Chap5
Transform analysis of LTI systems

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Chap 5 Transform analysis of LTI systems

- 5.1 Sinusoidal response of LTI systems
- 5.3 Distortion of signals passing through LTI systems
- 5.4 Ideal and practical filters
- 5.5 Frequency response for rational system functions
- 5.6 Dependence of frequency response on poles and zeros
- 5.7 Design of simple filters by pole-zero placement
- 5.8 Relationship between magnitude and phase responses
- 5.10 Invertibility and minimum-phase systems

Eigenfunctions of LTI systems

Eigenfunction $e^{j\omega n}$

$$x[n] = e^{j\omega n} \overset{\mathcal{H}}{\longmapsto} y[n] = H(e^{j\omega})e^{j\omega n}, \text{ all } n$$

$$H(e^{j\omega}) \triangleq H(z)|_{z=e^{j\omega}} = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

Uniqueness

$$y[n] = H(e^{j\omega})x[n] \text{ if and only if } x[n] = e^{j\omega n}, \text{ all } n$$

Frequency response

$$H(e^{j\omega}) = \underbrace{|H(e^{j\omega})|}_{\text{Magnitude response or gain}} e^{j\underbrace{\angle H(e^{j\omega})}_{\text{Phase response}}} = H_R(e^{j\omega}) + jH_I(e^{j\omega})$$

Magnitude response or gain

Phase response

Filtering

$$x[n] = Ae^{j(\omega n + \phi)} \overset{\mathcal{H}}{\longmapsto} y[n] = A|H(e^{j\omega})|e^{j[\omega n + \phi + \angle H(e^{j\omega})]}$$

Example of first-order difference equation

$$y[n] = ay[n-1] + bx[n]$$

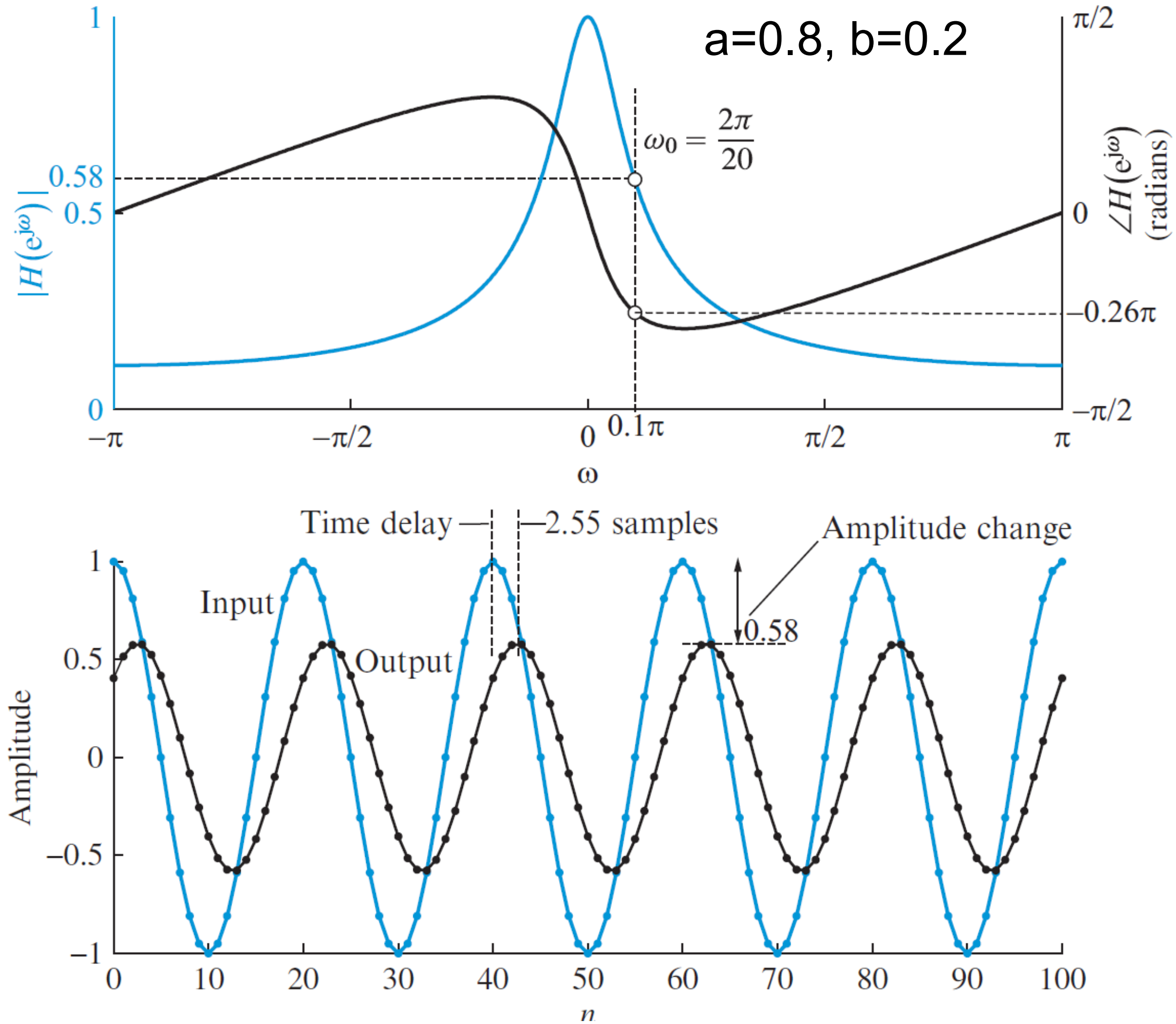
$$H(e^{j\omega}) = \frac{b}{1 - ae^{-j\omega}}$$

$$|H(e^{j\omega})| = \frac{|b|}{\sqrt{1 - 2a\cos\omega + a^2}},$$

$$\angle H(e^{j\omega}) = \angle b - \tan^{-1}\frac{a\sin\omega}{1 - a\cos\omega}.$$

At $\omega = 2\pi/20$, gain is 0.58 and phase shift is -0.26π.

(or $-0.26\pi/\omega = -2.55$ samples)

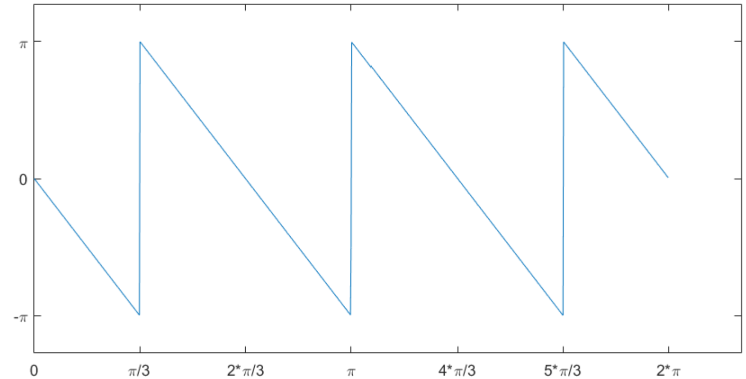


Continuous and principal phase functions

Phase ambiguity

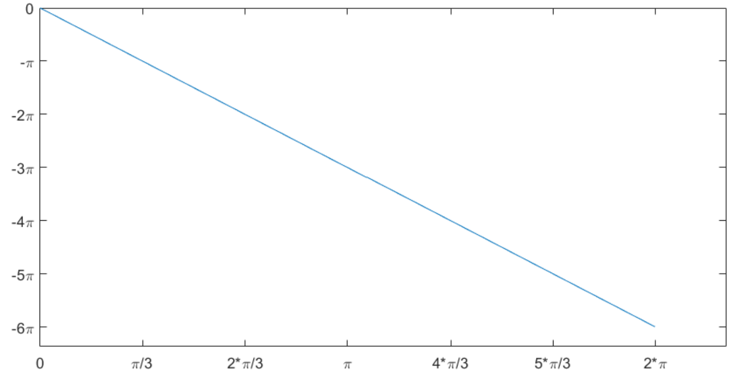
$$H\left(e^{j\omega}\right) = \left|H\left(e^{j\omega}\right)\right| e^{j\angle H\left(e^{j\omega}\right)} = \left|H\left(e^{j\omega}\right)\right| e^{j\left[\angle H\left(e^{j\omega}\right)+2m\pi\right]}$$

Principal phase



$$-\pi < \mathrm{ARG}\left[H\left(e^{j\omega}\right)\right] \leq \pi$$

Continuous phase
(integral of group delay)



$$\Psi(\omega)$$

Steady-state and transient response

Causal system and practical input $h[n] = 0, n < 0$ $x[n] = e^{j\omega n}u[n]$

Response $\lim_{n\to\infty} y[n] = H(e^{j\omega})e^{j\omega n} = y_{ss}[n]$

$$y[n] = \sum_{k=0}^{n} h[k]x[n-k] = \sum_{k=0}^{n} h[k]e^{j\omega(n-k)}$$

$$= \left(\sum_{k=0}^{\infty} h[k]e^{-j\omega k}\right)e^{j\omega n} - \left(\sum_{k=n+1}^{\infty} h[k]e^{-j\omega k}\right)e^{j\omega n}$$

$$= \underbrace{H(e^{j\omega})e^{j\omega n}}_{y_{ss}[n]} - \underbrace{\left(\sum_{k=n+1}^{\infty} h[k]e^{-j\omega k}\right)e^{j\omega n}}_{y_{tr}[n]}.$$

Steady-state

Transient

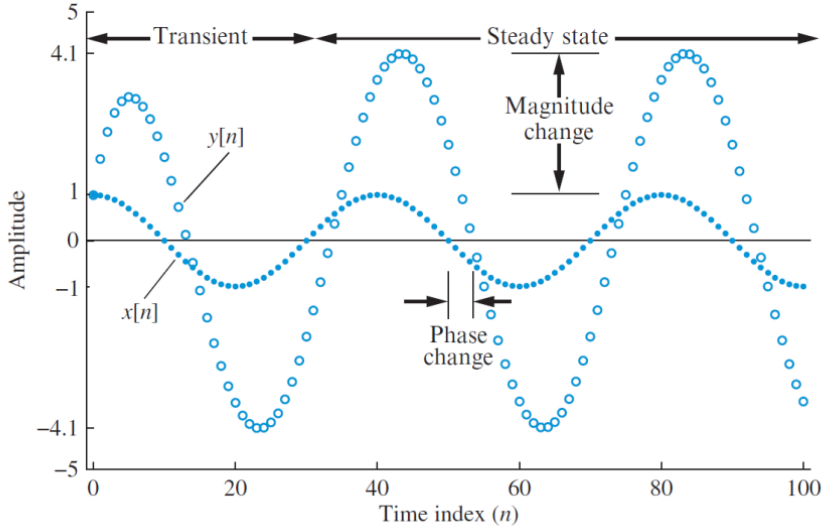


Figure 5.3 Transient and steady-state responses for sinusoidal excitation $x[n] = \cos(0.05\pi n)$.

Distortionless response system

$$y[n] = Gx[n - n_d], \quad G > 0$$

Maintain the “shape”

$$Y(e^{j\omega}) = Ge^{-j\omega n_d} X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = Ge^{-j\omega n_d}$$

→ $$|H(e^{j\omega})| = G$$ Constant gain

→ $$\angle H(e^{j\omega}) = -\omega n_d$$ Linear phase

Magnitude distortion

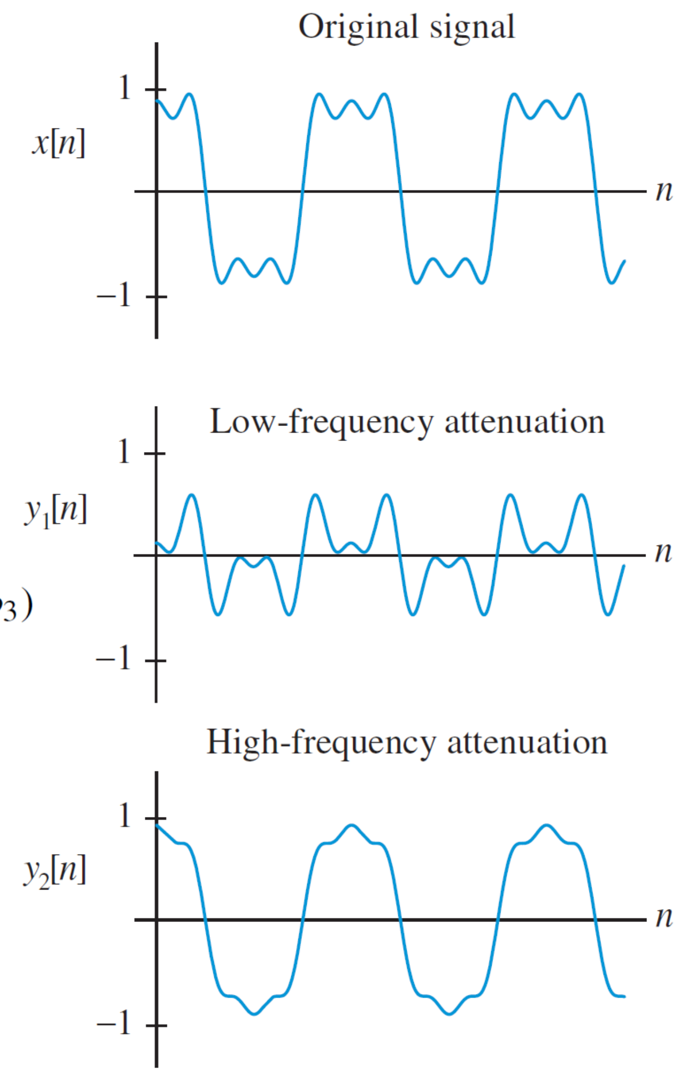
$$|H(e^{j\omega})| \neq G$$

Example:

$$x[n] = \cos(\omega_0 n) - \tfrac{1}{3}\cos(3\omega_0 n) + \tfrac{1}{5}\cos(5\omega_0 n)$$

$$y_i[n] = c_1\cos(\omega_0 n + \phi_1) + c_2\cos(3\omega_0 n + \phi_2) + c_3\cos(5\omega_0 n + \phi_3)$$

Signal	c_1	c_2	c_3	ϕ_1	ϕ_2	ϕ_3	Amplitude
$x[n]$	1	$-1/3$	$1/5$	0	0	0	original
$y_1[n]$	$1/4$	$-1/3$	$1/5$	0	0	0	highpass
$y_2[n]$	1	$-1/6$	$1/10$	0	0	0	lowpass



Phase or delay distortion

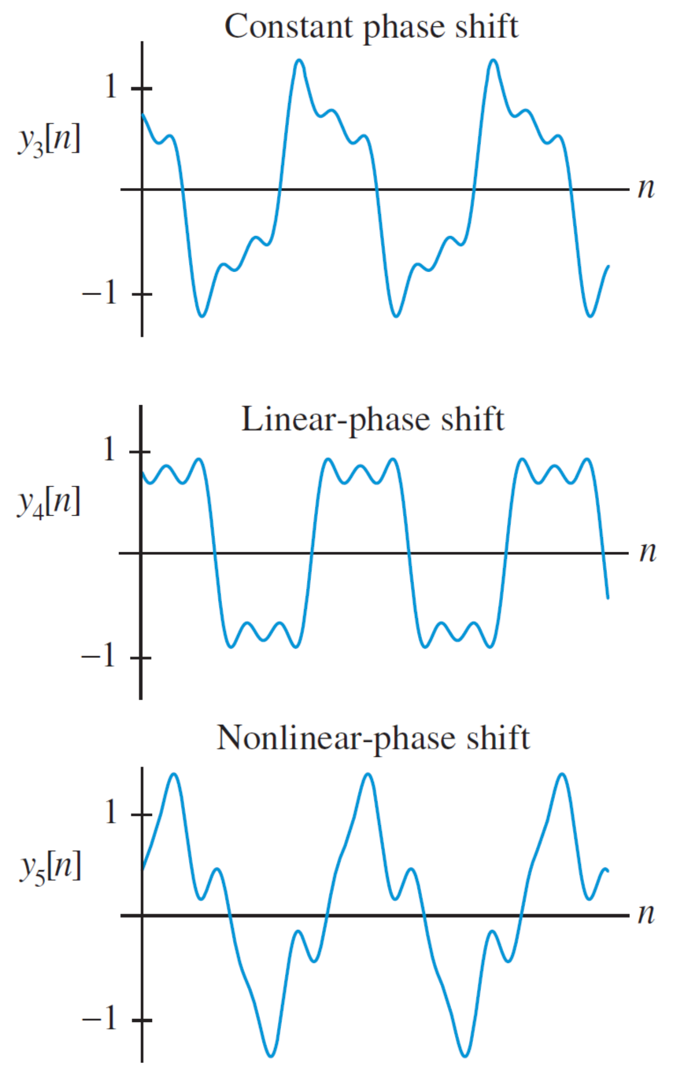
$$\angle H(e^{j\omega}) \neq -\omega n_d$$

$$y[n] = A_x |H(e^{j\omega})| \cos[\omega n + \phi_x + \angle H(e^{j\omega})]$$
$$= A_x |H(e^{j\omega})| \cos\left\{\omega\left[n + \frac{\phi_x}{\omega} + \frac{\angle H(e^{j\omega})}{\omega}\right]\right\}$$

Phase delay $$\tau_{pd}(\omega) \triangleq -\frac{\angle H(e^{j\omega})}{\omega}$$

Example:

Signal	c_1	c_2	c_3	ϕ_1	ϕ_2	ϕ_3	Phase shift
$x[n]$	1	$-1/3$	$1/5$	0	0	0	zero
$y_3[n]$	1	$-1/3$	$1/5$	$\pi/6$	$\pi/6$	$\pi/6$	constant
$y_4[n]$	1	$-1/3$	$1/5$	$-\pi/4$	$-3\pi/4$	$-5\pi/4$	linear
$y_5[n]$	1	$-1/3$	$1/5$	$-\pi/3$	$\pi/4$	$\pi/7$	nonlinear



Group delay

Group delay

$$\tau_{\mathrm{gd}}(\omega) \triangleq -\frac{\mathrm{d}\Psi(\omega)}{\mathrm{d}\omega}$$

$$\Psi(\omega) = -\int_0^{\omega} \tau_{\mathrm{gd}}(\theta)\mathrm{d}\theta + \Psi(0)$$

Example of modulation:

$$x[n] = s[n]\cos\omega_{\mathrm{c}}n$$

$s[n]$ is a lowpass signal with maximum frequency $\omega_{\mathrm{m}} \ll \omega_{\mathrm{c}}$

$$\Downarrow$$

$$\Psi(\omega) \approx \Psi(\omega_{\mathrm{c}}) + \left.\frac{\mathrm{d}\Psi(\omega)}{\mathrm{d}\omega}\right|_{\omega=\omega_{\mathrm{c}}}(\omega - \omega_{\mathrm{c}})$$

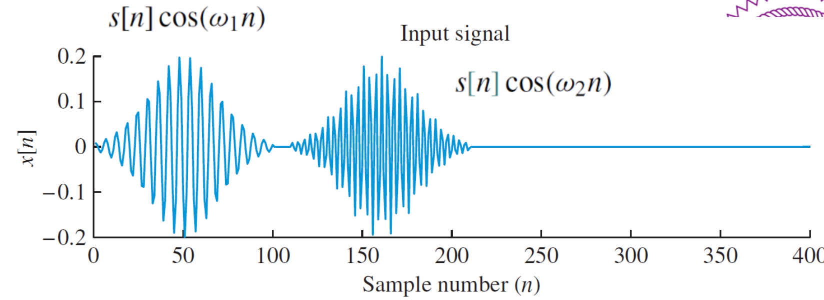
$$= -\tau_{\mathrm{pd}}(\omega_{\mathrm{c}})\omega_{\mathrm{c}} - \tau_{\mathrm{gd}}(\omega_{\mathrm{c}})(\omega - \omega_{\mathrm{c}}),$$

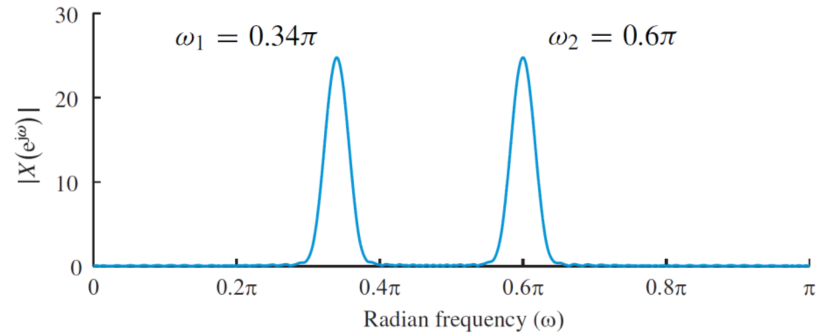
$$\Downarrow$$

$$y[n] \approx \left|H(\mathrm{e}^{\mathrm{j}\omega_{\mathrm{c}}})\right| s[n - \tau_{\mathrm{gd}}(\omega_{\mathrm{c}})]\cos\{\omega_{\mathrm{c}}[n - \tau_{\mathrm{pd}}(\omega_{\mathrm{c}})]\}$$

Group delay for the envelop s[n] (“group”)

Example of magnitude and group-delay distortions

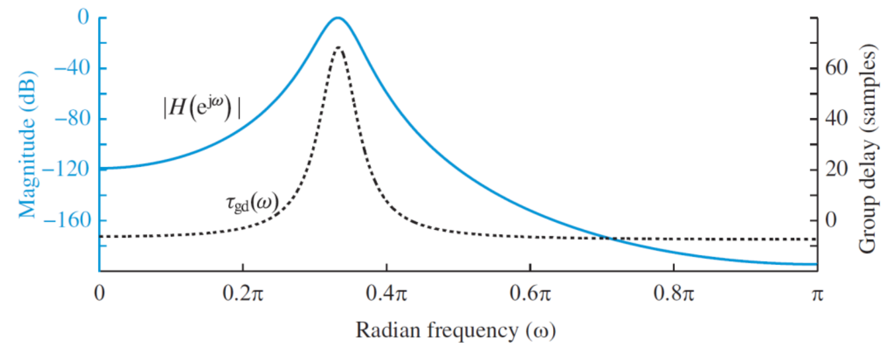




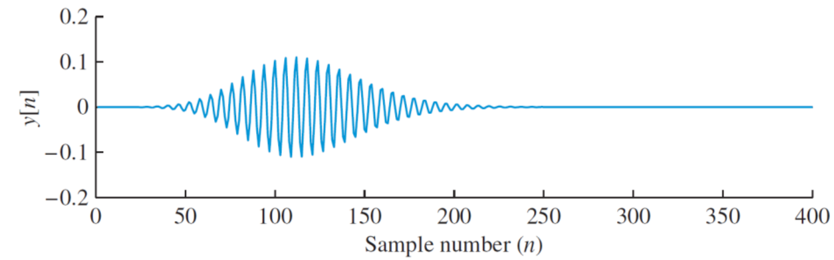
Input signal: $$s(t) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2}\frac{(t-\mu)^2}{\sigma^2}\right\}$$

Bandpass filter: $$H(z) = \frac{b_0}{\left[1 - 2r\cos(\omega_0)\, z^{-1} + r^2 z^{-2}\right]^K}$$

$$\omega_0 = \pi/3, \text{ and } K = 8$$

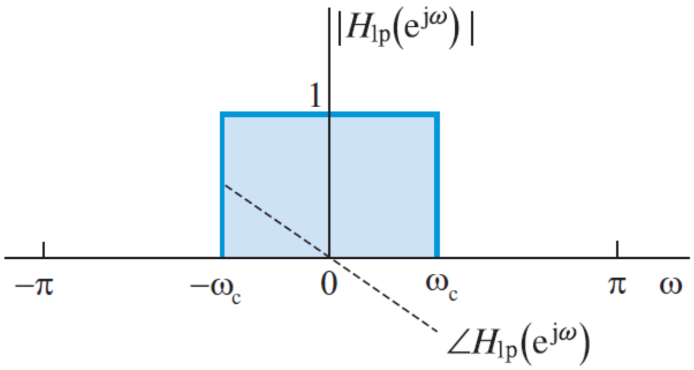


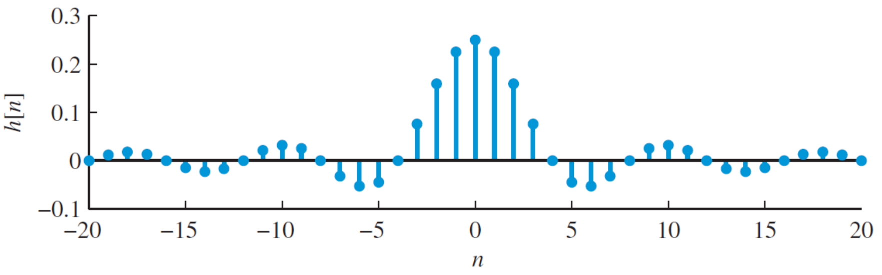
Output signal:



Ideal (frequency-selective) filters (1/2)

Lowpass filter





$$H_{\mathrm{lp}}(e^{j\omega}) = \begin{cases} e^{-j\omega n_d}, & |\omega| < \omega_c \\ 0. & \omega_c < |\omega| \le \pi \end{cases}$$

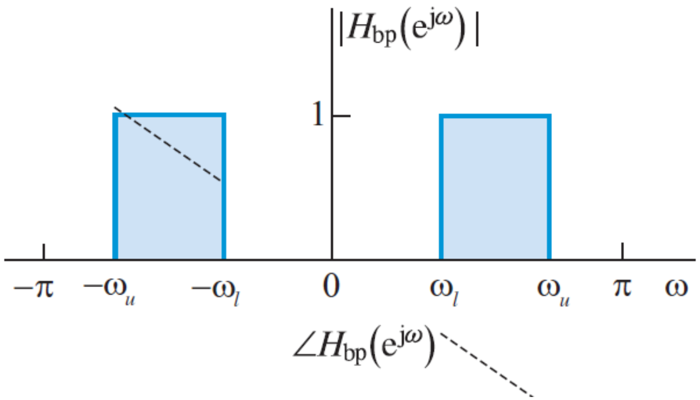
$$h_{\mathrm{lp}}[n] = \frac{\sin \omega_c (n - n_d)}{\pi (n - n_d)}$$

$$\sum_{n=-\infty}^{\infty} |h_{\mathrm{lp}}[n]| = \infty$$

Ideal filters are unstable and thus not practical.

Ideal (frequency-selective) filters (2/2)

Bandpass filter



$$H(e^{j\omega}) = \begin{cases} e^{-j\omega n_d}, & \omega_\ell \le |\omega| \le \omega_u \\ 0, & \text{otherwise} \end{cases}$$

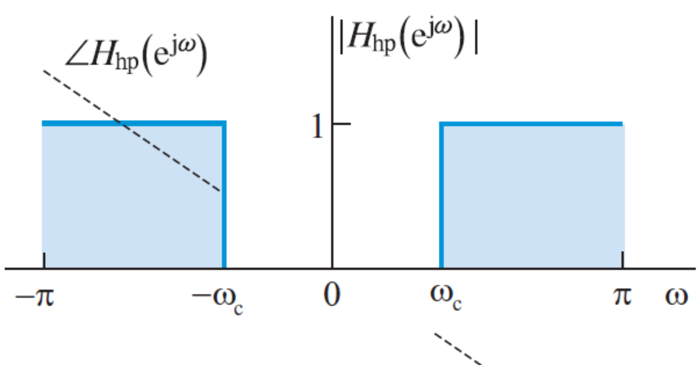
$$h_{bp}[n] = 2\frac{\sin \omega_c (n - n_d)}{\pi (n - n_d)} \cos \omega_0 n$$

$$\omega_c = (\omega_u - \omega_\ell)/2$$

$$\omega_0 = (\omega_u + \omega_\ell)/2$$

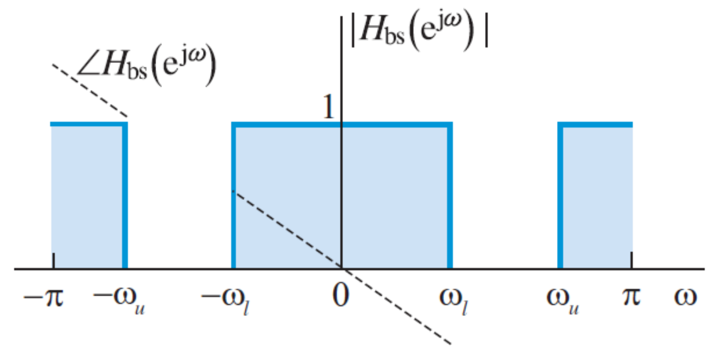
Ideal bandpass filters are modulated lowpass filters.

Highpass filter



$$h_{hp}[n] = \delta[n] - h_{lp}[n]$$

Bandstop filter



$$h_{bs}[n] = \delta[n] - h_{bp}[n]$$

Practical filter

Good filters should have

- Small ripples in passband
- Low gain in stopband
- Narrow width in transition-band

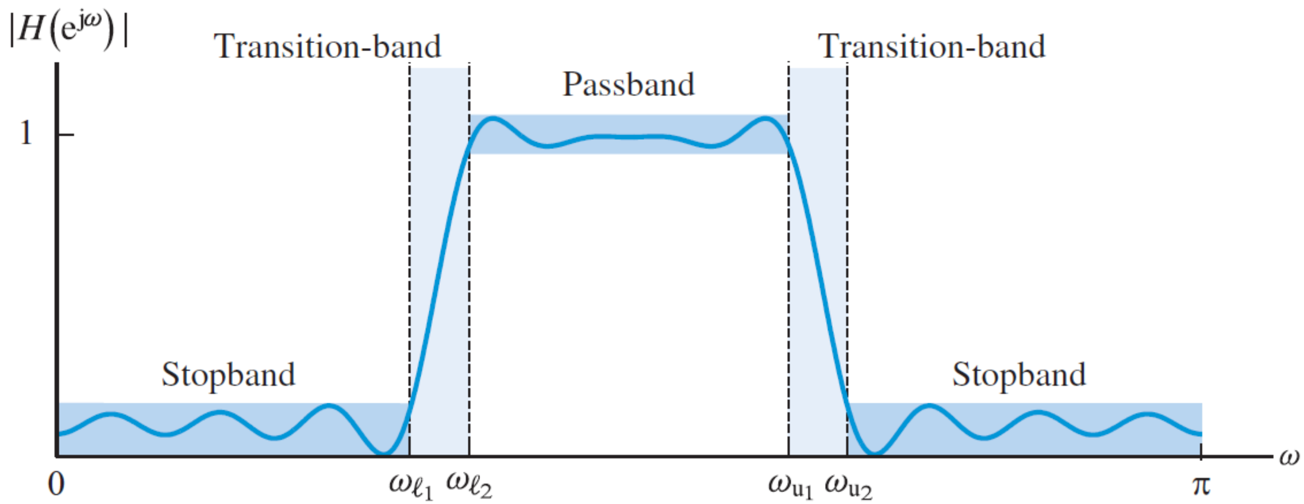


Figure 5.11 Typical characteristics of a practical bandpass filter.

Example of practical lowpass filter:

$$\hat{h}_{lp}[n] = \begin{cases} \dfrac{\sin \omega_c (n - n_d)}{\pi (n - n_d)}, & 0 \le n \le M - 1 \\ 0. & \text{otherwise} \end{cases}$$

Frequency response for rational system functions

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

zeros

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{B(z)}{A(z)} \qquad H(e^{j\omega}) = b_0 \left. \frac{\prod_{k=1}^{M} (1 - z_k z^{-1})}{\prod_{k=1}^{N} (1 - p_k z^{-1})} \right|_{z=e^{j\omega}}$$

poles

$$|H(e^{j\omega})| = |b_0| \prod_{k=1}^{M} \left|1 - z_k e^{-j\omega}\right| \Big/ \prod_{k=1}^{N} \left|1 - p_k e^{-j\omega}\right|,$$

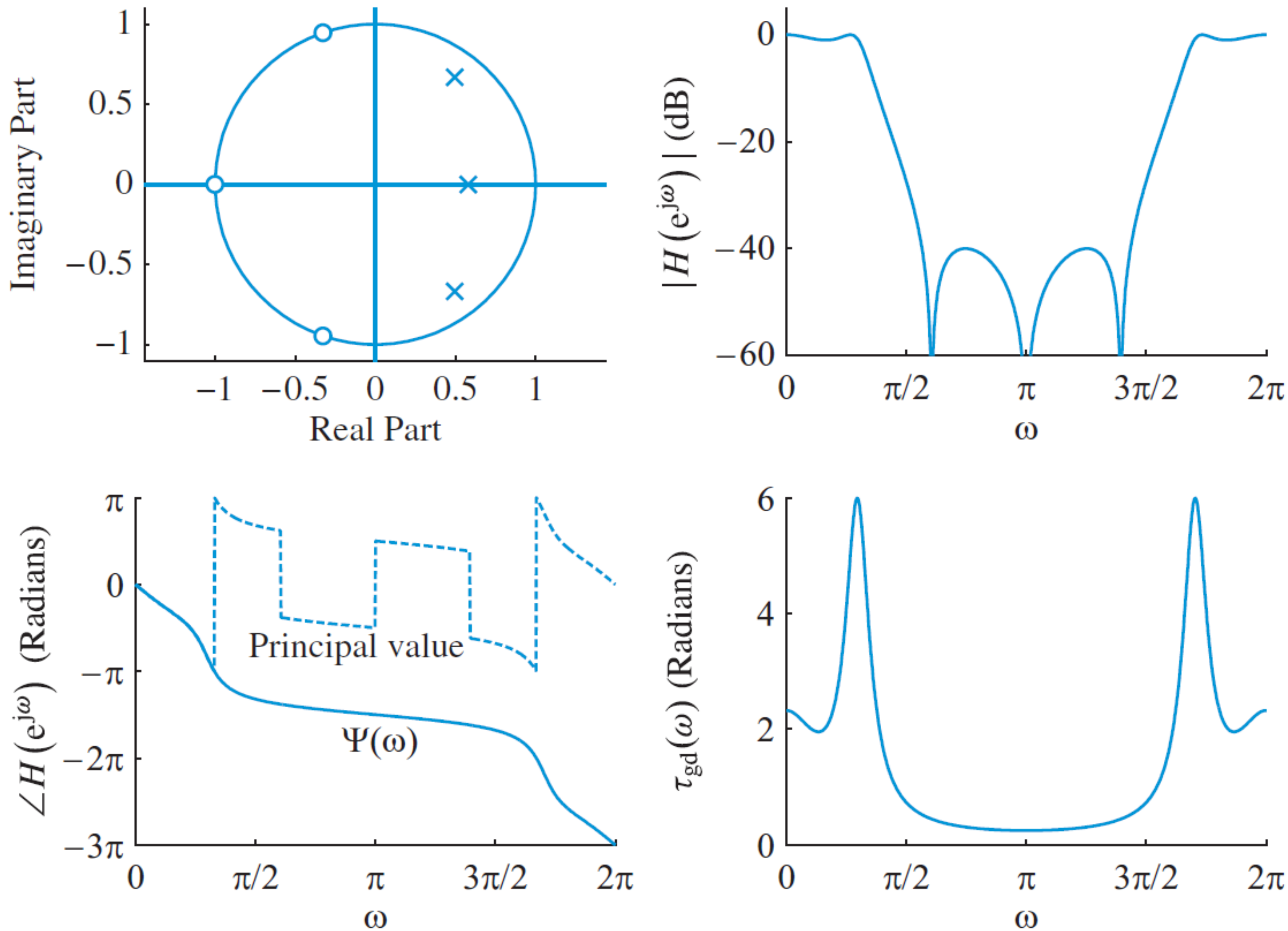
$$\angle H(e^{j\omega}) = \angle b_0 + \sum_{k=1}^{M} \angle(1 - z_k e^{-j\omega}) - \sum_{k=1}^{N} \angle(1 - p_k e^{-j\omega}),$$

$$\tau_{gd}(\omega) = \sum_{k=1}^{M} \frac{d}{d\omega} \left[\angle(1 - z_k e^{-j\omega})\right] - \sum_{k=1}^{N} \frac{d}{d\omega} \left[\angle(1 - p_k e^{-j\omega})\right]$$

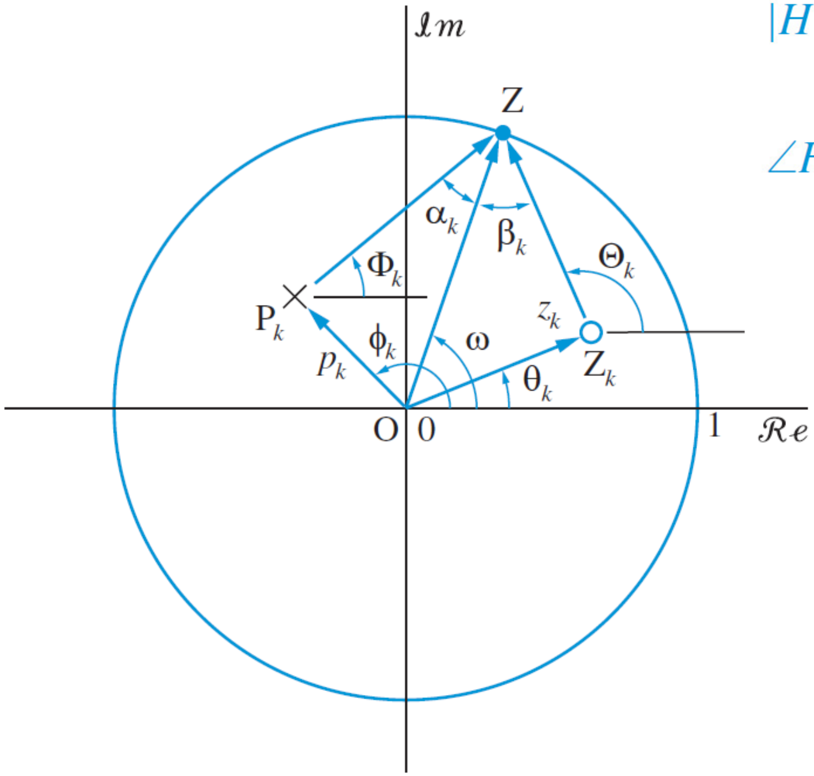
Detailed analytical expressions available

Example

$$H(z) = \frac{1 + 1.655z^{-1} + 1.655z^{-2} + z^{-3}}{1 - 1.57z^{-1} + 1.264z^{-2} - 0.4z^{-3}}$$



Geometrical evaluation from poles and zeros



$$|H(e^{j\omega})| = |b_0| \frac{\prod_{k=1}^{M} Q_k(\omega)}{\prod_{k=1}^{N} R_k(\omega)},$$

$$\angle H(e^{j\omega}) = \angle b_0 + \omega(N-M) + \sum_{k=1}^{M} \Theta_k(\omega) - \sum_{k=1}^{N} \Phi_k(\omega)$$

$Q_k(\omega)$ = distance of kth zero from $z = e^{j\omega}$,

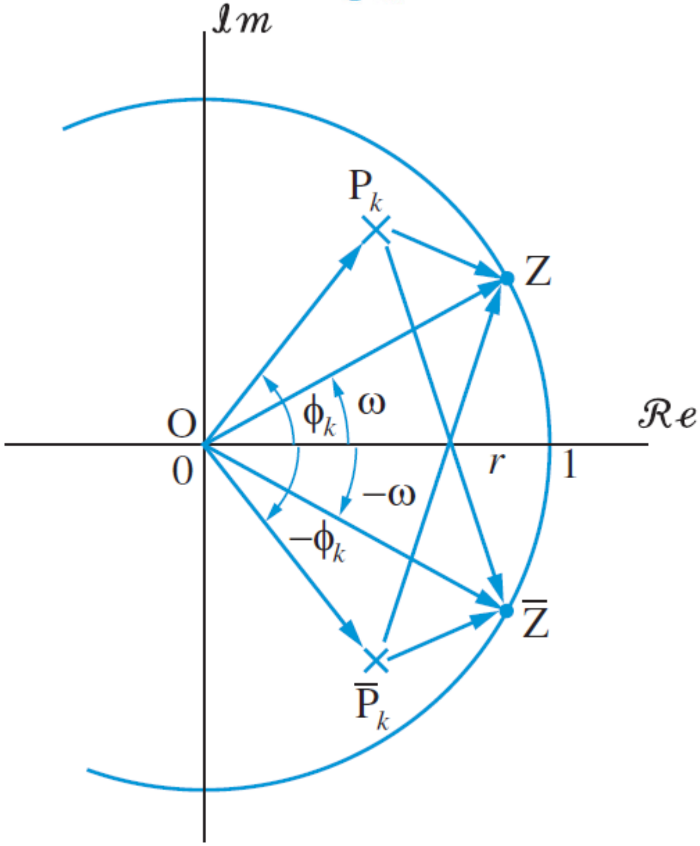
$R_k(\omega)$ = distance of kth pole from $z = e^{j\omega}$,

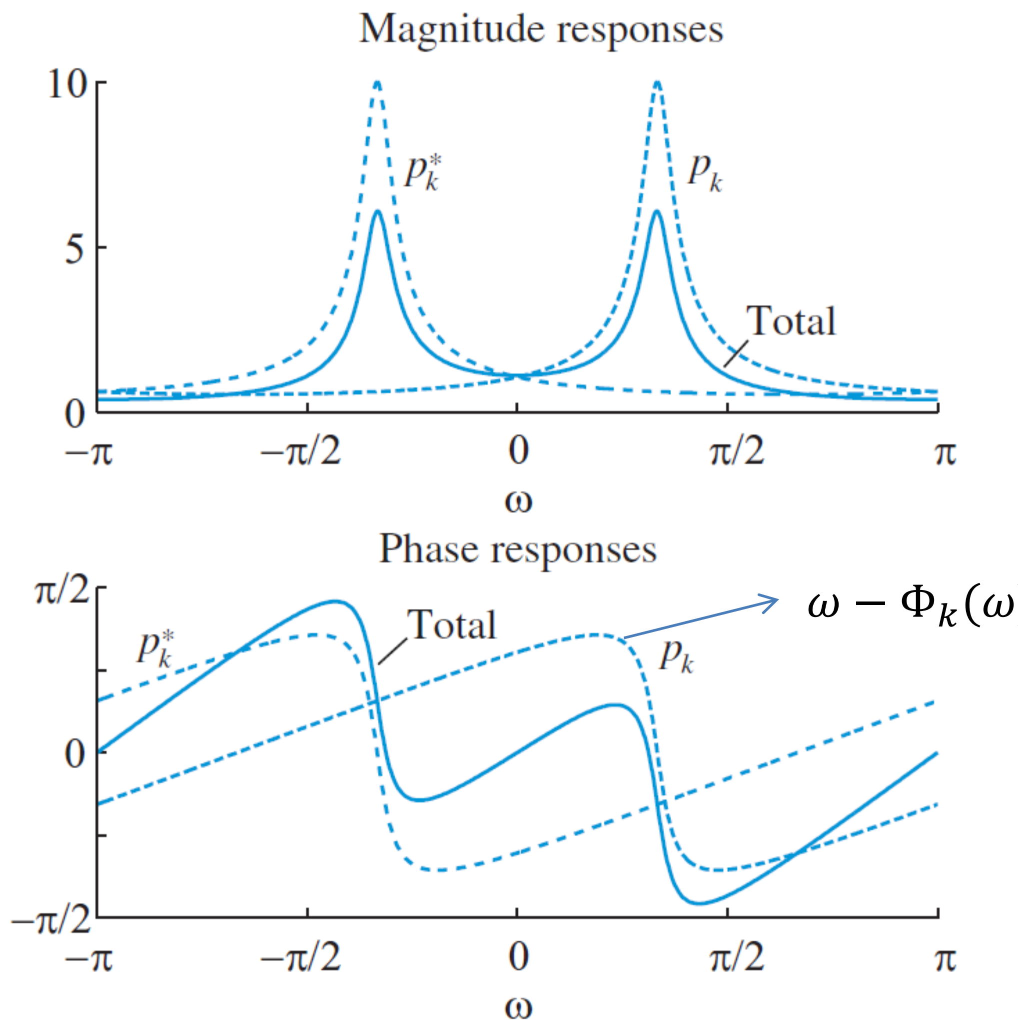
$\Theta_k(\omega)$ = angle of kth zero with the real axis,

$\Phi_k(\omega)$ = angle of kth pole with the real axis.

Example of complex conjugate poles

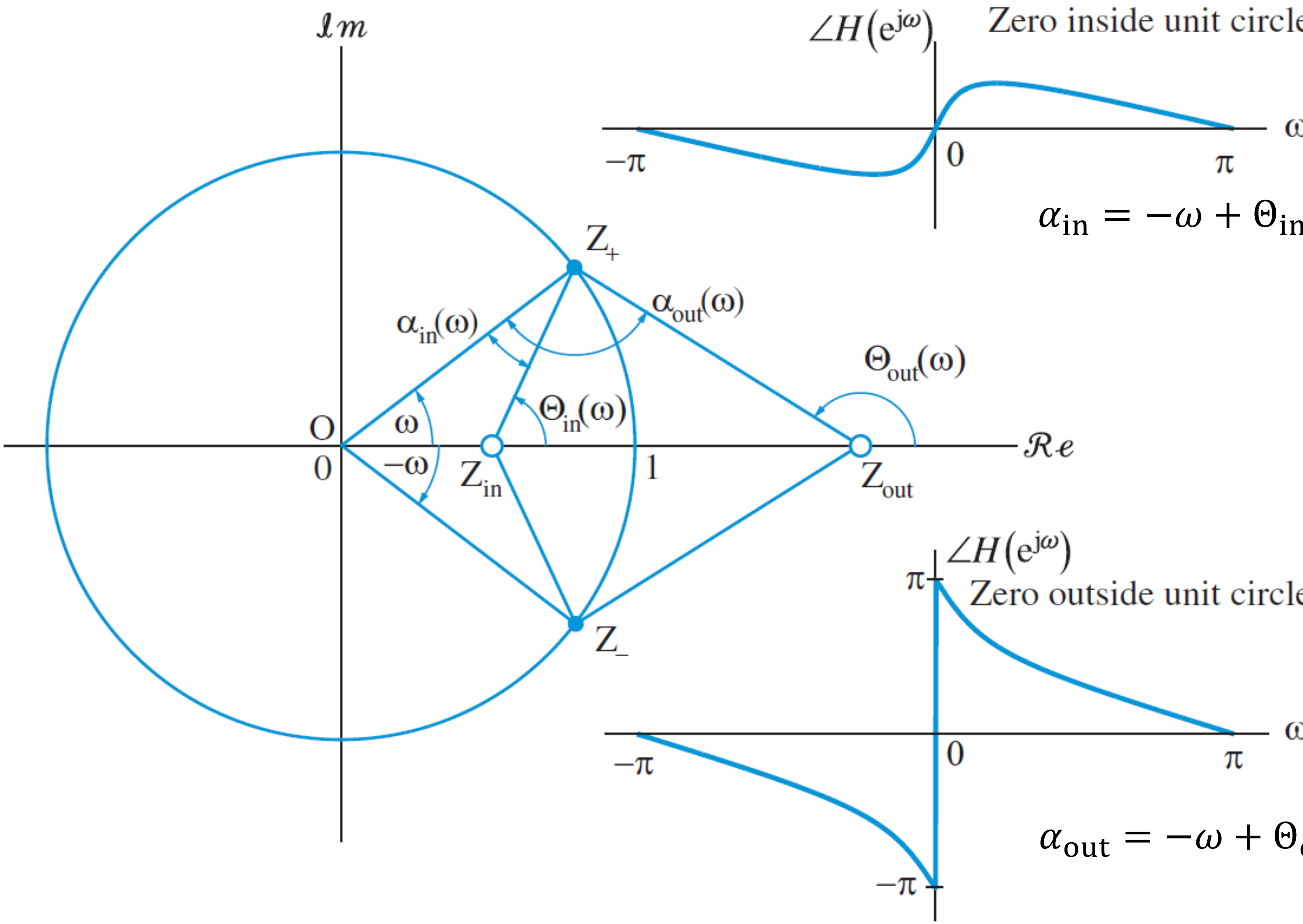
$p_k = 0.9e^{\pm j\pi/3}$





$\omega - \Phi_k(\omega)$

Example of two zeros



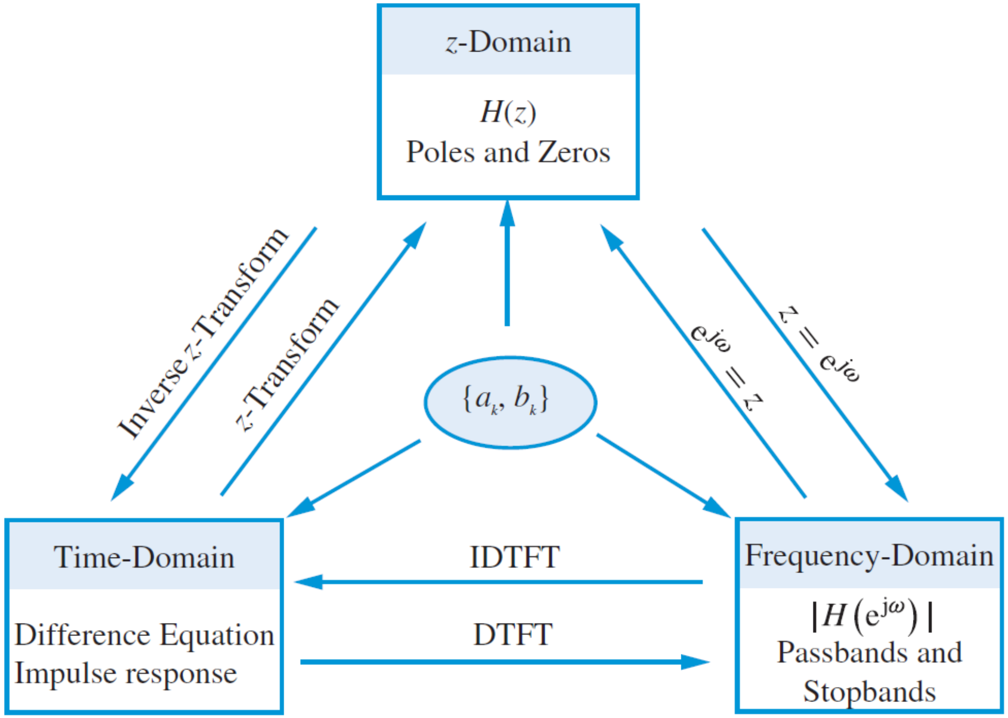
$$\alpha_{\text{in}} = -\omega + \Theta_{\text{in}}(\omega)$$

$$\alpha_{\text{out}} = -\omega + \Theta_{\text{out}}(\omega)$$

Design of simple filters by pole-zero placement

Guidelines:

- Place a zero at $\theta = \omega_0$ on the unit circle to suppress magnitude at $\omega = \omega_0$.
- Place a pole at $\phi = \omega_0$ inside the unit circle to enhance magnitude at $\omega = \omega_0$.
- Place complex conjugate pairs for zeros and poles to assure real coefficients.
- May introduce zeros and poles at z=0 to make N=M.

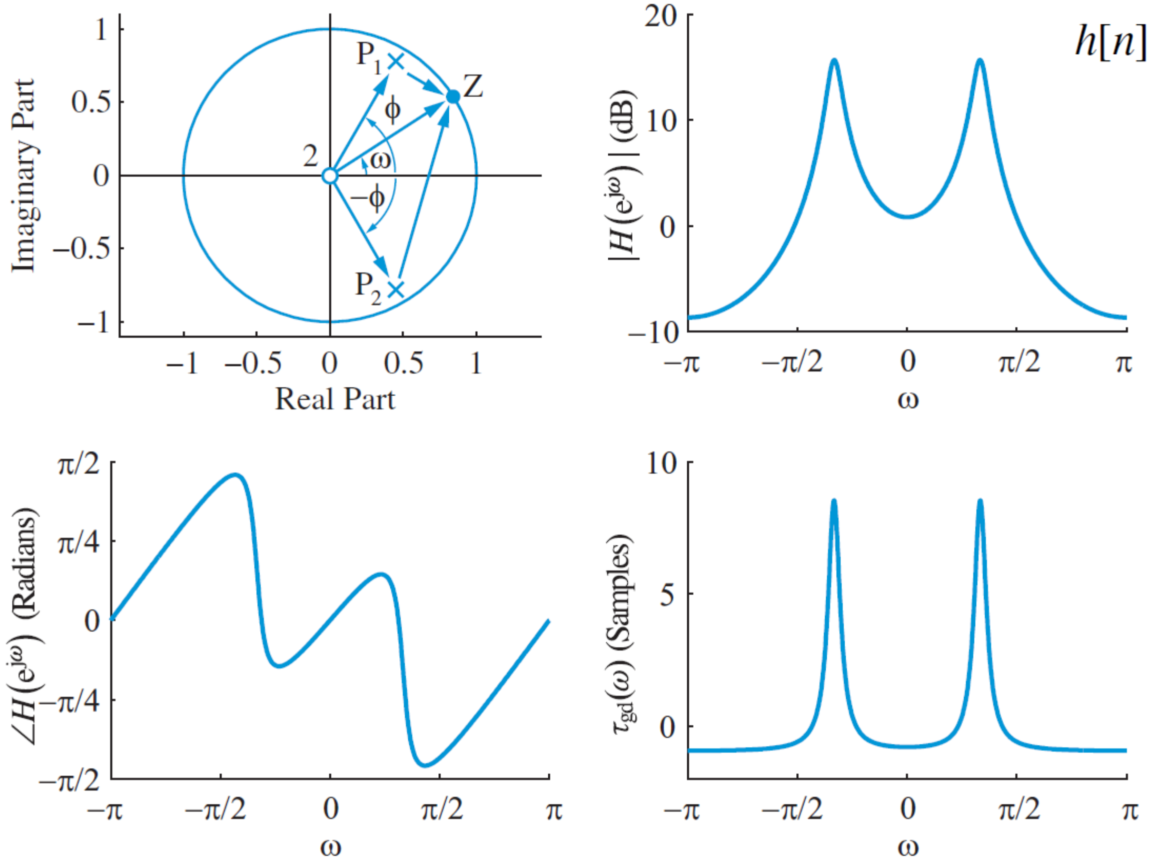


Example: Discrete-time resonator

$$H(z) = \frac{b_0}{\left(1 - re^{j\phi}z^{-1}\right)\left(1 - re^{-j\phi}z^{-1}\right)} = \frac{b_0}{1 - (2r\cos\phi)z^{-1} + r^2z^{-2}}$$

$$p_{1,2} = re^{\pm j\phi}$$

$$h[n] = b_0 r^n \frac{\sin[(n+1)\phi]}{\sin\phi} u[n]$$



Example: Notch filter

$$H(z) = b_0[1 - (2\cos\phi)z^{-1} + z^{-2}]$$

FIR (two zeros)

$$G(z) = b_0 \frac{1 - (2\cos\phi)z^{-1} + z^{-2}}{1 - (2r\cos\phi)z^{-1} + r^2 z^{-2}}$$

IIR (plus two poles)

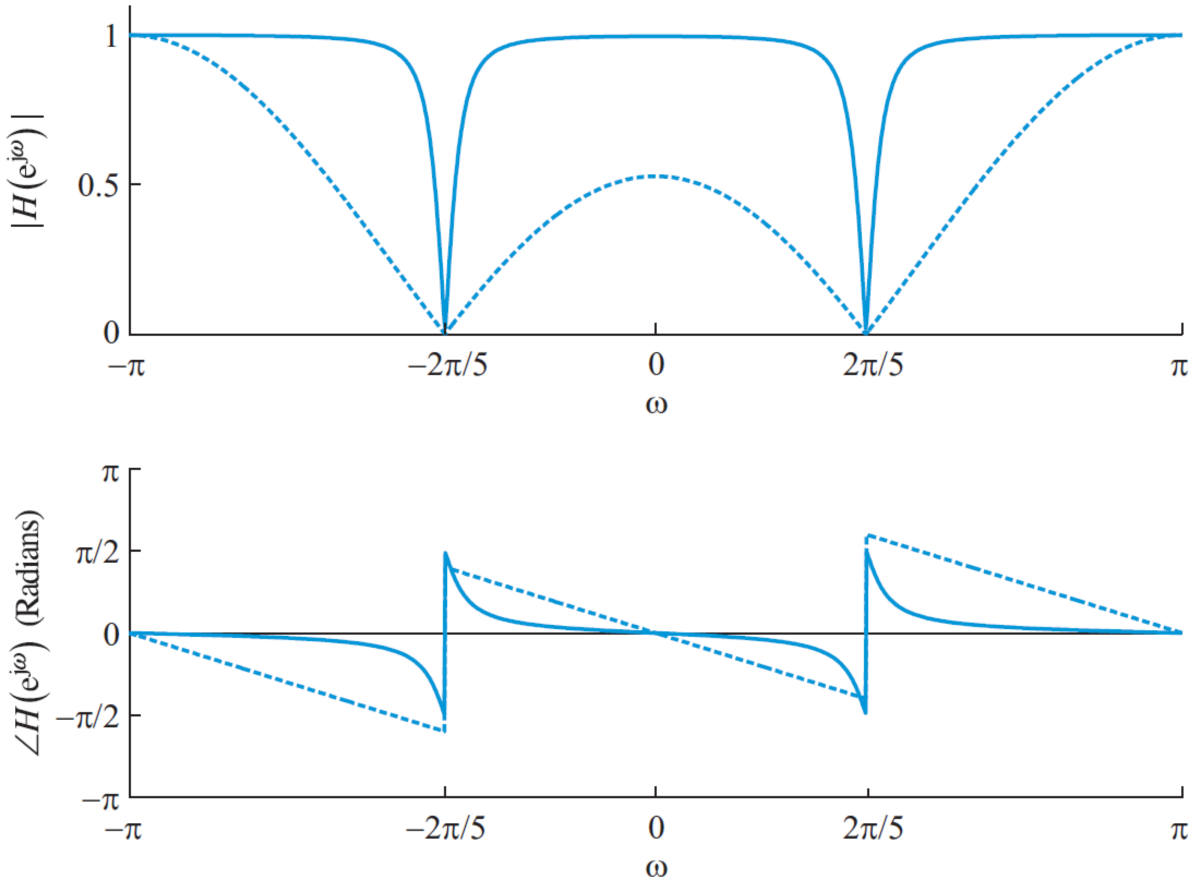


Figure 5.24 Magnitude and phase response of a second-order FIR notch filter (dashed line) and a second-order IIR notch filter with $r = 0.9$ and $\phi = 2\pi/5$.

Example: Comb filter

$$G(z) \triangleq H(z^L) = \sum_{n=-\infty}^{\infty} h[n] z^{-nL}$$

$$G(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega L n} = H(e^{j\omega L})$$

$$g[n] = \begin{cases} h[n/L], & n = 0, \pm L, \pm 2L, \dots \\ 0. & \text{otherwise} \end{cases}$$

Example:

$$H(z) = 1 - z^{-1}$$

$$G(z) = 1 - z^{-L}$$

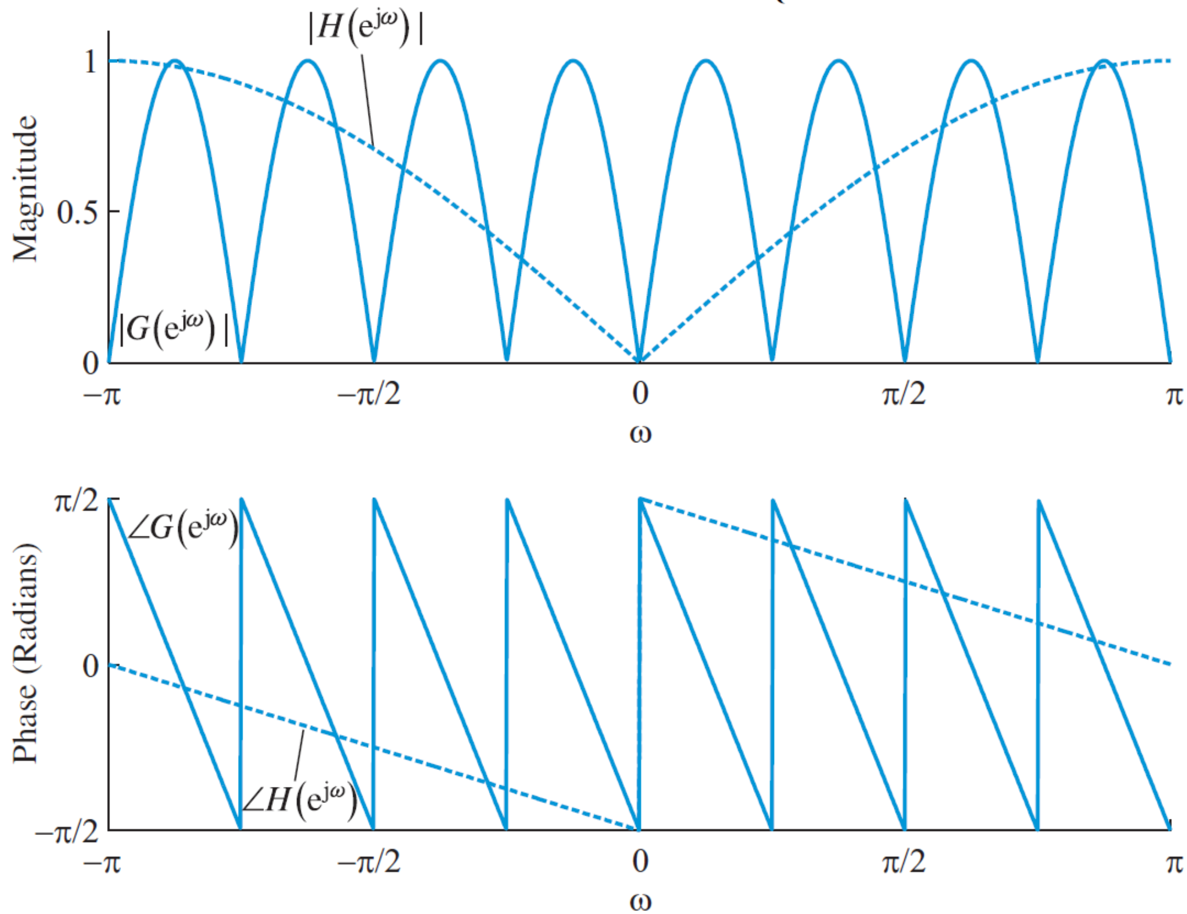


Figure 5.25 Magnitude and phase response of a first-order difference filter $H(e^{j\omega})$ and the corresponding comb filter $G(e^{j\omega})$ for $L = 8$. Note that $G(e^{j\omega})$ is periodic with period $2\pi/8$ radians and that both filters have a linear-phase response.

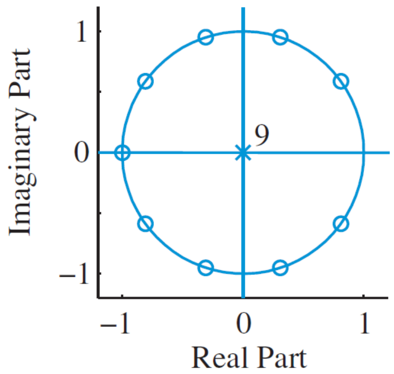
Example: Moving average filter

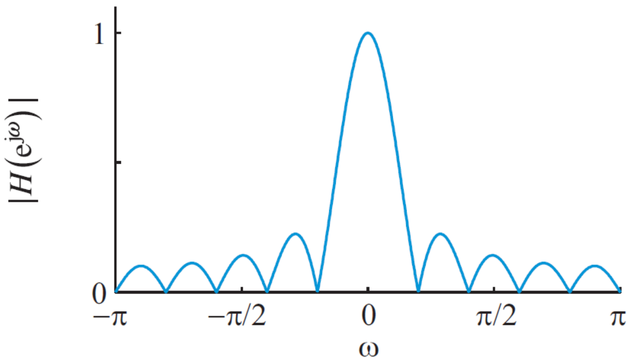
$$y[n] = \frac{1}{M} \sum_{k=0}^{M-1} x[n-k]$$

$$H(z) = \frac{1}{M} \sum_{k=0}^{M-1} z^{-k} = \frac{1}{M} \frac{1-z^{-M}}{1-z^{-1}} = \frac{1}{M} \frac{z^M - 1}{z^{M-1}(z-1)}$$

M-1 zeros
+ M-1 poles at z=0

$M = 10$

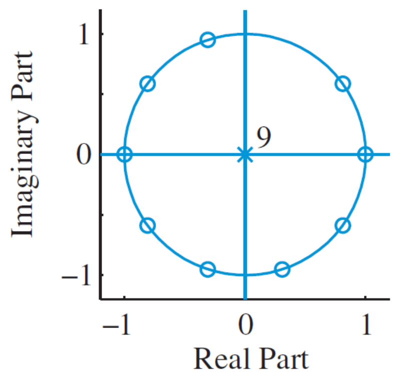


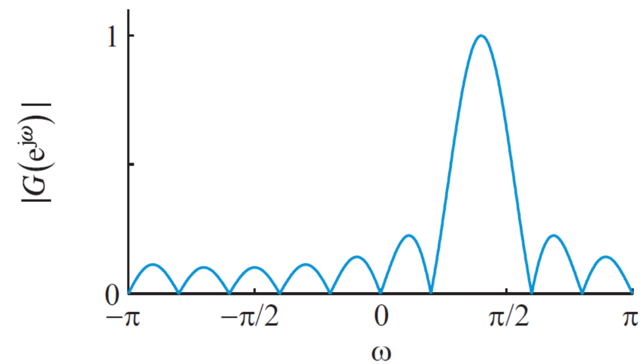


Example: Bandpass filter

$$e^{j\omega_m k} h[k] \overset{\text{DTFT}}{\longleftrightarrow} H(e^{j[\omega-\omega_m]})$$

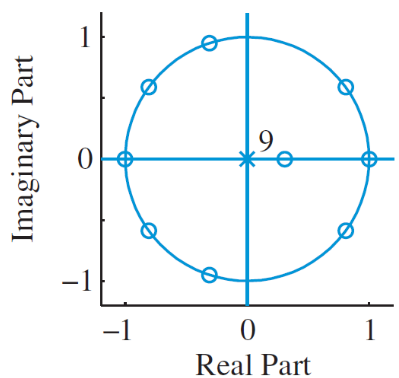
Complex bandpass filter

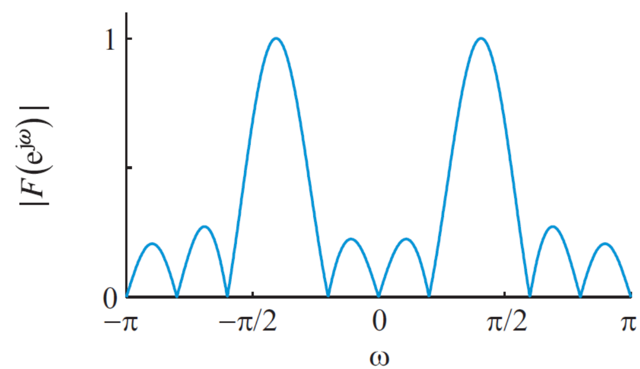




$$h[k]\cos\omega_m k \overset{\text{DTFT}}{\longleftrightarrow} \frac{1}{2}H(e^{j[\omega-\omega_m]}) + \frac{1}{2}H(e^{j[\omega+\omega_m]})$$

Real bandpass filter





Example: Highpass filter

Generate a highpass filter from a prototype lowpass one:

$$g[n] = (-1)^n h[n] \xleftrightarrow{\text{DTFT}} G(e^{j\omega}) = H(e^{j[\omega-\pi]})$$

Example of different equation:

$$y[n] = -\sum_{k=1}^{N}(-1)^n a_k y[n-k] + \sum_{k=0}^{M}(-1)^n b_k x[n-k]$$

Magnitude response cannot uniquely identify phase response

Assume

$$R(z) = H(z)H^*(1/z^*), \qquad \text{complex } h[n]$$
$$= H(z)H(1/z). \qquad \text{real } h[n]$$

⇩

$$R(z)|_{z=e^{j\omega}} = |H(e^{j\omega})|^2 = H(e^{j\omega})H^*(e^{j\omega})$$

Consider

$$H_1(z) = \left(1 - az^{-1}\right)\left(1 - bz^{-1}\right),$$
$$H_2(z) = \left(1 - az^{-1}\right)(1 - bz),$$
$$H_3(z) = (1 - az)\left(1 - bz^{-1}\right),$$
$$H_4(z) = (1 - az)(1 - bz).$$

Have the same magnitude response since their R(z) are identical

$$R(z) = H(z)H(1/z) = \left(1 - az^{-1}\right)\left(1 - bz^{-1}\right)(1 - az)(1 - bz)$$

Minimum-phase systems

Definition A causal and stable LTI system with a causal and stable inverse.
⇒ All zeros and poles are inside unit circle.

Invertibility

$$H_{\text{inv}}(z) = \frac{1}{H(z)} = \frac{A(z)}{B(z)}$$

Decomposition Any rational system function can be decomposed into a minimum-phase system and an all-pass system.

$$H(z) = H_{\min}(z) H_{\text{ap}}(z)$$

Example:

$$H(z) = H_1(z)\left(z^{-1} - a^*\right)$$

zero: $z = 1/a^*$, where $|a| < 1$

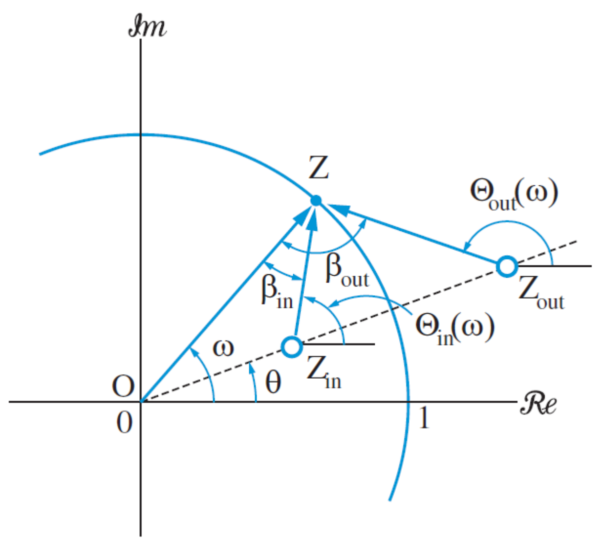
$$\Rightarrow \quad H(z) = H_1(z)\left(1 - az^{-1}\right)\frac{\left(z^{-1} - a^*\right)}{\left(1 - az^{-1}\right)}$$

(match a pole at reciprocal conjugate)

Minimum-phase systems

Minimum delay property

A minimum-phase system has the minimum phase-lag and the minimum group delay among all systems with the same magnitude response.



$$\angle H\left(e^{j\omega}\right) = -\omega + \Theta_k(\omega) = \beta_k$$