Chap3
The z-transform

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Chap 3 The z-transform

- 3.1 Motivation
- 3.2 The z-transform
- 3.3 The inverse z-transform
- 3.4 Properties of the z-transform
- 3.5 System function of LTI systems
- 3.6 LTI systems characterized by LCCDE

Eigenfunctions of LTI systems

Consider a complex exponential sequence as the system input:

$$x[n] = z^n, \quad \text{for all } n$$

The system response becomes:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] z^{n-k} = \left(\sum_{k=-\infty}^{\infty} h[k] z^{-k} \right) z^n, \quad \text{for all } n$$

Therefore, z^n is an eigenfunction of the system with an eigenvalue $H(z)$:

$$H(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-k}$$

System or transfer function $H(z)$ is the z-transform of impulse response $h(n)$.

The z-transform is a powerful tool

- Helps to understand, analyze, and design LTI systems
- Provides insight of input/output signals

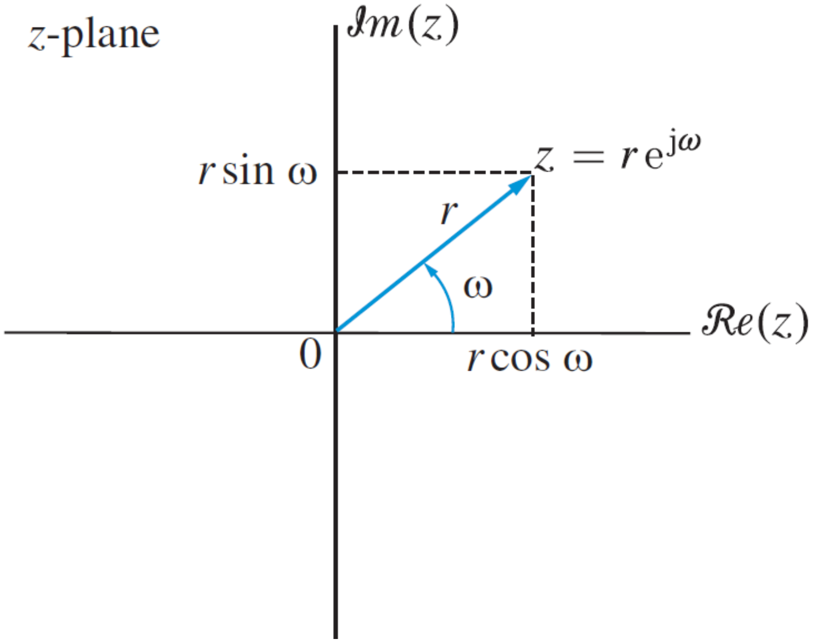
$$x[n] = \sum_k c_k z_k^n, \quad \text{for all } n$$

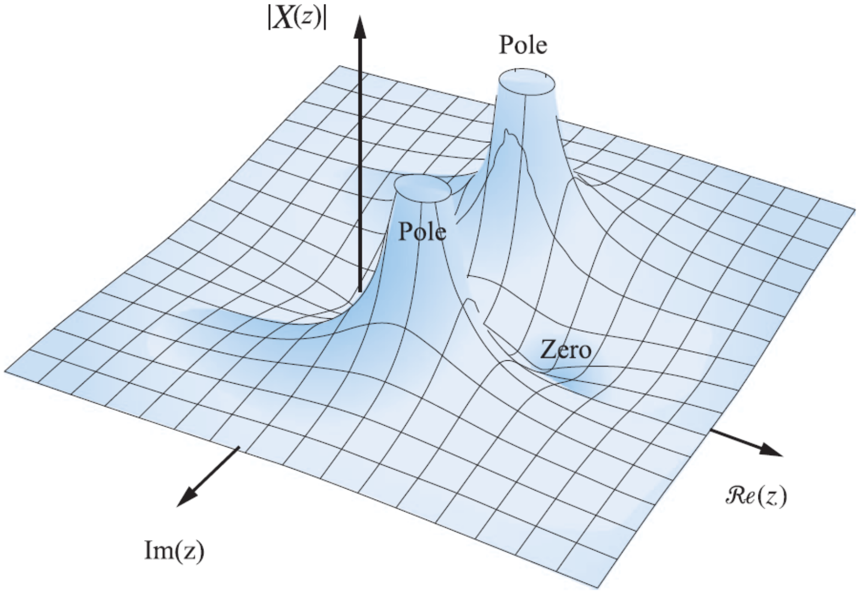
$$\Downarrow$$

$$y[n] = \sum_k c_k H(z_k) z_k^n, \quad \text{for all } n$$

The z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$





Examples

Impulse signal

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = z^0 = 1. \quad \text{ROC: All } z$$

ROC: region of convergence

$$x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

Square-pulse signal

$$\Downarrow$$

$$X(z) = \sum_{n=0}^{M} 1 z^{-n} = \frac{1 - z^{-(M+1)}}{1 - z^{-1}}. \quad \text{ROC: } z \neq 0$$

Causal exponential sequence

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}. \quad \text{ROC: } |z| > |a|$$

$$x[n] = a^n u[n]$$

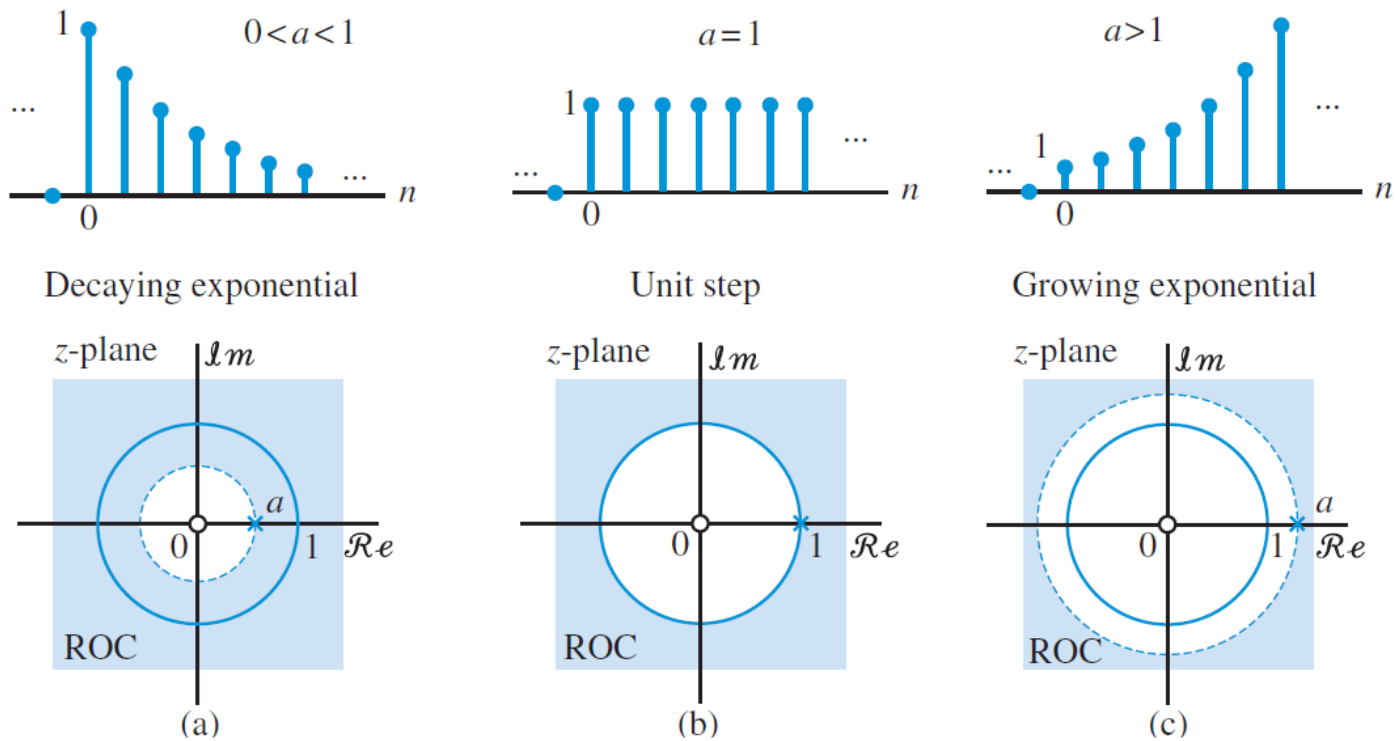


Figure 3.3 Pole-zero plot and region of convergence of a causal exponential sequence $x[n] = a^n u[n]$ with (a) decaying amplitude ($0 < a < 1$), (b) fixed amplitude (unit step sequence), and (c) growing amplitude ($a > 1$).

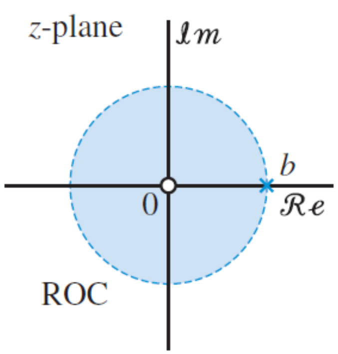
Anticausal exponential sequence

$$y[n] = -b^n u[-n-1] = \begin{cases} 0, & n \geq 0 \\ -b^n, & n < 0 \end{cases}$$

$$Y(z) = -\sum_{n=-\infty}^{-1} b^n z^{-n} = -b^{-1}z(1 + b^{-1}z + b^{-2}z^2 + \cdots)$$

$$Y(z) = \frac{-bz^{-1}}{1-b^{-1}z} = \frac{1}{1-bz^{-1}} = \frac{z}{z-b}. \quad \text{ROC: } |z| < |b|$$

The unique specification of a sequence requires both the z-transform and its ROC.



The inverse z-transform

Formal formula

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} \mathrm{d}z$$

Empirical way

$$x[n] = \sum_{k=1}^{N} A_k (p_k)^n \overset{\mathcal{Z}}{\longleftrightarrow} X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}}$$

Partial fraction expansion

+ ROC definition

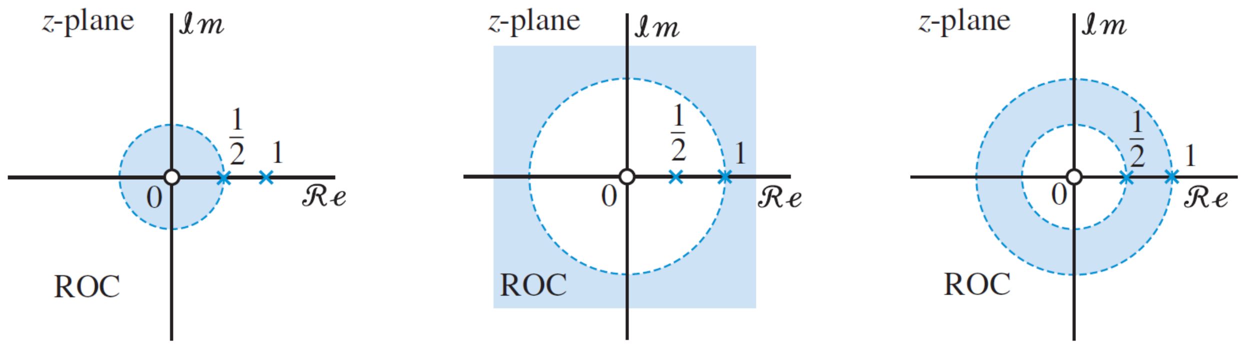
Example

$$z + 1 = A_1(z - 0.5) + A_2(z - 1)$$
$$\Rightarrow A_1 = 4, A_2 = -3$$

$$X(z) = \frac{1 + z^{-1}}{(1 - z^{-1})(1 - 0.5z^{-1})} = \frac{A_1}{1 - z^{-1}} + \frac{A_2}{1 - 0.5z^{-1}}$$

ROC: $|z| > 1$

$$x[n] = 4u[n] - 3\left(\frac{1}{2}\right)^n u[n]. \quad \text{(causal)}$$



ROC: $|z| < 0.5$

ROC: $0.5 < |z| < 1$

$$x[n] = -4u[-n-1] + 3\left(\frac{1}{2}\right)^n u[-n-1]. \quad \text{(anticausal)}$$

$$x[n] = -4u[-n-1] - 3\left(\frac{1}{2}\right)^n u[n]. \quad \text{(two-sided)}$$

Partial fraction expansion

$$X(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

$$\Downarrow \quad A_k = (1 - p_k z^{-1}) X(z)|_{z=p_k}$$

$$X(z) = \sum_{k=0}^{M-N} C_k z^{-k} + \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}}$$

Finite-length sequence Exponential sequence

Properties of the z-transform

	Property	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1.	Linearity	$a_1x_1[n] + a_2x_2[n]$	$a_1X_1(z) + a_2X_2(z)$	At least $R_{x_1} \bigcap R_{x_2}$
2.	Time shifting	$x[n-k]$	$z^{-k}X(z)$	R_x except $z = 0$ or ∞
3.	Scaling	$a^n x[n]$	$X(a^{-1}z)$	$\lvert a\rvert R_x$
4.	Differentation	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x
5.	Conjugation	$x^*[n]$	$X^*(z^*)$	R_x
6.	Real-part	$\text{Re}\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	At least R_x
7.	Imaginary part	$\text{Im}\{x[n]\}$	$\frac{1}{2}[X(z) - X^*(z^*)]$	At least R_x
8.	Folding	$x[-n]$	$X(1/z)$	$1/R_x$
9.	Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least $R_{x_1} \bigcap R_{x_2}$
10.	Initial-value theorem	$x[n] = 0$ for $n < 0$	$x[0] = \lim_{z\to\infty} X(z)$	

System function of LTI systems

ROC of Y(z) is the ROC overlap of H(z) and X(z).

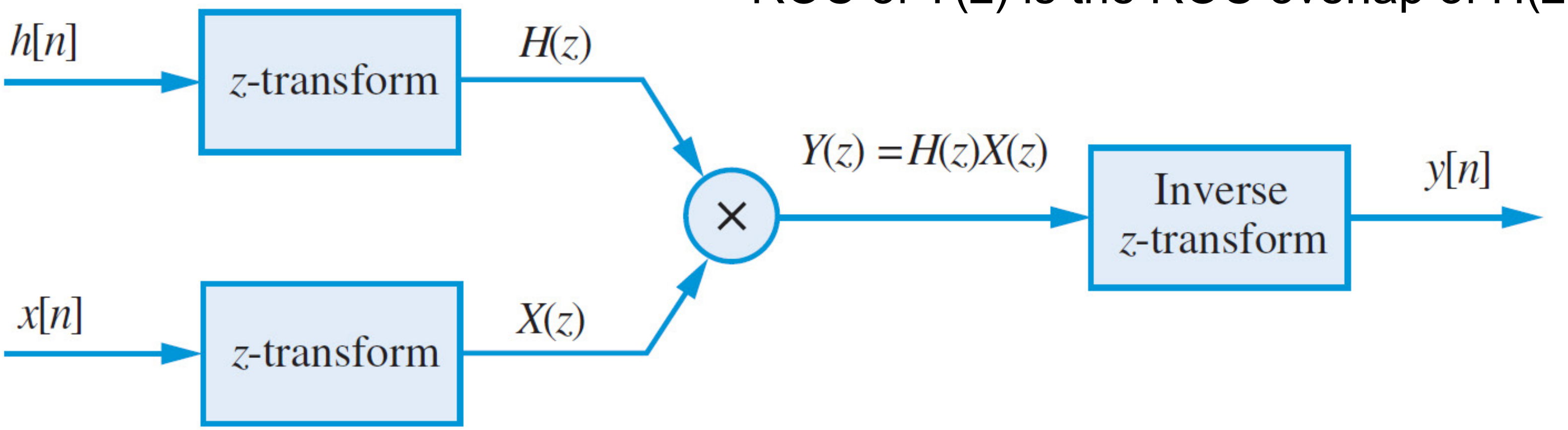


Figure 3.7 Procedure for the analytical computation of the output of an LTI system using the convolution theorem of the z-transform.

Causality and stability

Result 3.5.1 A system function $H(z)$ with the ROC that is the exterior of a circle, extending to infinity, is a necessary condition for a discrete-time LTI system to be causal but not a sufficient one.

$$\text{ROC: } |z| > r$$

Causality

Result 3.5.2 A LTI system is stable if and only if the ROC of the system function $H(z)$ includes the unit circle $|z| = 1$.

Stability

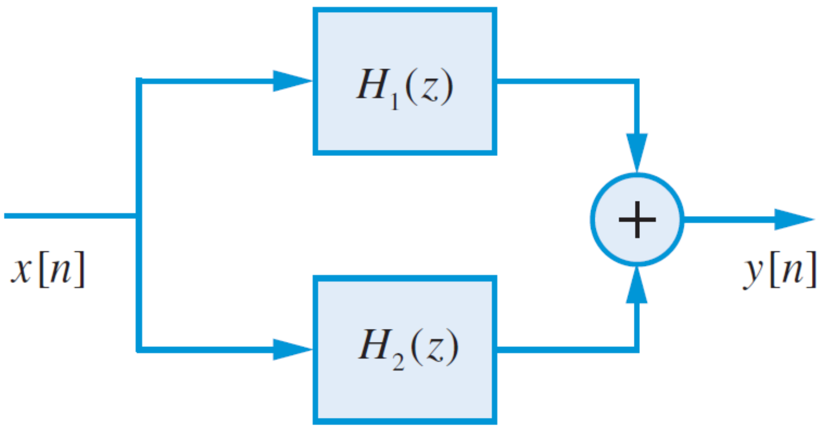
$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty \quad \Rightarrow |H(z)| \le \sum_{n=-\infty}^{\infty} |h[n]z^{-n}| < \infty \quad \text{for } |z| = 1$$

Result 3.5.3 An LTI system with rational $H(z)$ is both causal and stable if and only if all poles of $H(z)$ are *inside* the unit circle *and* its ROC is on the exterior of a circle, extending to infinity.

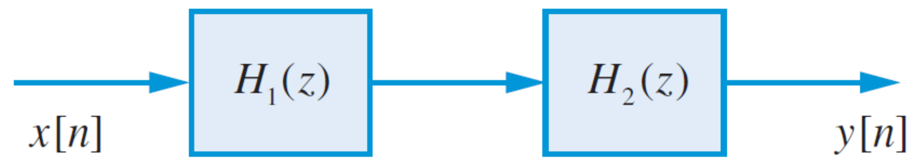
Causal and stable system

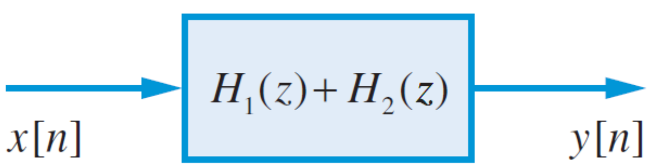
System function algebra

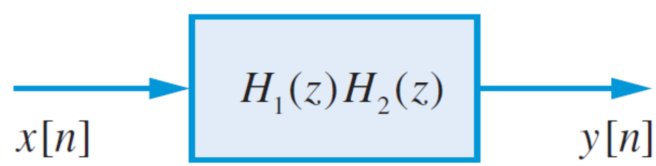
$$h[n] = h_1[n] + h_2[n]$$



$$h[n] = h_1[n] * h_2[n]$$







LTI systems characterized by LCCDE

Causal linear constant-coefficient difference equation:

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k] \Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

$$H(z) = \sum_{k=0}^{M-N} C_k z^{-k} + \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}}$$

$$h[n] = \sum_{k=0}^{M-N} C_k \delta[n-k] + \sum_{k=1}^{N} A_k (p_k)^n u[n]$$

Finite impulse response (FIR) if N=0

Infinite impulse response (IIR) if exists