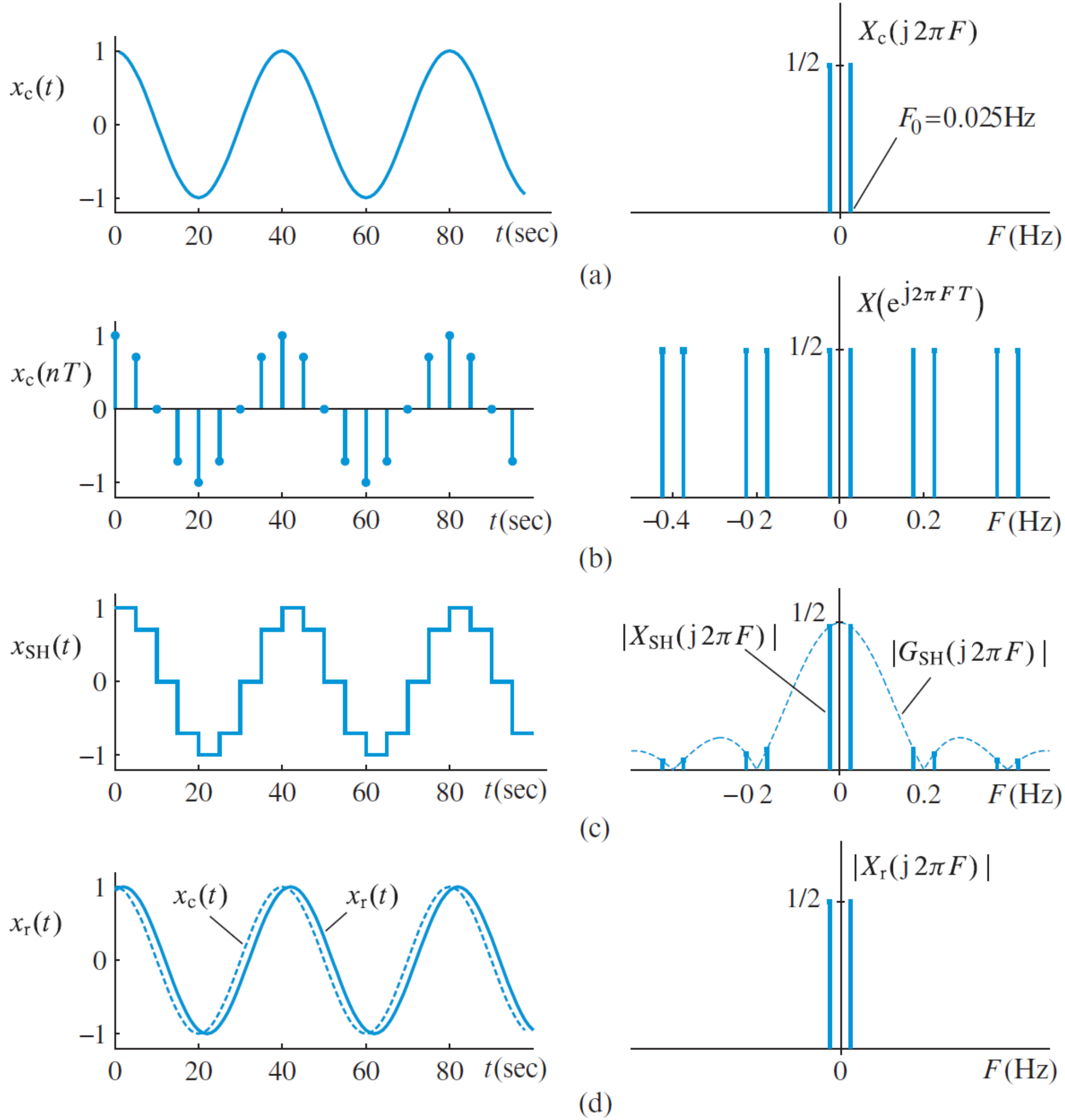
Reconstruction filter
(anti-imaging and equalization)

$$H_r(j\Omega) = \begin{cases} \frac{\Omega T/2}{\sin(\Omega T/2)} e^{j\Omega T/2}, & |\Omega| < \pi/T \\ 0. & \text{otherwise} \end{cases}$$

Frequency domain of S/H reconstruction



Example of S/H reconstruction



Sampling of bandpass signals

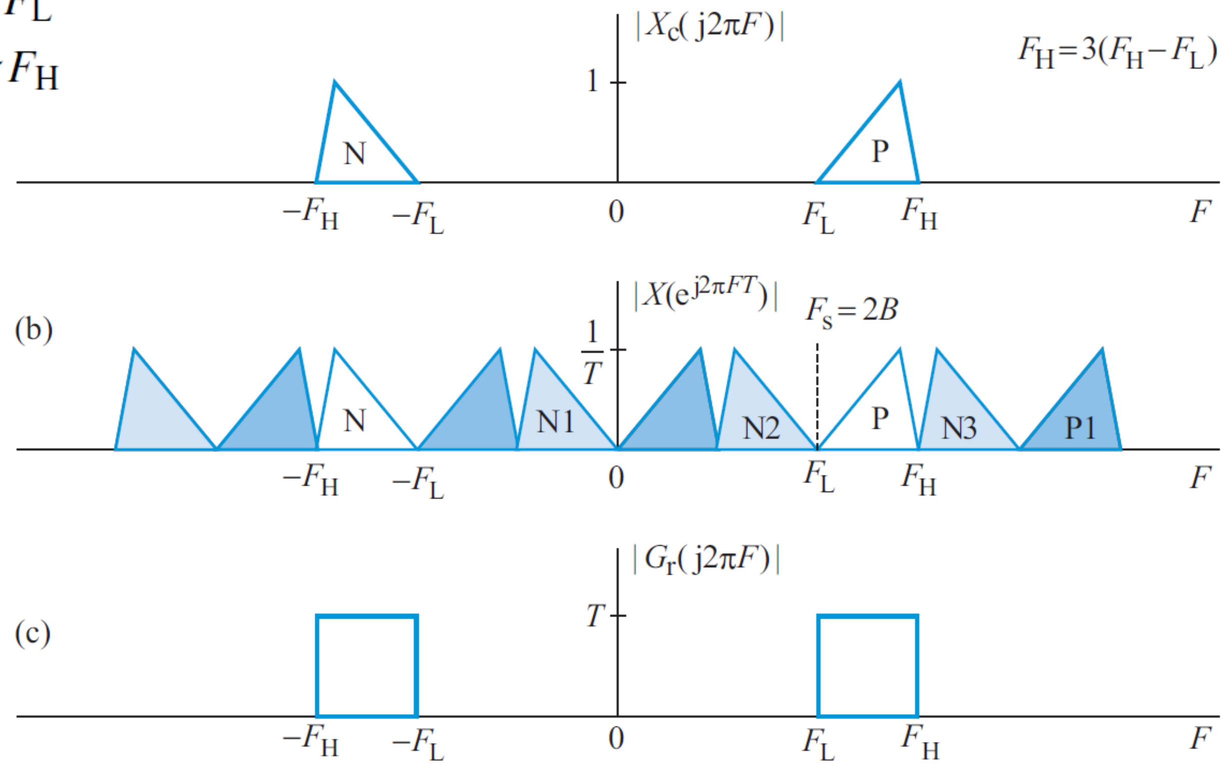
$$X_c(j\Omega) = \begin{cases} 0, & |\Omega| \le \Omega_L = 2\pi F_L \\ 0, & |\Omega| \ge \Omega_H = 2\pi F_H \end{cases}$$

$$B \triangleq F_H - F_L$$

$$2B \le F_s \le 4B$$

Sufficient sampling rate for perfect reconstruction

Ideal bandpass interpolator



2-D transform for image processing

2D CTFT

$$S_c(F_x, F_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s_c(x, y) e^{-j2\pi(xF_x + yF_y)} dxdy,$$

$$s_c(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_c(F_x, F_y) e^{j2\pi(xF_x + yF_y)} dF_x dF_y,$$

2D sampling

$$s[m, n] \triangleq s_c(m\Delta x, n\Delta y)$$

2D DTFT

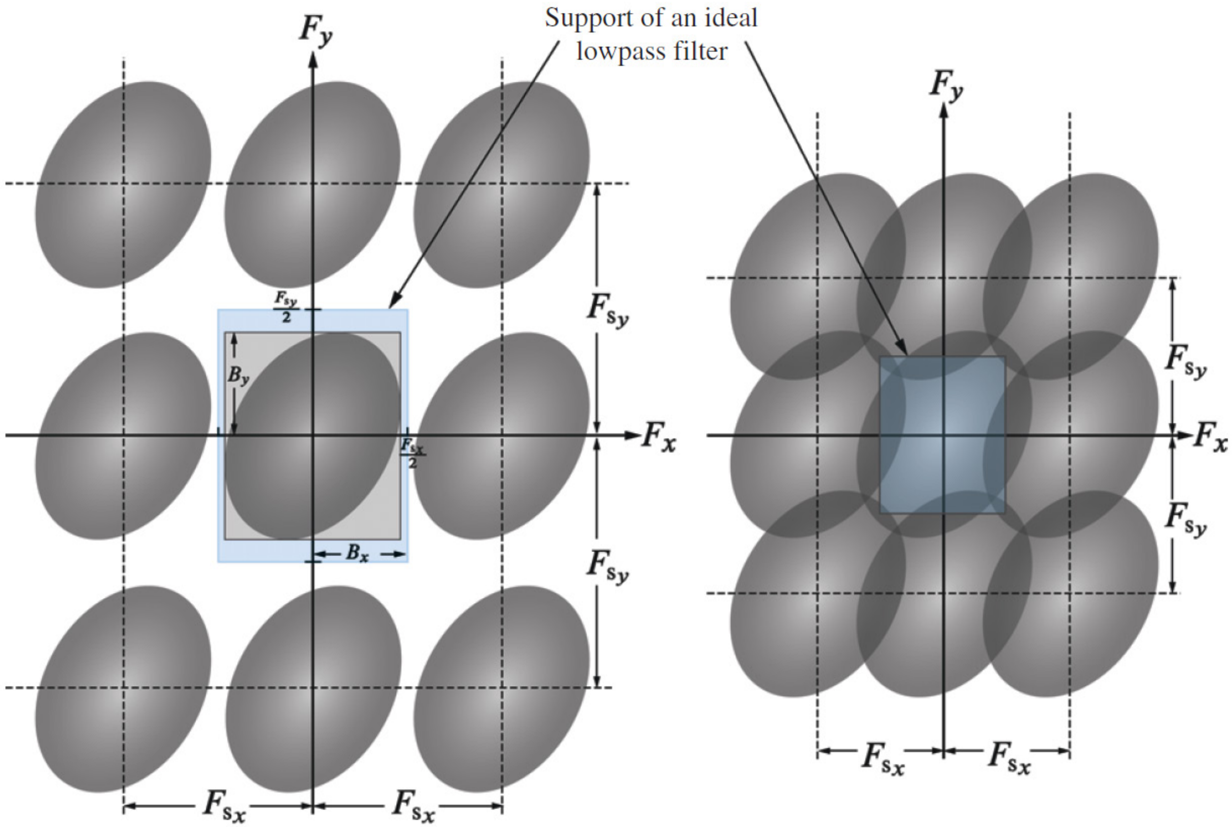
$$\tilde{S}(F_x, F_y) \triangleq \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} s[m, n] e^{j2\pi(m\Delta x F_x + n\Delta y F_y)}$$

$$= \frac{1}{\Delta x \Delta y} \sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} S_c(F_x - kF_{s_x}, F_y - \ell F_{s_y})$$

2-D sampling theorem

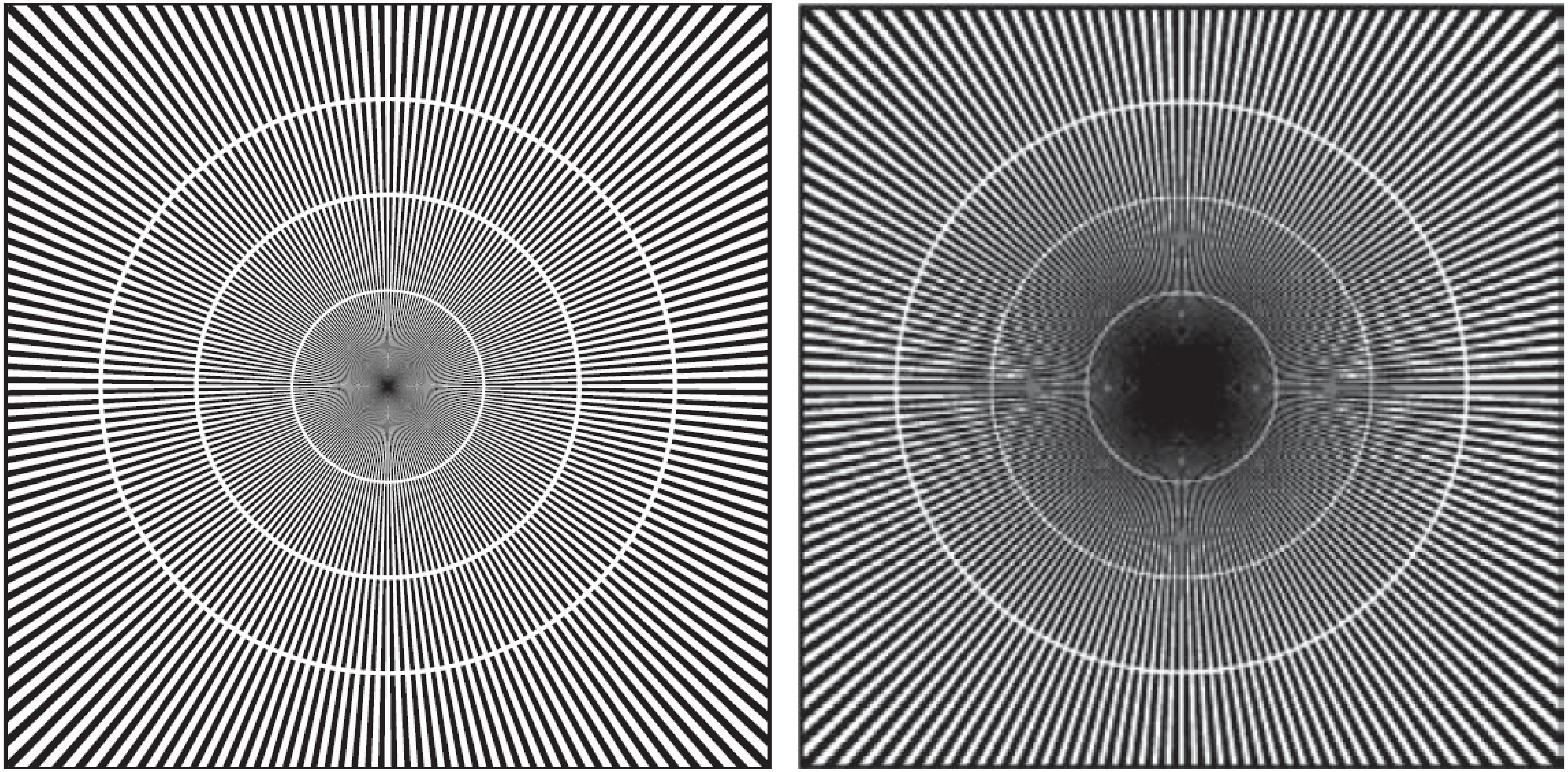
Bandlimited signal $S_c(F_x, F_y) = 0 \quad \text{for } |F_x| > B_x \text{ and } |F_y| > B_y$

Sampling frequency $F_{s_x} \geq 2B_x \quad \text{and} \quad F_{s_y} \geq 2B_y$



(a) No aliasing (b) Aliasing

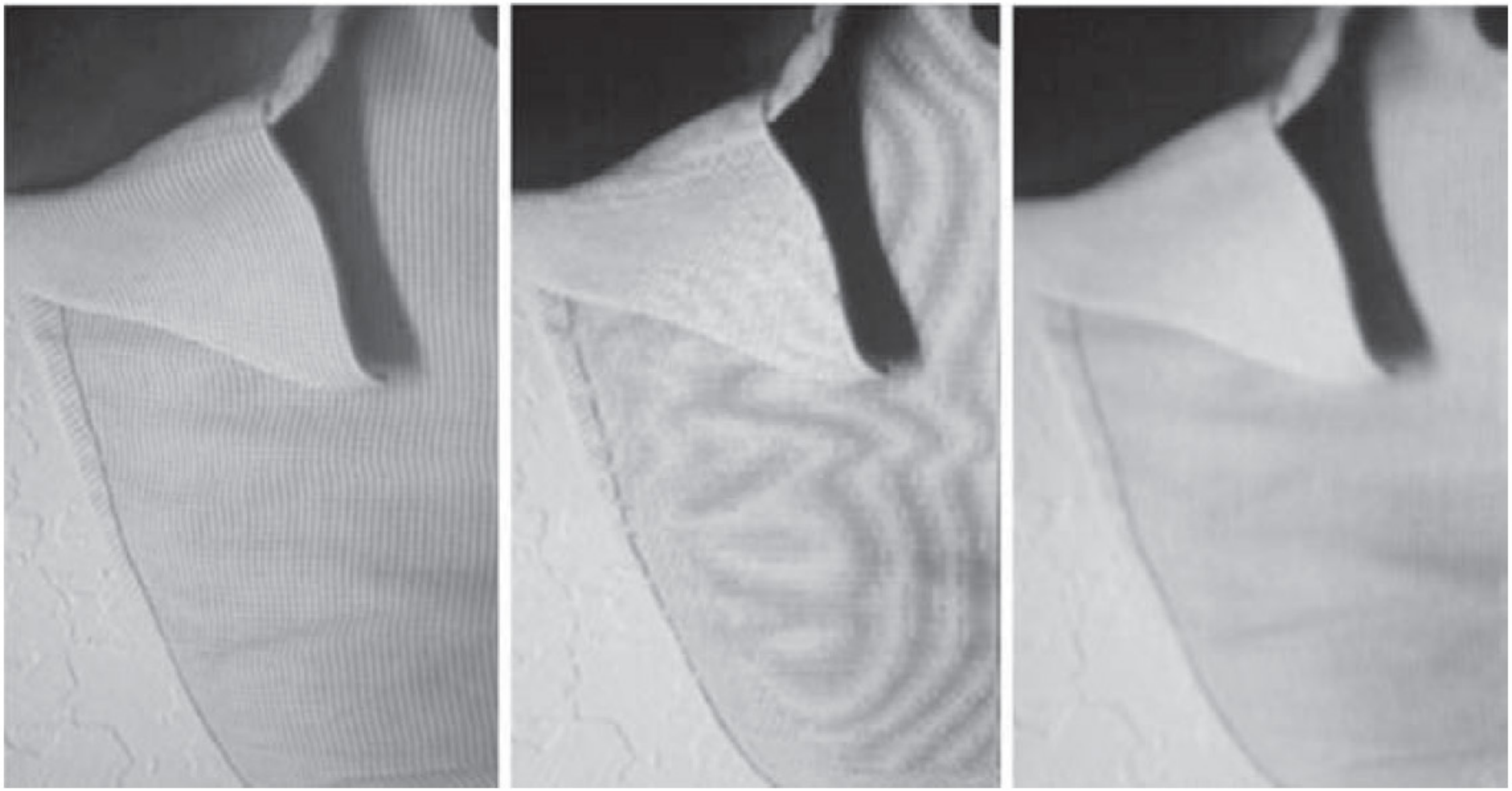
Moiré pattern due to aliasing



(a) (b)

Figure 6.35 Moiré pattern due to aliasing: (a) original pattern, (b) 72 dpi pattern.

Tradeoff between resolution and aliasing



(a) (b) (c)

Figure 6.36 Aliasing in resampled images (digital aliasing): (a) original image, (b) resampled without pre-filtering, and (c) resampled with pre-filtering.