Chap14
Random Signal Processing

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Chap 14 Random Signal Processing

- 14.1 Estimation of mean, variance, and covariance
- 14.2 Spectral analysis of stationary processes

Properties of estimators

Bias The bias of an estimator $\hat{\theta}$ of a parameter θ is defined by

$$B(\hat{\theta}) \triangleq \mathrm{E}(\hat{\theta}) - \theta$$

$$\hat{\theta} = \hat{\theta}(X_1, X_2, \dots, X_N)$$

A random variable

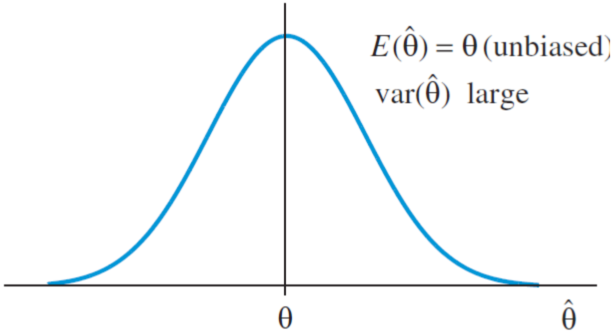
Variance

$$\mathrm{var}(\hat{\theta}) = \mathrm{E}\{[\hat{\theta} - \mathrm{E}(\hat{\theta})]^2\}$$

Mean square error

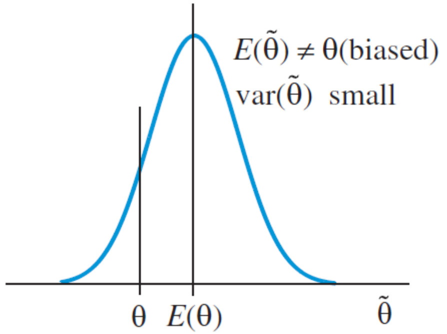
$$\mathrm{mse}(\hat{\theta}) = \mathrm{E}[(\hat{\theta} - \theta)^2]$$

$$\mathrm{mse}(\hat{\theta}) = \mathrm{var}(\hat{\theta}) + B^2(\hat{\theta})$$



Consistency

$$\hat{\theta} \to \theta \text{ as } N \to \infty$$



Sample mean

Sample mean
(consistent)

$$\hat{m} = \frac{1}{N}\sum_{k=1}^{N} x_k \qquad \hat{m} = \frac{1}{N}\sum_{k=1}^{N} x_k \triangleq \frac{1}{N}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{x}$$

$$u = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

Bias

$$\mathrm{E}(\hat{m}) = \frac{1}{N}\sum_{k=1}^{N} \mathrm{E}(x_k) = m$$

(unbiased)

Variance

$$\mathrm{var}(\hat{m}) = \frac{1}{N^2}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{u}$$

(if mutually uncorrelated)

$$\mathrm{var}(\hat{m}) = \frac{\sigma^2}{N}$$

(If positively/ negatively correlated)

$$\mathrm{var}(\hat{m}) \nearrow / \searrow$$

Sample variance

Sample variance
(consistent)

$$\hat{\sigma}^2 \triangleq \frac{1}{N}\sum_{k=1}^{N}(x_k - \hat{m})^2$$

Bias

$$E(\hat{\sigma}^2) = \frac{N-1}{N}\sigma^2$$

(biased/
asymptotically unbiased)

Variance

$$\text{var}(\hat{\sigma}^2) = \frac{N-1}{N^3}\left[(N-1)m_4 - (N-3)\sigma^4\right]$$

(for Gaussian RV) $$\text{var}(\hat{\sigma}^2) = \frac{2(N-1)}{N^2}\sigma^4$$

Sample covariance

Sample covariance
(consistent)

$$\hat{\sigma}_{xy} \triangleq \frac{1}{N} \sum_{k=1}^{N} (x_k - \hat{m}_x)(y_k - \hat{m}_y)$$

Sample correlation coefficient

$$\hat{\rho}_{xy} \triangleq \frac{\hat{\sigma}_{xy}}{\hat{\sigma}_x \hat{\sigma}_y} = \frac{\sum_{k=1}^{N} (x_k - \hat{m}_x)(y_k - \hat{m}_y)}{\sqrt{\sum_{k=1}^{N} (x_k - \hat{m}_x)^2 \sum_{k=1}^{N} (y_k - \hat{m}_y)^2}}$$

(the properties are more difficult to find)

PSD estimation
(only meaningful for stationary processes)

- Use of PSD estimation includes
  - Measure PSD of designed optimum filters
  - Detect narrowband signals in wideband noise
  - Estimate impulse response using white noise excitation

$$S_{yx}(\omega) = H(\mathrm{e}^{\mathrm{j}\omega})S_{xx}(\omega)$$

  - Extract physical information

Ergodicity

Ergodic

We can estimate ensemble average by time average of a single sequence. But the sequence has to be a representative one, e.g. not all zeros.

Mean-ergodic

$$\langle x[n] \rangle_N = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \xrightarrow[N \to \infty]{} m_x$$

$$\hat{m}[n] = \frac{1}{N} \sum_{k=1}^{N} x[n, \zeta_k]$$

(may be not practical)

correlation-ergodic

$$\langle x[n+\ell] x[n] \rangle_N = \frac{1}{N} \sum_{n=0}^{N-1} x[n+\ell] x[n] \xrightarrow[N \to \infty]{} r_x[\ell]$$

Estimation by time averages

Estimation of mean
(consistent)

$$\hat{m} = \frac{1}{N}\sum_{n=0}^{N-1} x[n] \qquad \mathrm{E}(\hat{m}) = \frac{1}{N}\sum_{n=0}^{N-1} \mathrm{E}(x[n]) = m$$

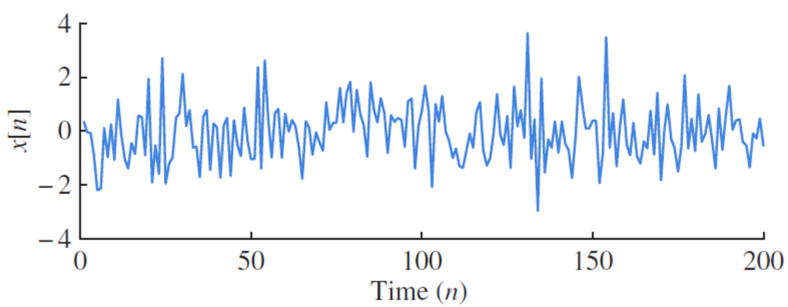
$$\mathrm{var}(\hat{m}) = \frac{\sigma^2}{N}\left[1 + 2\sum_{\ell=1}^{N-1}\left(1 - \frac{\ell}{N}\right)\rho[\ell]\right]$$

(smaller for negatively correlated, i.e. $\rho[l] < 0$ for $l \neq 0$)

Estimation of ACRS
(consistent if $l \ll N$)

$$\hat{r}[\ell] = \frac{1}{N}\sum_{n=0}^{N-\ell-1} x[n]x[n+\ell], \; 0 \le \ell \le N-1$$

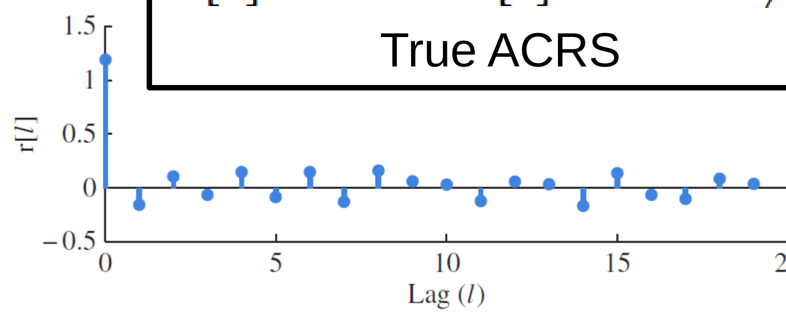
$$(x[0], x[\ell]), (x[1], x[\ell+1]), \ldots, (x[N-\ell-1], x[N-1])$$



A realization sequence for WGN

$c[0] = \sigma^2$ and $c[\ell] = 0$ for $\ell \neq 0$

True ACRS



Estimated ACRS

Periodogram

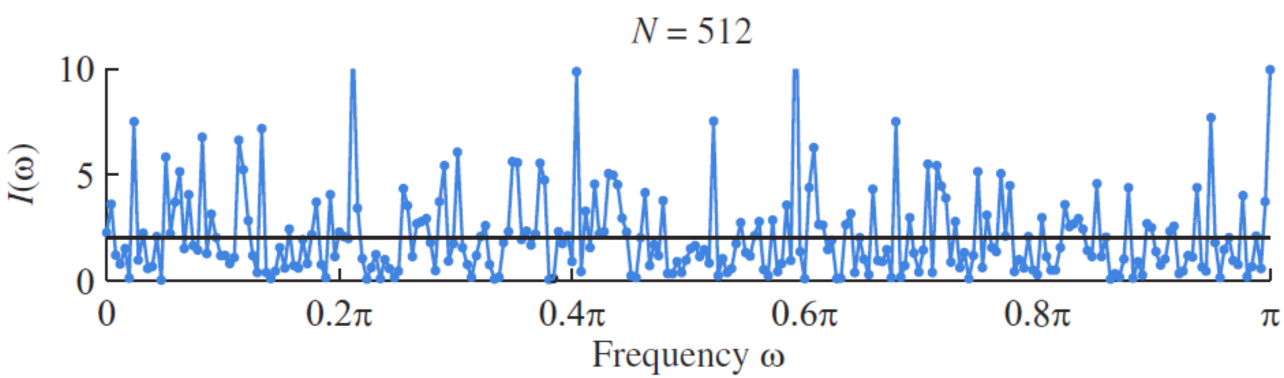
Periodogram
(PSD estimation)

$$I(\omega) \triangleq \sum_{\ell=-(N-1)}^{N-1} \hat{r}[\ell] e^{-j\omega\ell} = \mathcal{F}\{x[n] * x[-n]\}$$

$$I(\omega) = \frac{1}{N} \left| \sum_{n=0}^{N-1} x[n] e^{-j\omega n} \right|^2$$

Example of WGN

N	128	256	384	512
Mean of $I(\omega)$	2.3581	2.0370	2.1205	2.0170
Variance of $I(\omega)$	5.6406	4.1541	3.6799	4.5522



Variance doesn't decrease as N increases!

Properties of periodogram

Mean
(asymptotically unbiased)

$$\lim_{N\to\infty} \mathrm{E}[I(\omega)] = S(\omega)$$

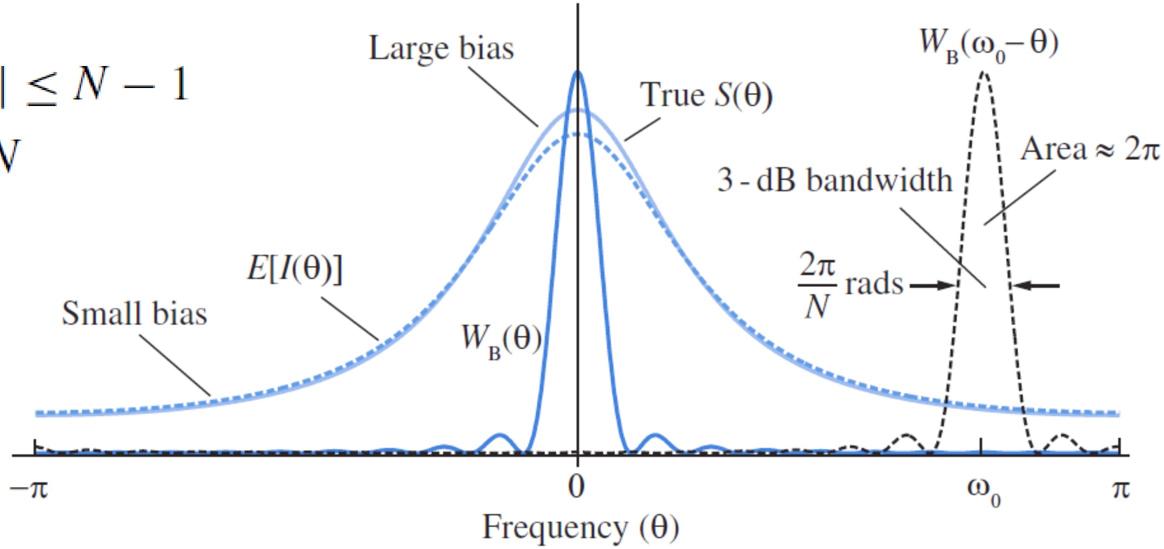
Variance
(holds for most processes)

$$\mathrm{var}[I(\omega)] \approx S^2(\omega), \quad 0 < \omega < \pi$$

Windowing effect

$$\mathrm{E}[I(\omega)] = \sum_{\ell=-(N-1)}^{N-1} w_B[\ell] r[\ell] \mathrm{e}^{-j\omega\ell}$$

$$w_B[\ell] \triangleq \begin{cases} 1 - \frac{|\ell|}{N}, & 0 \le |\ell| \le N-1 \\ 0, & |\ell| \ge N \end{cases}$$



Modified periodogram

Data window

$$\tilde{I}(\omega) \triangleq \frac{1}{N}\left|\sum_{n=0}^{N-1} \underline{w[n]}x[n]e^{-j\omega n}\right|^2$$

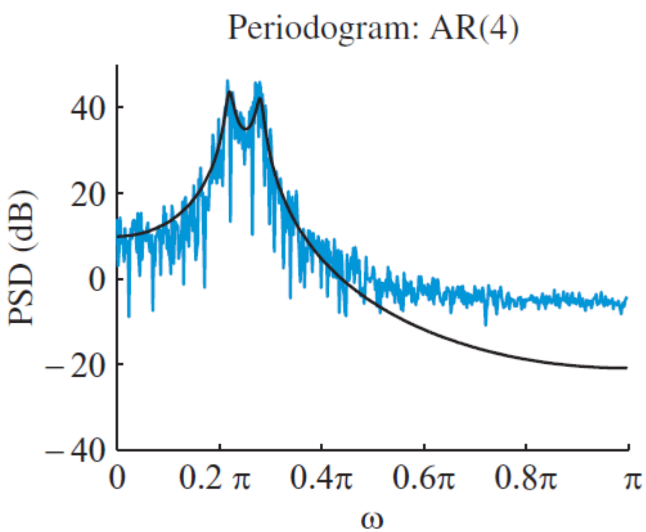
Correlation window

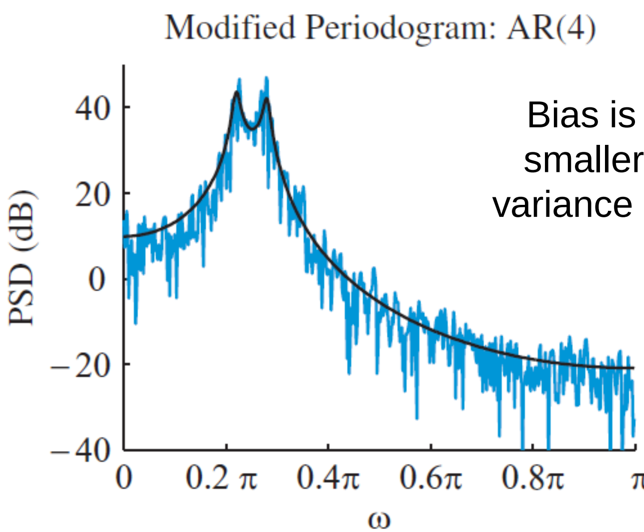
$$E[\tilde{I}(\omega)] = \sum_{\ell=-(N-1)}^{N-1} \underline{w_c[\ell]}r[\ell]e^{-j\omega\ell} = W_c(\omega) \otimes S(\omega)$$

makes larger lags (which are less accurate) weighted less

Condition for asymptotically unbiased

$$\sum_{n=0}^{N-1} w^2(n) = N$$





Bias is improved for smaller leakage; but variance is not reduced.

Methods for reducing variance (1/2)

Blackman-Tukey estimator
(shorter window)

$$\hat{S}_{\mathrm{BT}}(\omega) \triangleq \sum_{\ell=-(L-1)}^{L-1} w_c[\ell]\hat{r}_x[\ell]e^{-j\omega\ell}$$

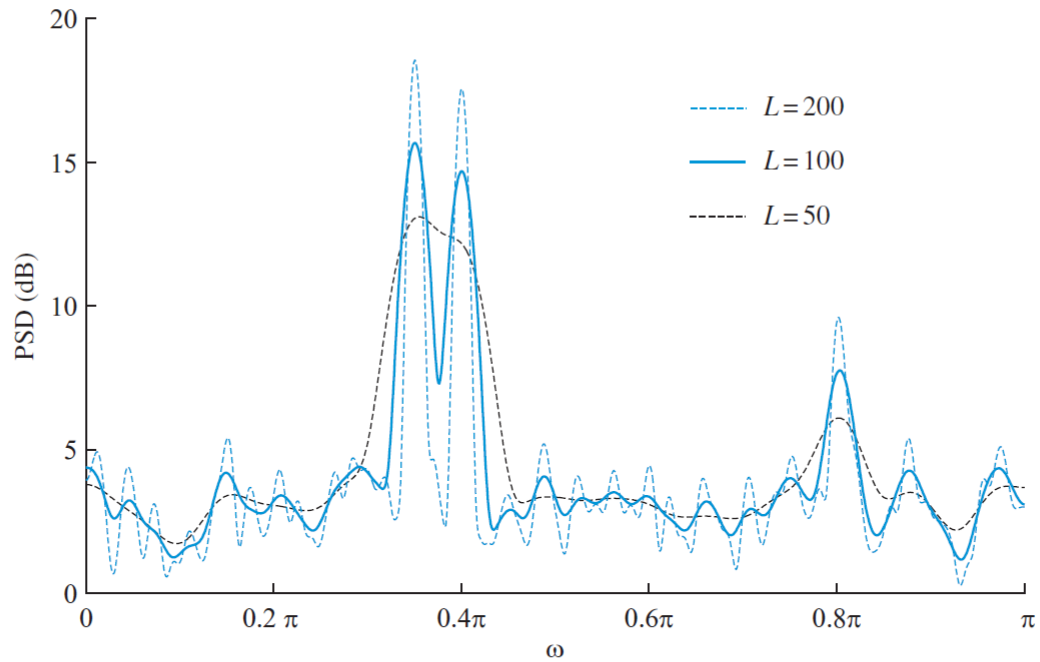
Reduced variance

$$\mathrm{var}[\hat{S}_{\mathrm{BT}}(\omega)] \simeq \left(\frac{1}{N}\sum_{\ell=-(L-1)}^{L-1} w_c^2[\ell]\right) S^2(\omega)$$

Bias-variance constraint

$$\text{Bias} \times \text{Variance} = \text{Constant}$$

Example
(3 sinusoids in WN)



Methods for reducing variance (2/2)

Welch estimator
(averaging multiple periodograms)

$$\tilde{I}_m(\omega) \triangleq \frac{1}{L}\left|\sum_{n=0}^{L-1} w[n]x_m[n]e^{-j\omega n}\right|^2$$

$$\hat{S}_W(\omega) = \frac{1}{M}\sum_{m=1}^{M} \tilde{I}_m(\omega)$$

Example
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