Chap4
Fourier representation of signals

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Chap 4 Fourier representation

- 4.1 Sinusoidal signals and their properties
- 4.2 Representation of continuous-time signals
- 4.3 Representation of discrete-time signals
- 4.4 Summary of Fourier series and transform
- 4.5 Properties of the discrete-time Fourier transform

Continuous-time sinusoids

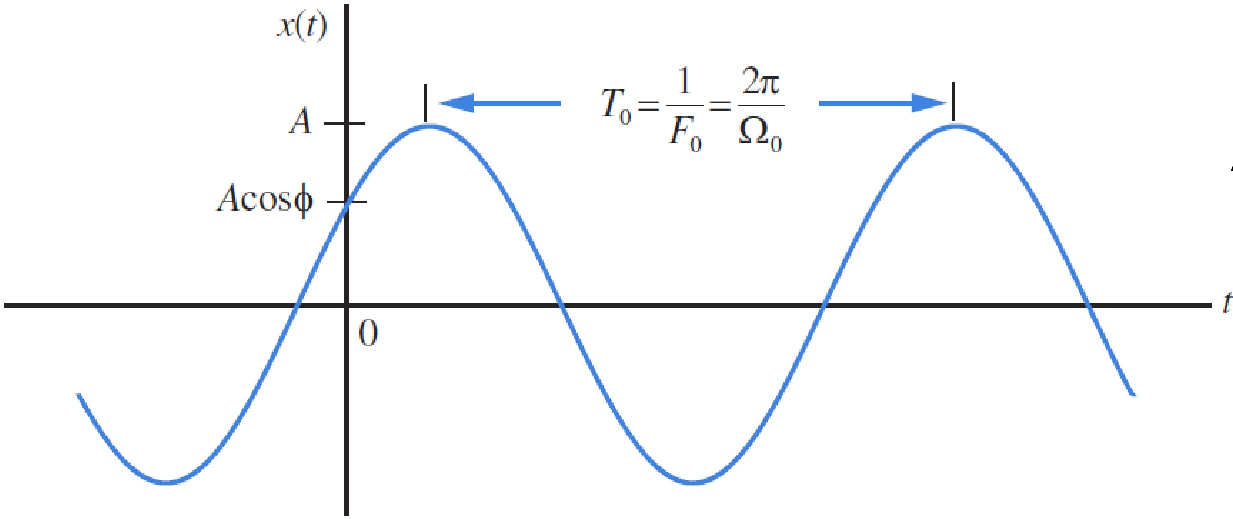
$$x(t) = A\cos(2\pi F_0 t + \theta)$$

$$T_0 = 1/F_0$$

Period

$$\Omega_0 = 2\pi F_0$$

Angular frequency



$$A\cos(\Omega_0 t + \theta) = \frac{A}{2}e^{j\theta}e^{j\Omega_0 t} + \frac{A}{2}e^{-j\theta}e^{-j\Omega_0 t}$$

Negative frequency

Harmonically related signals

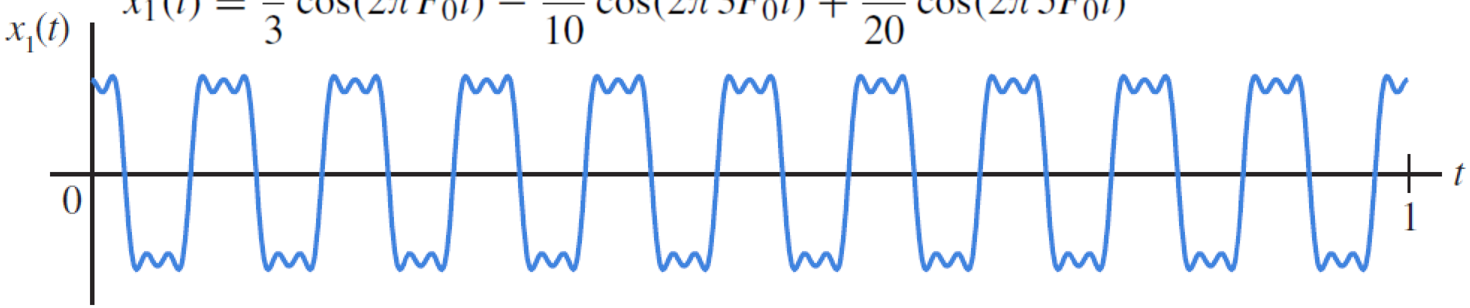
$$\Omega_0 = 2\pi / T_0 = 2\pi F_0$$ Fundamental frequency

$$s_k(t) = e^{jk\Omega_0 t} = e^{j2\pi k F_0 t}, \quad k = 0, \pm 1, \pm 2, \ldots$$

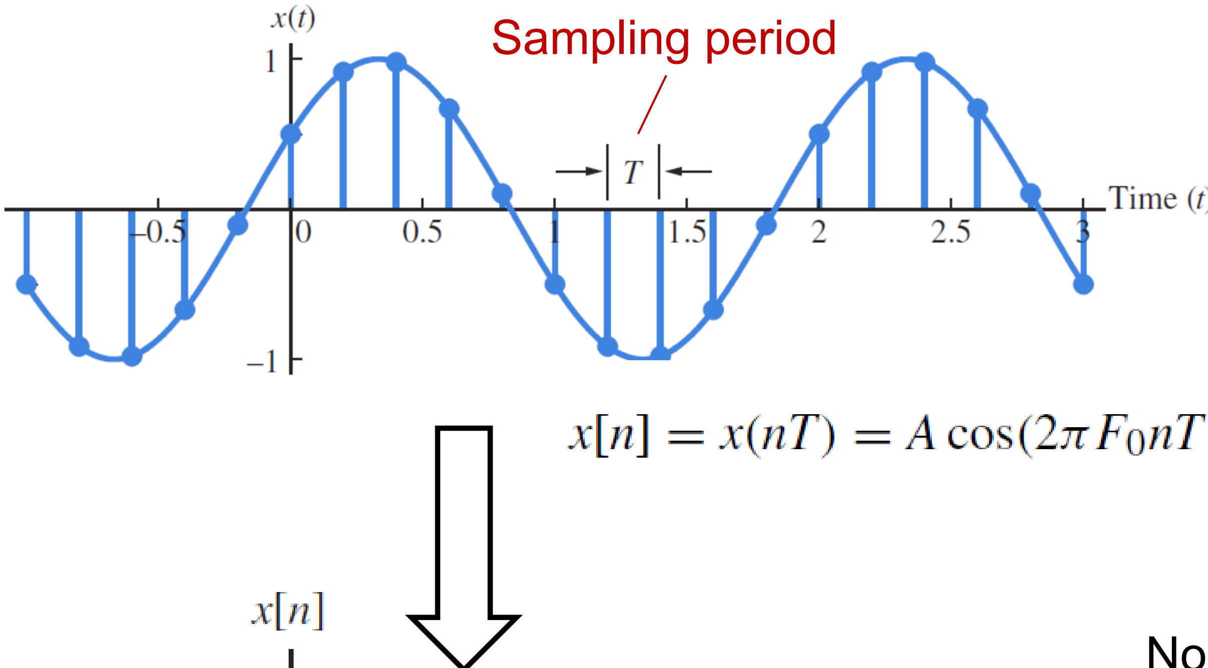
$$\int_{T_0} s_k(t) s_m^*(t) dt = \int_{T_0} e^{jk\Omega_0 t} e^{-jm\Omega_0 t} dt = \begin{cases} T_0, & k = m \\ 0, & k \neq m \end{cases}$$

Orthogonality property

$$x_1(t) = \frac{1}{3}\cos(2\pi F_0 t) - \frac{1}{10}\cos(2\pi 3 F_0 t) + \frac{1}{20}\cos(2\pi 5 F_0 t)$$



Discrete-time sinusoids



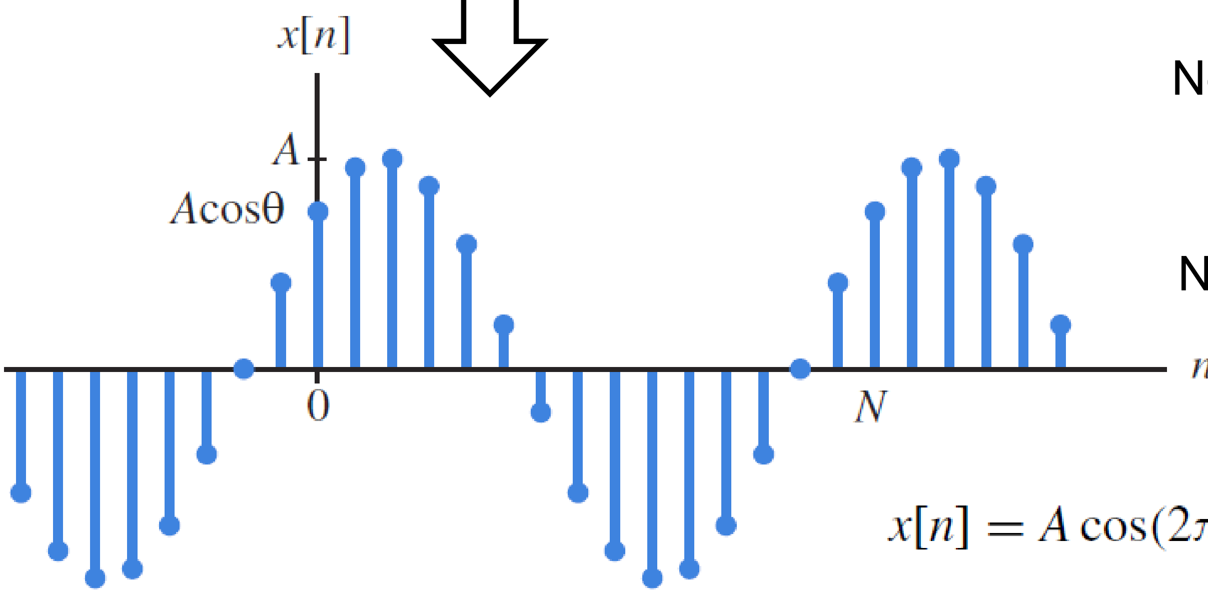
$$x[n] = x(nT) = A\cos(2\pi F_0 nT + \theta) = A\cos\left(2\pi \frac{F_0}{F_s} n + \theta\right)$$

$$f \triangleq \frac{F}{F_s} = FT$$

Normalized/relative frequency

$$\omega \triangleq 2\pi f = 2\pi \frac{F}{F_s} = \Omega T$$

Normalized angular frequency



$$x[n] = A\cos(2\pi f_0 n + \theta) = A\cos(\omega_0 n + \theta)$$

Periodicity

LTI system $x[n] = e^{j\omega n} \overset{\mathcal{H}}{\longmapsto} y[n] = H(e^{j\omega})e^{j\omega n}$ $z = e^{j\omega}$

Periodicity in time

$$x[n+N] = A\cos(2\pi f_0 n + 2\pi f_0 N + \theta) = A\cos(2\pi f_0 n + \theta) = x[n]$$

$$\Leftrightarrow \quad 2\pi f_0 N = 2\pi k \qquad f_0 = \frac{F_0}{F_s} = \frac{k}{N} = \frac{1/T_0}{1/T} = \frac{T}{T_0}$$

Result 4.1.1 The sequence $x[n] = A\cos(2\pi f_0 n + \theta)$ is periodic if and only if $f_0 = k/N$, that is, f_0 is a rational number. If k and N are a pair of prime numbers, then N is the fundamental period of $x[n]$.

Periodicity in frequency

$$A\cos[(\omega_0 + k2\pi)n + \theta] = A\cos(\omega_0 n + kn2\pi + \theta) = A\cos(\omega_0 n + \theta)$$

Result 4.1.2 The sequence $x[n] = A\cos(\omega_0 n + \theta)$ is periodic in ω_0 with fundamental period 2π and periodic in f_0 with fundamental period one.

Harmonically related signals

$$s_k[n] = \mathrm{e}^{\mathrm{j}\omega_k n} = \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}kn} \qquad \omega_k = 2\pi k/N$$

$$s_k[n+N] = s_k[n], \qquad \text{(periodic in time)}$$

$$s_{k+N}[n] = s_k[n]. \qquad \text{(periodic in frequency)}$$

$$\sum_{n=\langle N\rangle} s_k[n]s_m^*[n] = \sum_{n=\langle N\rangle} \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}kn}\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}mn} = \begin{cases} N, & k=m \\ 0, & k\neq m \end{cases}$$

Orthogonality property

Continuous-time Fourier series (CTFS)

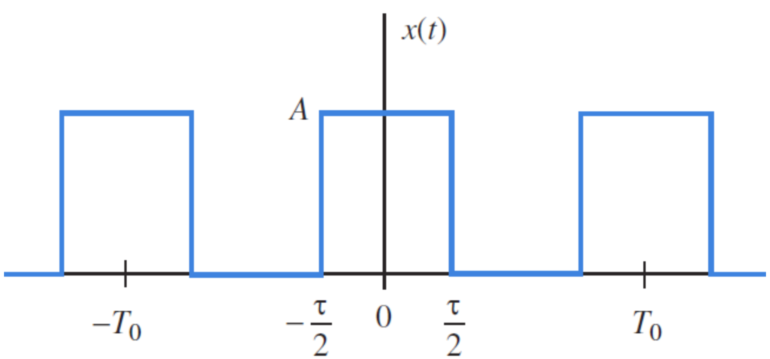
Fourier Synthesis Equation

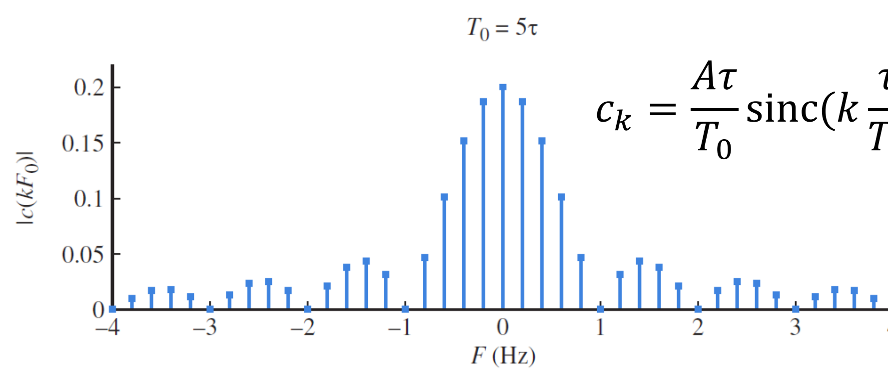
Fourier Analysis Equation

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\Omega_0 t} \quad \overset{\text{CTFS}}{\longleftrightarrow} \quad c_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\Omega_0 t} dt.$$

Continuous-time signals with period $T_0 = 2\pi/\Omega_0$

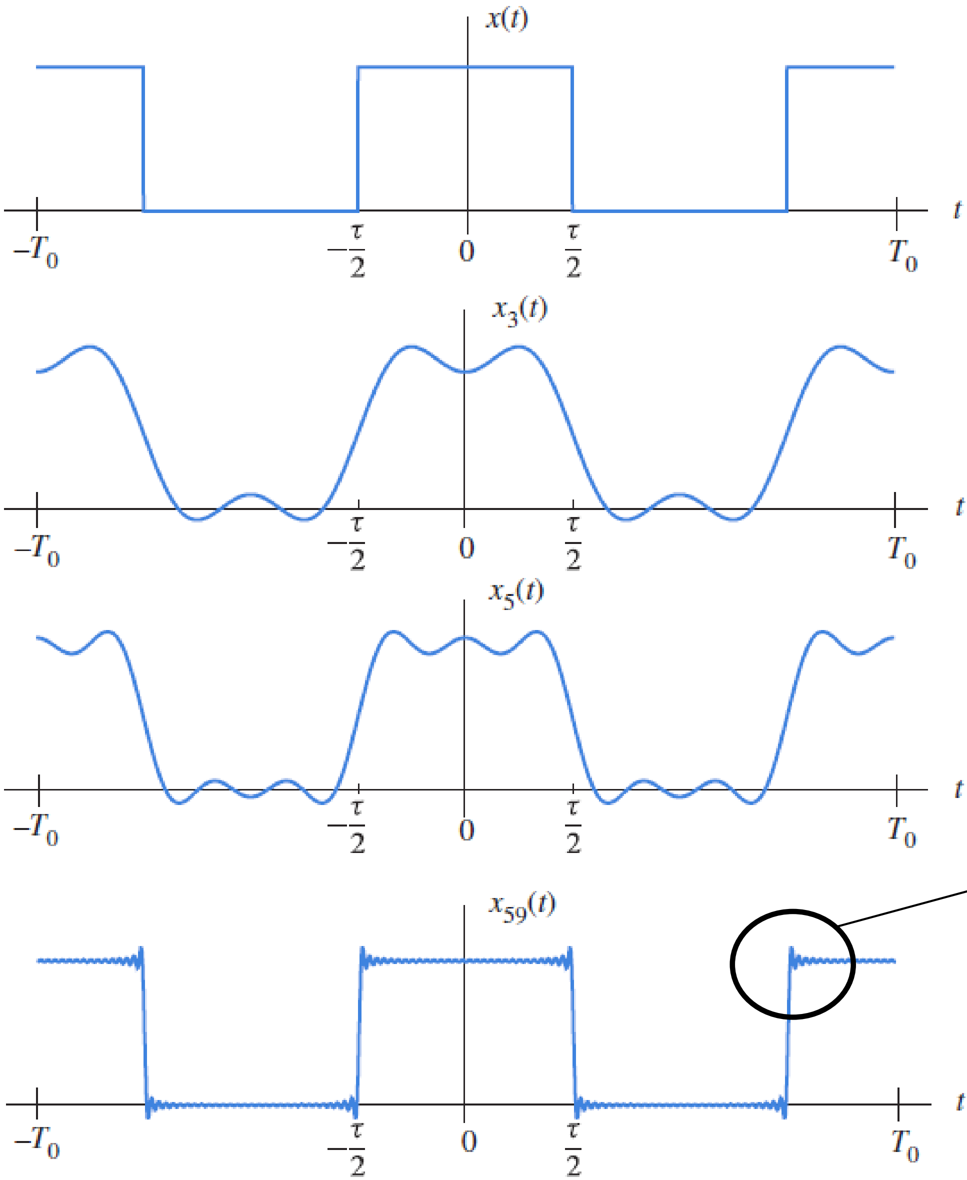
Discrete Fourier series coefficients



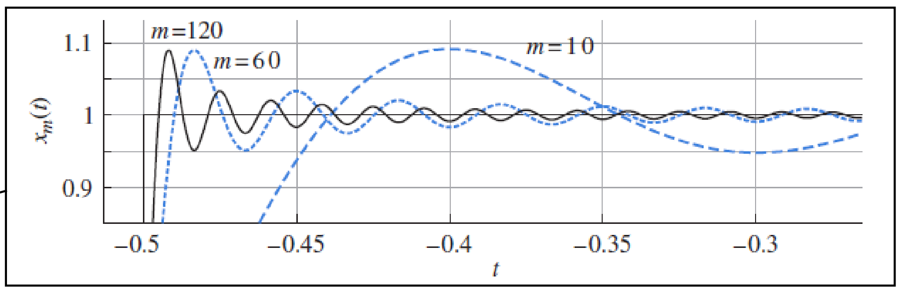


$$c_k = \frac{A\tau}{T_0}\text{sinc}(k\frac{\tau}{T_0})$$

Gibbs phenomenon



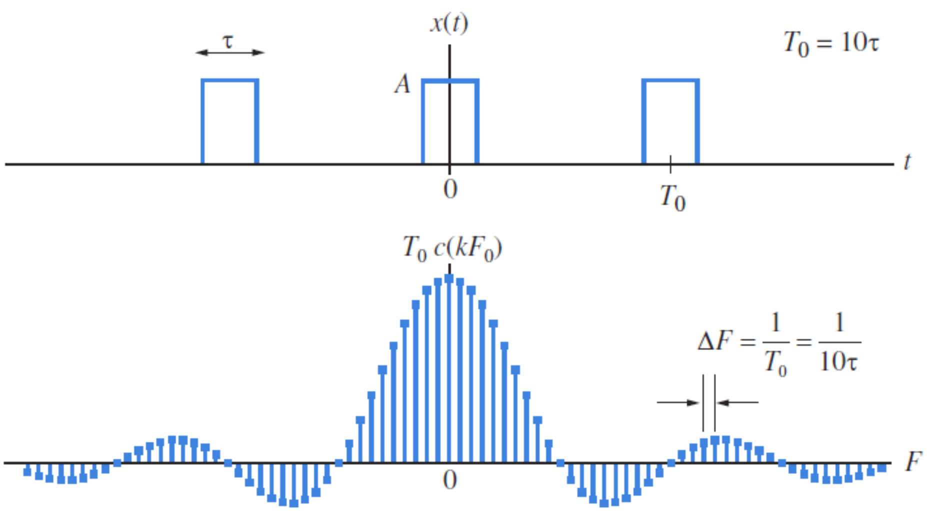
$$x_m(t) = \sum_{k=-m}^{m} c_k e^{jk\Omega_0 t}$$



Ripples' energy decreases, but their overshoot amplitudes do not.

From Fourier series to Fourier transform

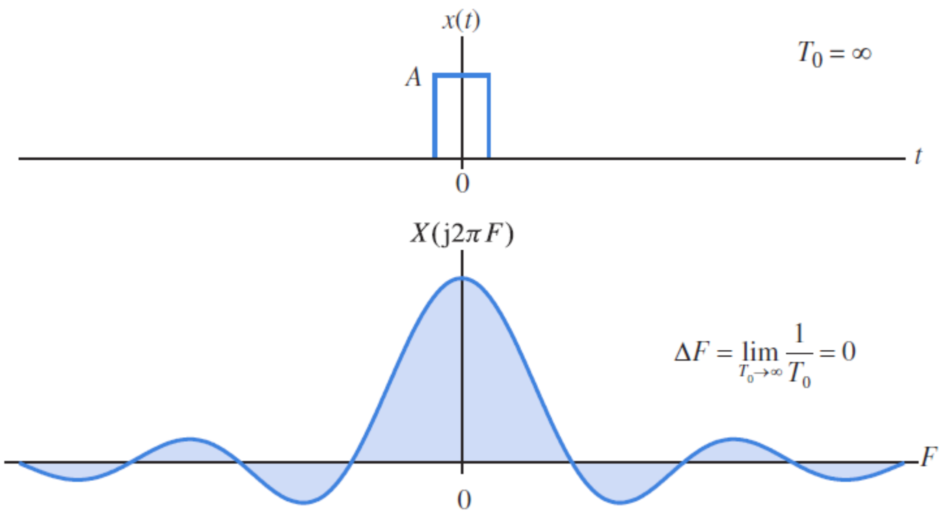
Continuous-time Fourier series



$$F = kF_0$$

$$F_0 = 1/T_0$$

Continuous-time Fourier transform



Continuous-time Fourier transform (CTFT)

Fourier Synthesis Equation **Fourier Analysis Equation**

$$x(t) = \int_{-\infty}^{\infty} X(j2\pi F)\, e^{j2\pi F t} dF \xleftrightarrow{\text{CTFT}} X(j2\pi F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi F t} dt,$$

Continuous-time aperiodic signals

Continuous transform spectrum

$$x(t) = \mathcal{F}^{-1}\{X(j2\pi F)\} \xleftrightarrow{\text{CTFT}} X(j2\pi F) = \mathcal{F}\{x(t)\}$$

Parseval's relation

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(j2\pi F)|^2 dF$$

Example of causal exponential signal

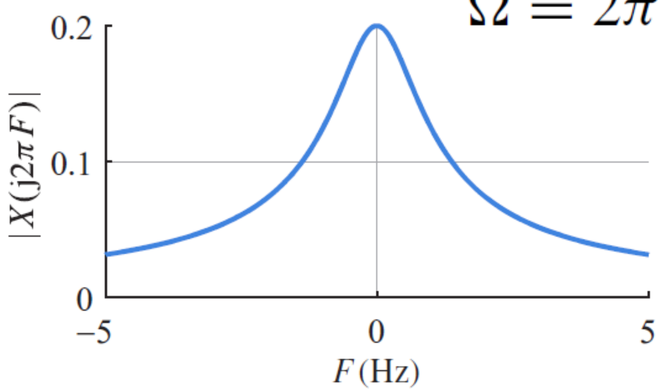
$$x(t) = \begin{cases} e^{-at}, & t > 0 \\ 0. & t < 0 \end{cases} \quad \overset{\mathcal{F}}{\Longrightarrow} \quad X(j2\pi F) = \frac{1}{a + j2\pi F} \quad \text{or}$$

$$X(j\Omega) = \frac{1}{a + j\Omega}. \quad a > 0$$

$$\Omega = 2\pi F$$

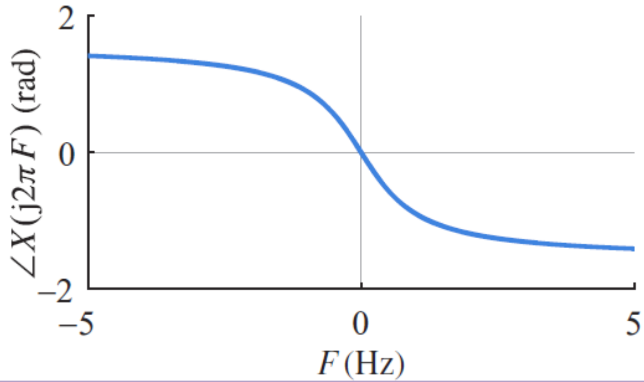
$$|X(j2\pi F)| = \frac{1}{\sqrt{a^2 + (2\pi F)^2}}$$

Magnitude

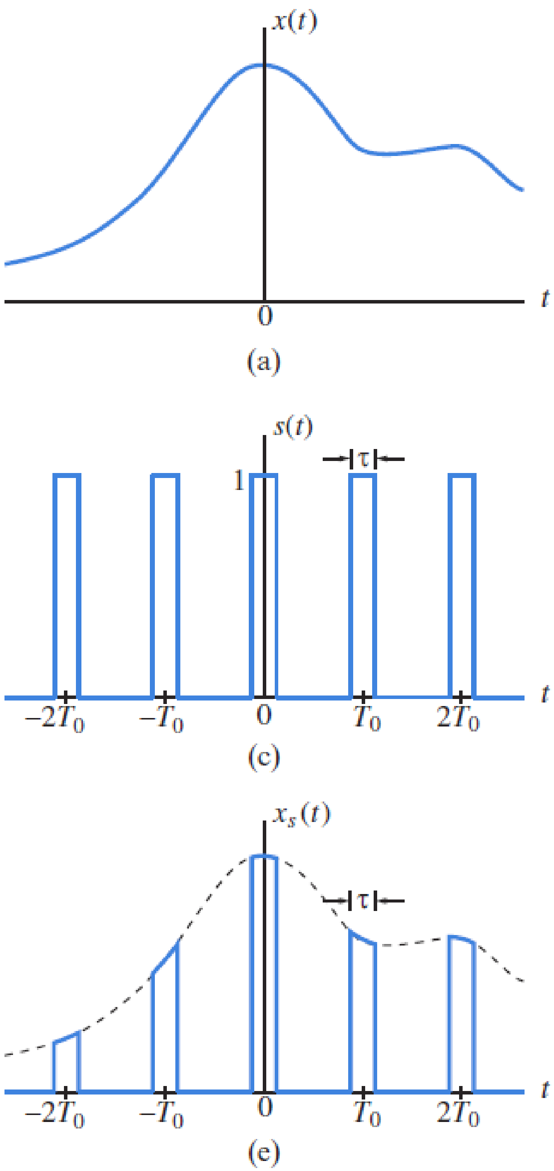


$$\angle X(j2\pi F) = -\tan^{-1}\left(2\pi \frac{F}{a}\right)$$

Phase



Sampling an aperiodic signal with a periodic one



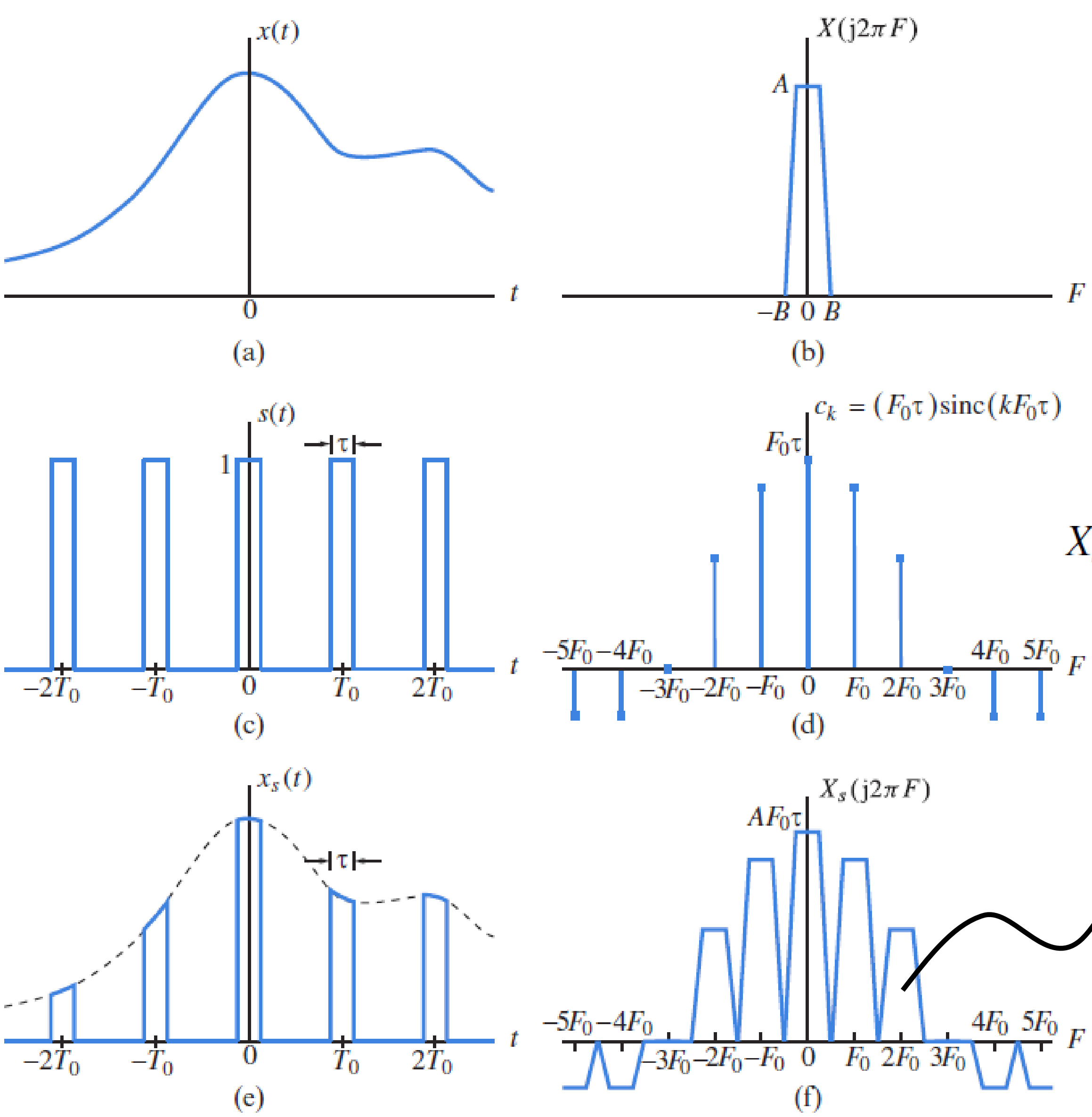
$$x_s(t) = x(t)s(t)$$

$$X_s(j2\pi F) = \int_{-\infty}^{\infty} x(t)\left[\sum_{k=-\infty}^{\infty} c_k e^{j2\pi F_0 kt}\right] e^{-j2\pi Ft} dt$$

$$= \sum_{k=-\infty}^{\infty} c_k \left[\int_{-\infty}^{\infty} x(t) e^{-j2\pi(F-kF_0)t} dt\right]$$

$$= \sum_{k=-\infty}^{\infty} c_k X[j2\pi(F-kF_0)].$$

Sampling an aperiodic signal with a periodic one



$$X_s(j2\pi F) = \sum_{k=-\infty}^{\infty} c_k X[j2\pi(F - kF_0)]$$

Information carried by k-th sinusoidal modulation

$x(t)$ can be perfectly recovered from $x_s(t)$ if

$$\begin{cases} X(j2\pi F) = 0 \text{ for } |F| > B; \\ F_0 > 2B. \end{cases}$$

Discrete-time Fourier series (DTFS)

Fourier Synthesis Equation

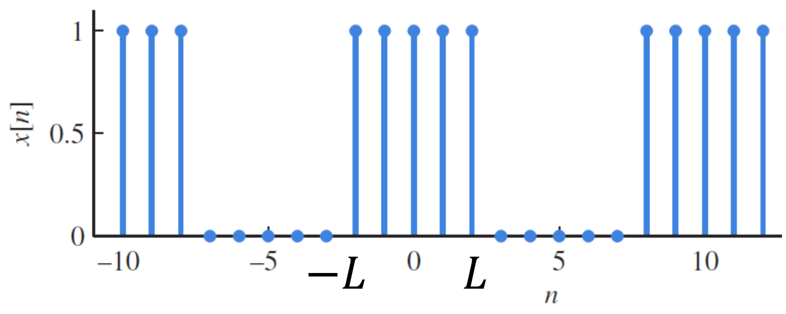
Fourier Analysis Equation

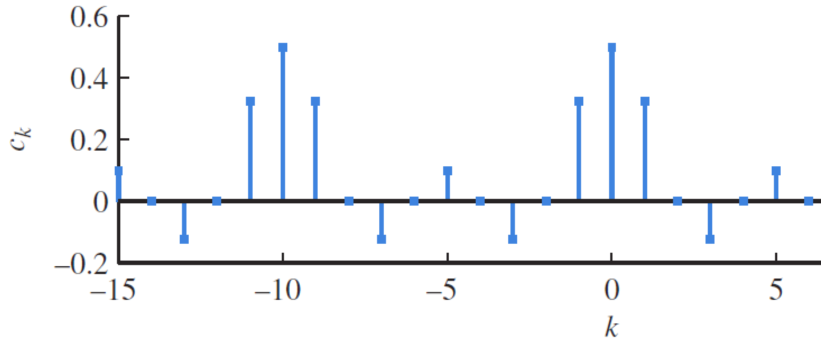
$$x[n] = \sum_{k=0}^{N-1} c_k e^{j\frac{2\pi}{N}kn} \quad \overset{\text{DTFS}}{\longleftrightarrow} \quad c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}.$$

Discrete-time signals with period N

Discrete Fourier series coefficients with period N

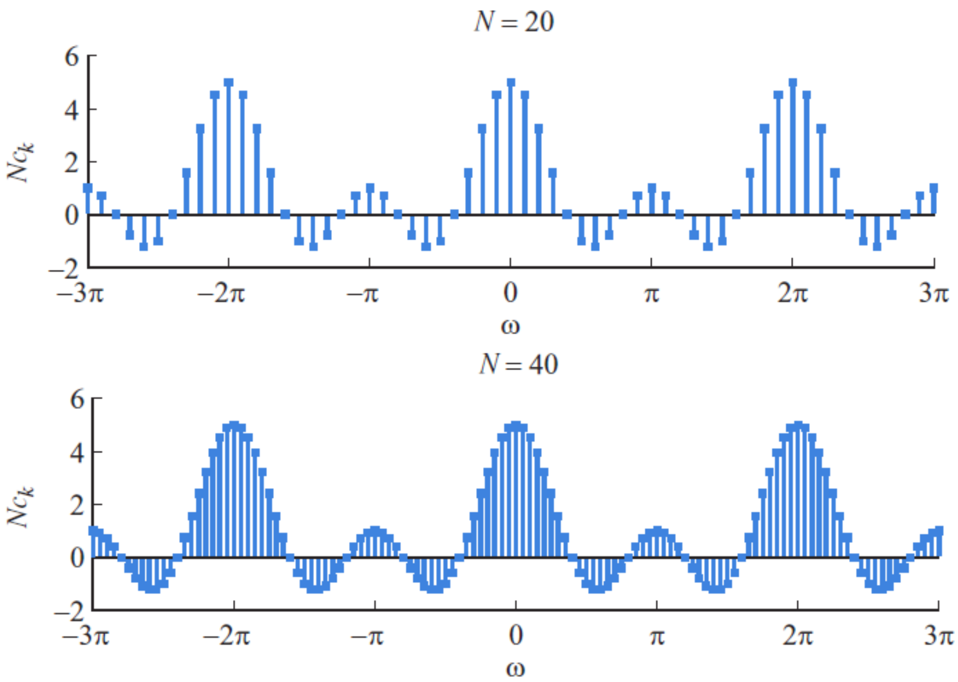
$$c_k = \frac{1}{N}\frac{\sin\left[\frac{2\pi}{N}k\left(L+\frac{1}{2}\right)\right]}{\sin\left(\frac{2\pi}{N}k\frac{1}{2}\right)}$$





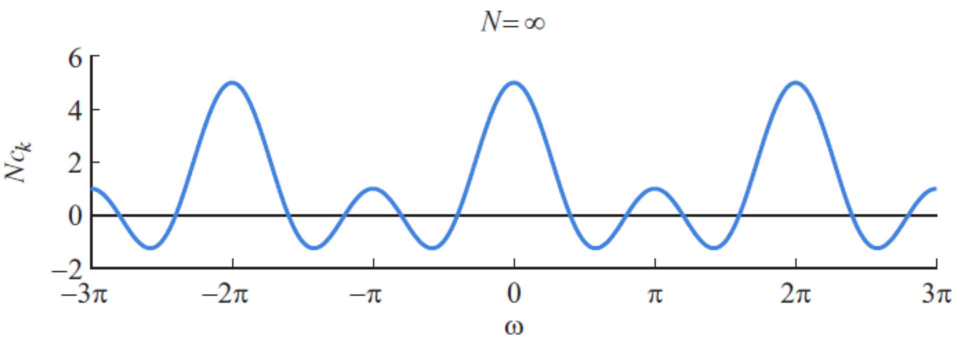
From Fourier series to Fourier transform

Discrete-time Fourier series



$$\omega_k = (2\pi/N)k$$

Discrete-time Fourier transform



Discrete-time Fourier transform (DTFT)

Fourier Synthesis Equation **Fourier Analysis Equation**

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega \xleftrightarrow{\text{DTFT}} X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}.$$

Discrete-time aperiodic signals

Continuous transform spectrum

Parseval's relation

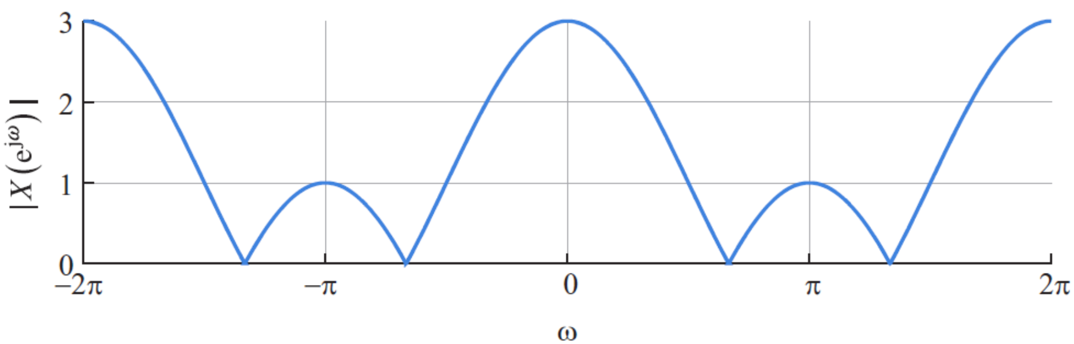
$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

Example of finite length pulse

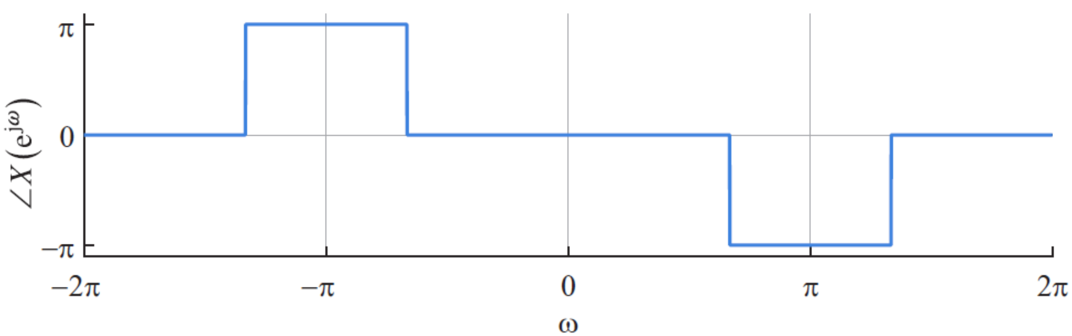
$$x[n] = \delta[n+1] + \delta[n] + \delta[n-1]$$

$$X(e^{j\omega}) = \sum_{n=-1}^{1} x[n]e^{-j\omega n} = e^{j\omega} + 1 + e^{-j\omega} = 1 + 2\cos(\omega)$$

$$|X(e^{j\omega})| = |1 + 2\cos(\omega)|$$

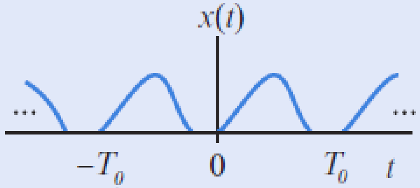
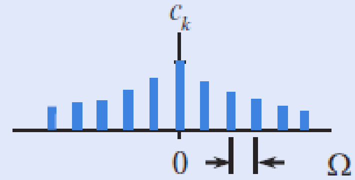
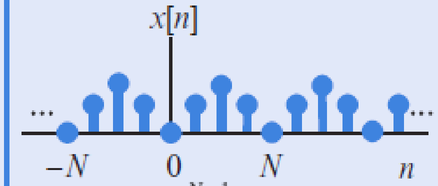
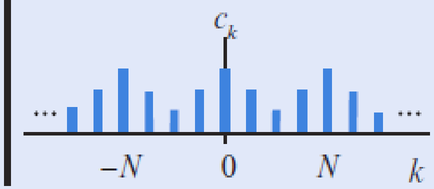
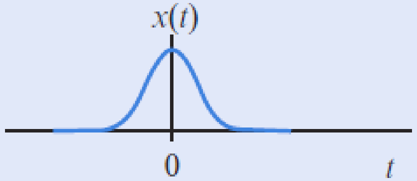
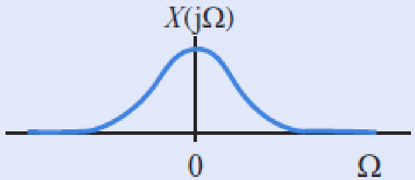
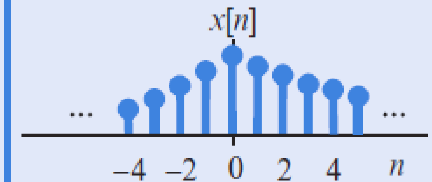
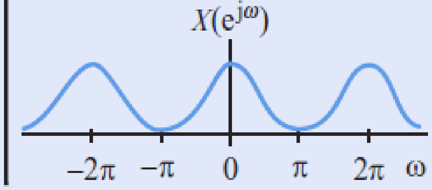


$$\angle X(e^{j\omega}) = \begin{cases} 0, & X(e^{j\omega}) > 0 \\ \pi. & X(e^{j\omega}) < 0 \end{cases}$$



Summary of Fourier series and transforms

Computable signals

		Continuous-time signals: Time-domain	Continuous-time signals: Frequency-domain	Discrete-time signals: Time-domain	Discrete-time signals: Frequency-domain
Periodic signals	Fourier series	 $c_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\Omega_0 t}dt$ CTFS →	 $\Omega_0 = \frac{2\pi}{T_0}$ ← ICTFS $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\Omega_0 t}$	 $c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$ DTFS →	 ← IDTFS $x[n] = \sum_{k=0}^{N-1} c_k e^{j\frac{2\pi}{N}kn}$
		Continuous and periodic	Discrete and aperiodic	Discrete and periodic	Discrete and periodic
Aperiodic signals	Fourier transforms	 $X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$ CTFT →	 ← ICTFT $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}d\Omega$	 $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ DTFT →	 ← IDTFT $x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$
		Continuous and aperiodic	Continous and aperiodic	Discrete and aperiodic	Continous and periodic

Real-world signals

Discretely sampled signals

The z-transform vs. DTFT

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \qquad X(z)|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = X(e^{j\omega})$$

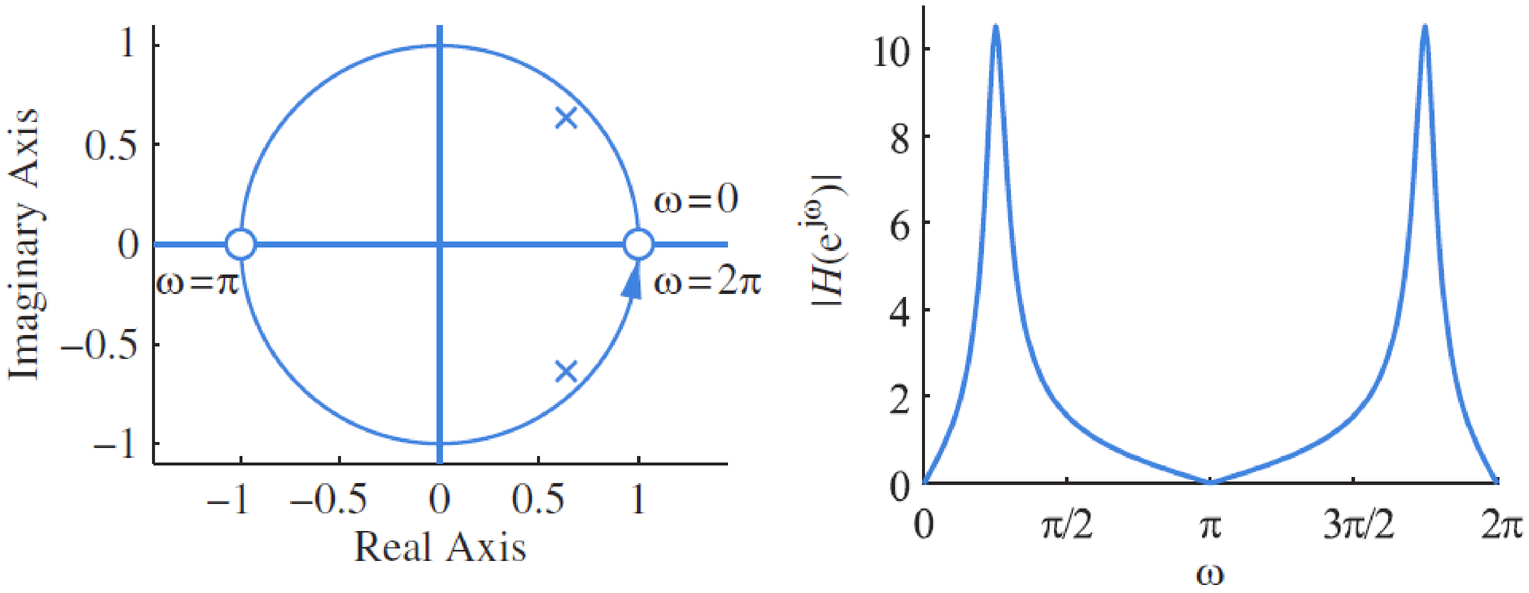


Figure 4.26 The relationship between the z-transform and the DTFT for a sequence with two complex-conjugate poles at $z = 0.9e^{j\pm\pi/4}$ and two zeros at $z = \pm 1$.

Symmetry properties of the DTFT

Sequence $x[n]$	Transform $X(e^{j\omega})$
	Complex signals
$x^*[n]$	$X^*(e^{-j\omega})$
$x^*[-n]$	$X^*(e^{j\omega})$
$x_R[n]$	$X_e(e^{j\omega}) \triangleq \frac{1}{2}\left[X(e^{j\omega}) + X^*(e^{-j\omega})\right]$
$jx_I[n]$	$X_o(e^{j\omega}) \triangleq \frac{1}{2}\left[X(e^{j\omega}) - X^*(e^{-j\omega})\right]$
$x_e[n] \triangleq \frac{1}{2}(x[n] + x^*[-n])$	$X_R(e^{j\omega})$
$x_o[n] \triangleq \frac{1}{2}(x[n] - x^*[-n])$	$jX_I(e^{j\omega})$

Sequence $x[n]$	Transform $X(e^{j\omega})$
	Real signals
Any real $x[n]$	$X(e^{j\omega}) = X^*(e^{-j\omega})$ $X_R(e^{j\omega}) = X_R(e^{-j\omega})$ $X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ $\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ $\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$
$x_e[n] = \frac{1}{2}(x[n] + x[-n])$ Even part of $x[n]$	$X_R(e^{j\omega})$ real part of $X(e^{j\omega})$ (even)
$x_o[n] = \frac{1}{2}(x[n] - x[-n])$ Odd part of $x[n]$	$jX_I(e^{j\omega})$ imaginary part of $X(e^{j\omega})$ (odd)

Example

$$x[n] = a^n u[n]$$
$$|a| < 1$$

$$X(e^{j\omega}) = \frac{1}{1 - a e^{-j\omega}}$$

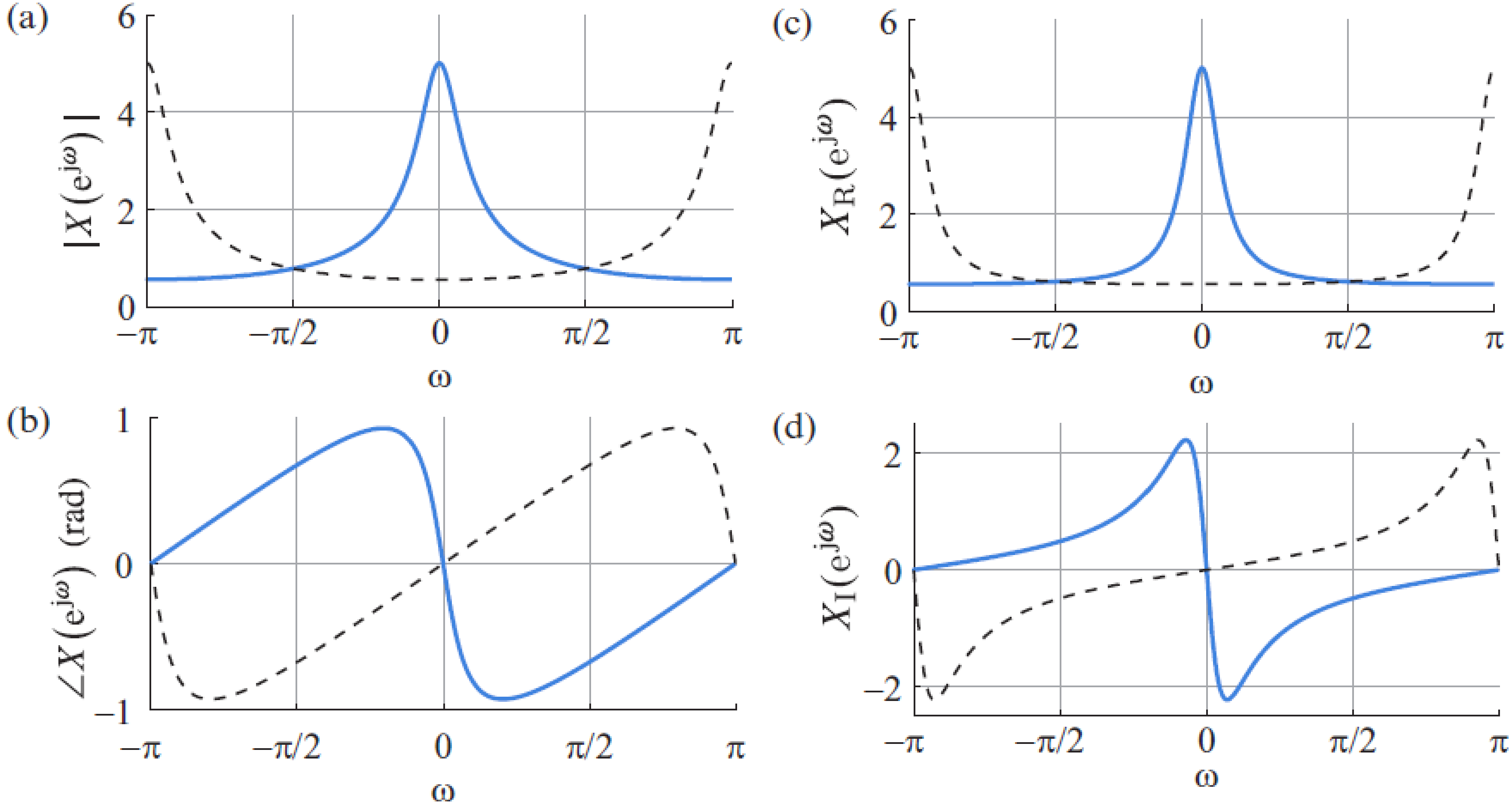
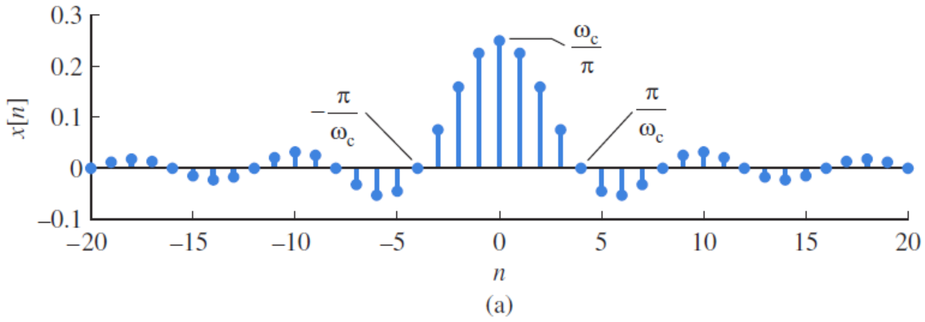


Figure 4.27 Plots of the magnitude (a), phase (b), real part (c), and imaginary part (d) of the DTFT for the sequence $x[n] = a^n u[n]$. The solid lines correspond to a lowpass sequence ($a = 0.8$) and the dashed lines to a highpass sequence ($a = -0.8$).

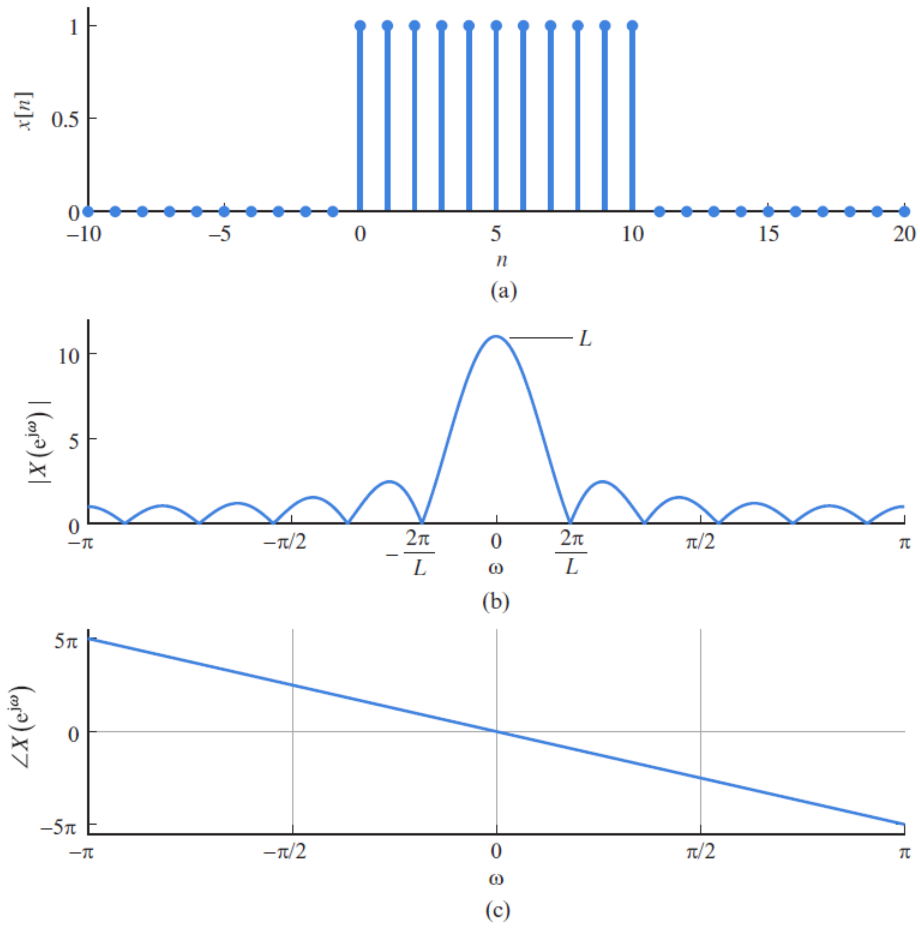
Example

$$x[n] = \begin{cases} A, & 0 \le n \le L-1 \\ 0. & \text{otherwise} \end{cases}$$



$$X(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0. & \omega_c < |\omega| < \pi \end{cases}$$

Ideal lowpass sequence



Rectangular pulse sequence