Chap13
Random Signals

Chao-Tsung Huang

National Tsing Hua University
Department of Electrical Engineering

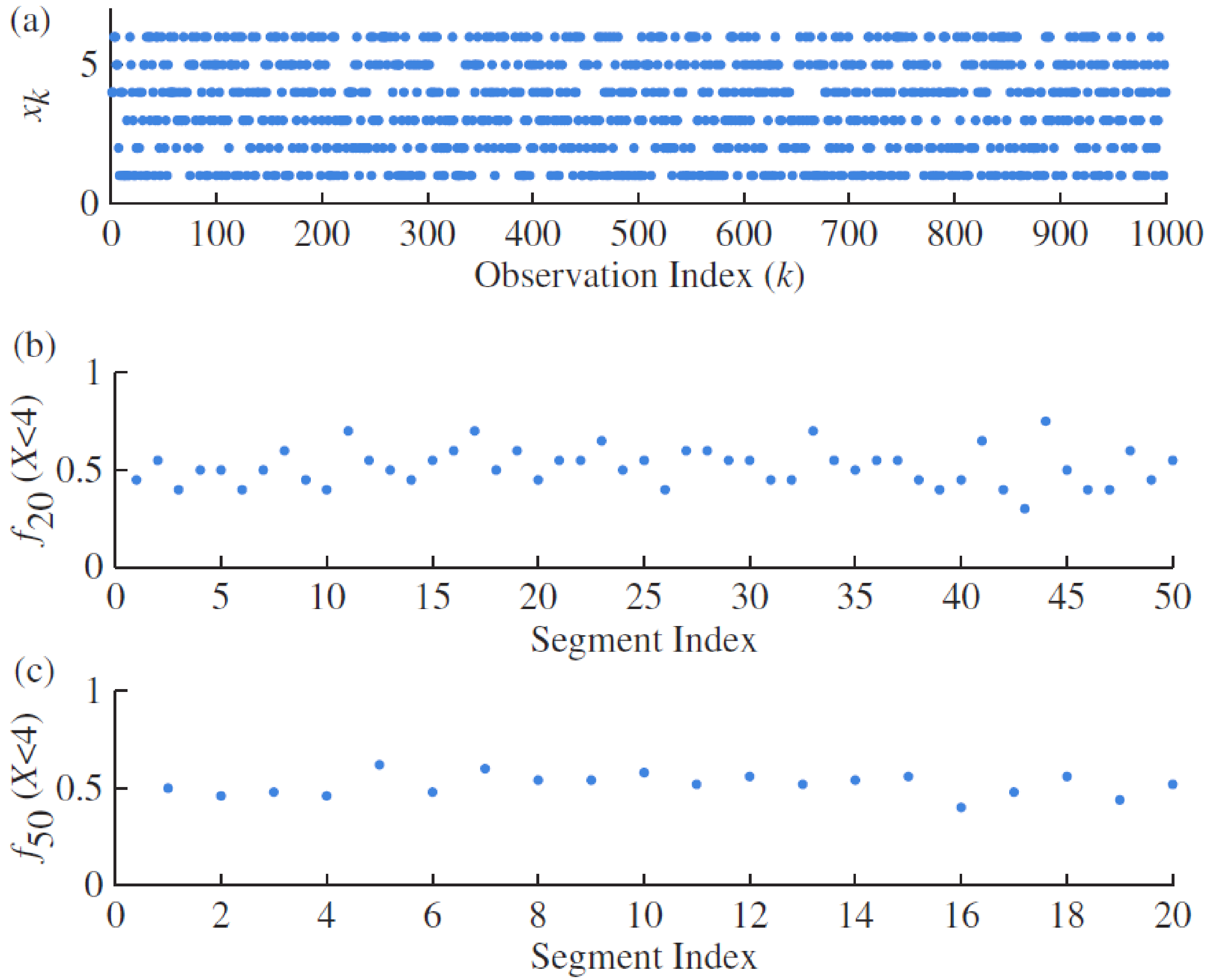
Chap 13 Random Signals

- 13.1 Probability models and random variables
- 13.2 Jointly distributed random variables
- 13.4 Random processes

Randomness and statistical regularity

Randomness
(statistical variability)

Outcome (realization): $x_k\ (\zeta)$
Sample space: S



Statistical regularity

Relative frequency

$$f_N(A) \triangleq \frac{N(A)}{N} = \frac{\text{Number of occurrences of event } A}{\text{Total number of trials}}$$

Random variables

Random variable

A function from S to real numbers

$$S = \{HHH, HHT, HTT, HTH, TTT, TTH, THH, THT\}.$$

$$\downarrow$$

$$X = \{0, 1, 2, 3\}$$

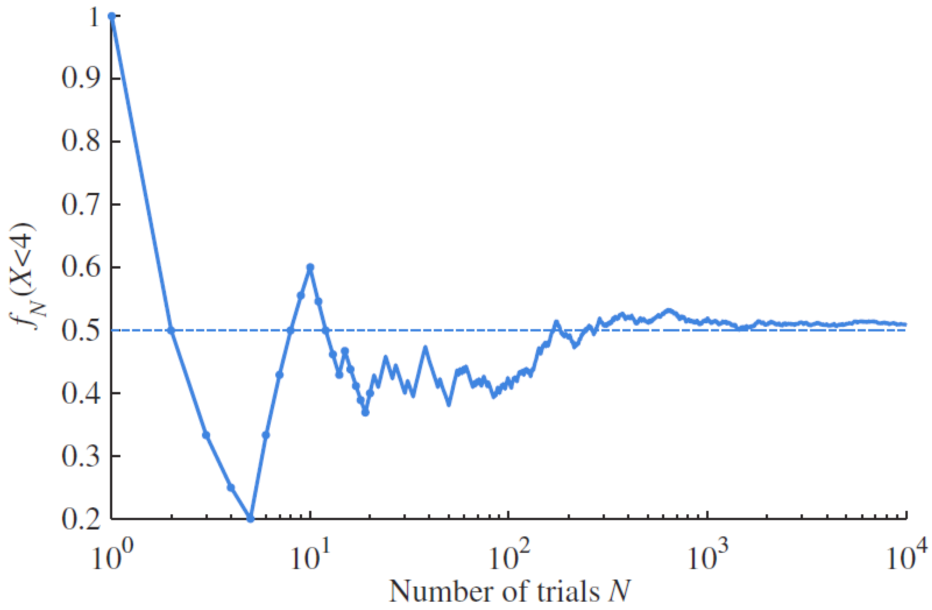
the total number of heads

Probability
(frequentist)

Example:

Event A: X<4

$$P(A) = \lim_{N\to\infty} \frac{N(A)}{N}$$



Probability distributions

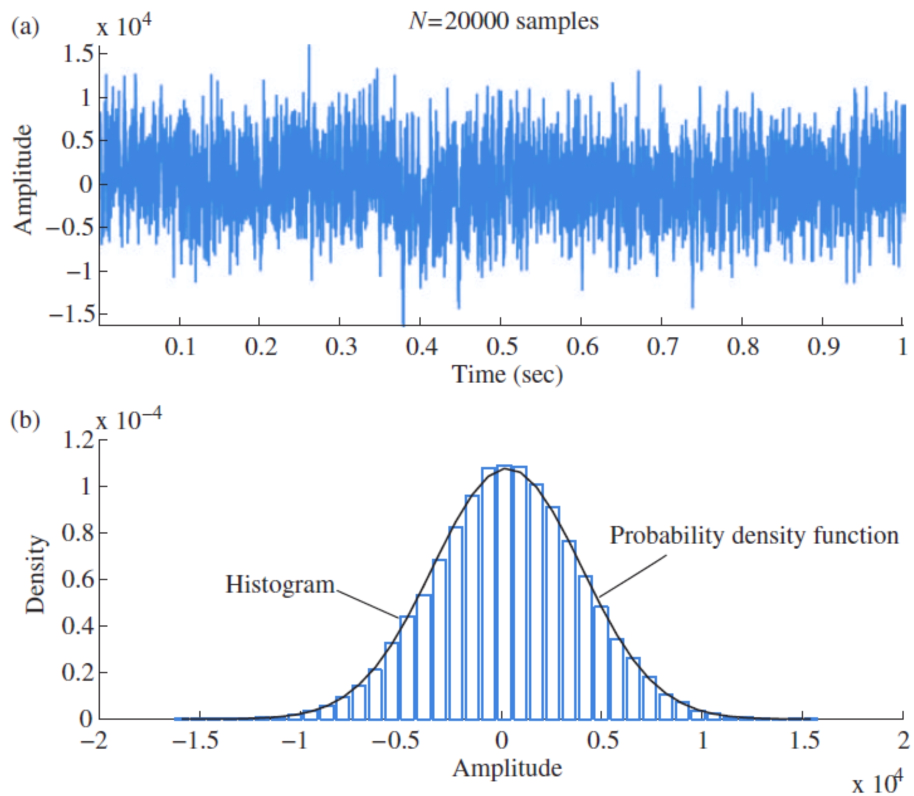
Probability distribution function (pdf): $f_X(x)$

$$\Pr(a_1 < x < a_2) = \int_{a_1}^{a_2} f_X(x)\mathrm{d}x$$

Cumulative distribution function (CDF) : $F_X(x)$

$$F_X(a) \triangleq \Pr(x \le a) = \int_{-\infty}^{a} f_X(x)\mathrm{d}x$$

Example: F-16 noise



Statistical averages

Arithmetic average of observations

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

Mean value

$$m_x \triangleq \mathrm{E}(X) \triangleq \int_{-\infty}^{\infty} x f_X(x)\mathrm{d}x$$

Variance

$$\sigma_x^2 \triangleq \mathrm{var}(X) \triangleq \mathrm{E}[(x - m_x)^2] = \int_{-\infty}^{\infty} (x - m_x)^2 f_X(x)\mathrm{d}x$$

$$= \mathrm{E}(X^2) - 2m_x\mathrm{E}(X) + m_x^2 = \mathrm{E}(X^2) - m_x^2$$

Standard deviation

$$\sigma_x \triangleq \sqrt{\mathrm{var}(X)}$$

Expectation

$$\mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\mathrm{d}x$$

Example of random variables

Uniform distribution

$$X \sim U(a, b)$$

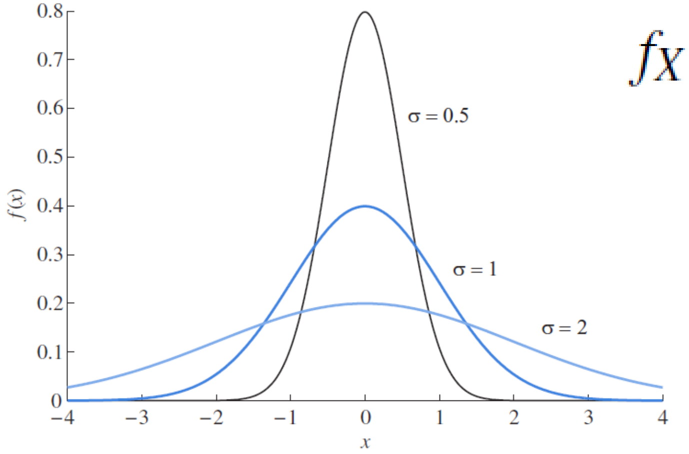
U(0,1) can be simulated by pseudo-random number generator.

$$f_X(x) = \begin{cases} 1/(b-a), & \text{if } a < x < b \\ 0. & \text{otherwise} \end{cases}$$

$$\text{var}(X) = \text{E}(X^2) - [\text{E}(X)]^2 = \frac{1}{12}(b-a)^2$$

Normal distribution

$$X \sim N(m, \sigma^2)$$



$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-m)^2/2\sigma^2}$$

1. Linear combination of normal distributions is still normally distributed.
2. Central limit theorem applies everywhere.
3. *N* can be simulated by inverse transform sampling method.

Jointly distributed random variables

$f_{X,Y}(x,y)$

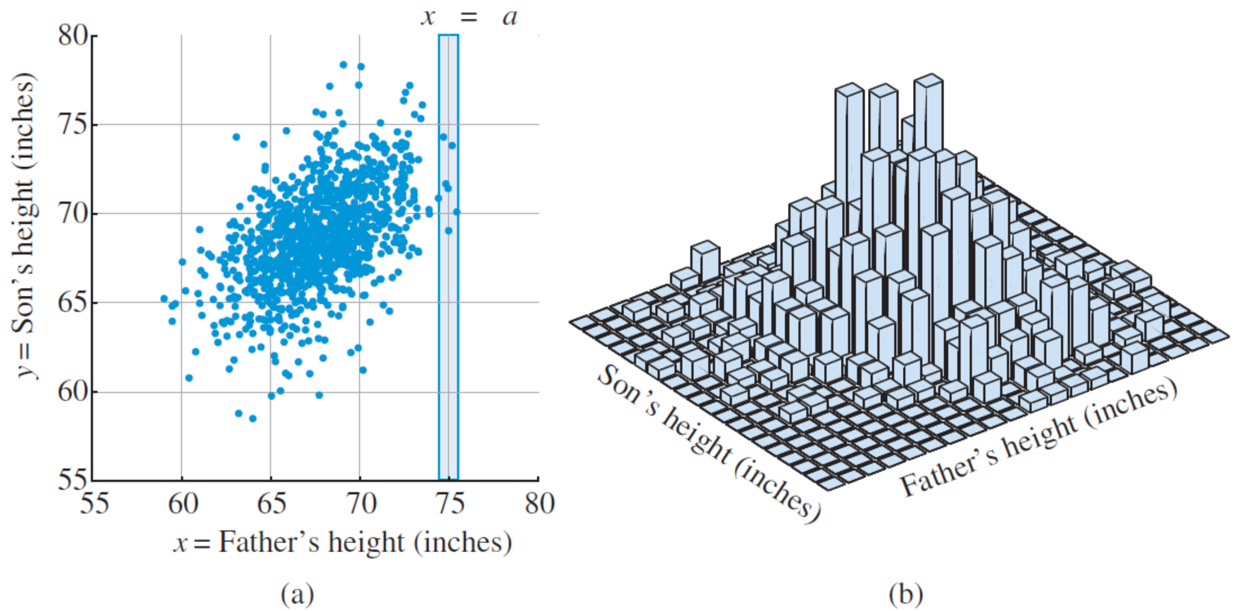


Figure 13.6 (a) A scatter plot of the heights of 1078 sons versus the heights of their fathers, and (b) the corresponding two-dimensional (2-D) histogram.

Marginal distributions

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\mathrm{d}y \quad \text{and} \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\mathrm{d}x$$

Expectation

$$\mathrm{E}[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f_{X,Y}(x,y)\mathrm{d}x\mathrm{d}y$$

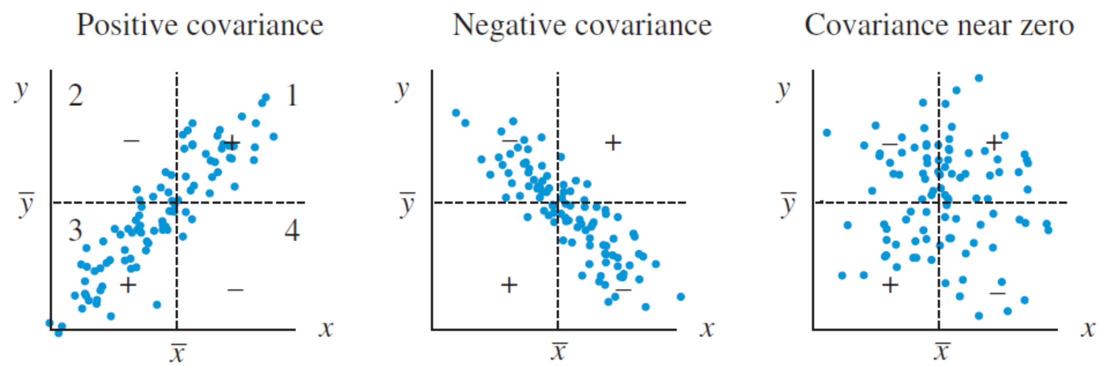
$$\mathrm{E}(aX + bY) = a\mathrm{E}(X) + b\mathrm{E}(Y)$$

Jointly distributed random variables

Conditional pdf $f_{Y|X}(Y|X=a)$ $\qquad f_{X,Y}(x,y) = f_{Y|X}(y|x)f_X(x) = f_{X|Y}(x|y)f_Y(y)$

Covariance $c_{xy} \triangleq \mathrm{cov}(X,Y) \triangleq \mathrm{E}[(X-m_x)(Y-m_y)]$

$$\mathrm{cov}(X,Y) = \mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y)$$



Correlation $r_{xy} \triangleq \mathrm{E}(XY)$

Independent $f_{Y|X}(y|x) = f_Y(y) \;\Rightarrow\; f_{X,Y}(x,y) = f_X(x)f_Y(y)$ ← not always true

Uncorrelated X, Y uncorrelated $\Leftrightarrow \mathrm{cov}(X,Y) = 0$ or $\mathrm{E}(XY) = \mathrm{E}(X)\mathrm{E}(Y)$

Jointly distributed random variables

Correlation coefficient
(measure linear relationship)

$$\tilde{X} \triangleq \frac{X - m_x}{\sigma_x} \quad \text{and} \quad \tilde{Y} \triangleq \frac{Y - m_y}{\sigma_y}$$

$$\rho_{xy} \triangleq \text{cov}(\tilde{X}, \tilde{Y}) = \frac{\text{cov}(X, Y)}{\sqrt{\text{var}(X)}\sqrt{\text{var}(Y)}} = \frac{c_{xy}}{\sigma_x \sigma_y} \qquad -1 \leq \rho_{xy} \leq 1$$

Covariance matrix

$$\boldsymbol{C_x} \triangleq \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1p} \\ c_{21} & c_{22} & \dots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{p1} & c_{p2} & \dots & c_{pp} \end{bmatrix} \qquad c_{ij} \triangleq \text{cov}(X_i, X_j)$$

Correlation matrix

$$\boldsymbol{R_x} \qquad r_{ij} \triangleq \text{E}(X_i X_j)$$

Linear combination of random variables

$$Y = \sum_{i=1}^{p} a_i X_i = \boldsymbol{a}^{\text{T}} \boldsymbol{x} \qquad \text{E}(Y) = \sum_{i=1}^{p} a_i \text{E}(X_i) = \boldsymbol{a}^{\text{T}} \boldsymbol{m_x}$$

$$\text{var}(Y) = \text{var}(\boldsymbol{a}^{\text{T}} \boldsymbol{x}) = \boldsymbol{a}^{\text{T}} \boldsymbol{C_x} \boldsymbol{a} \geq 0$$

C_x is nonnegative definite

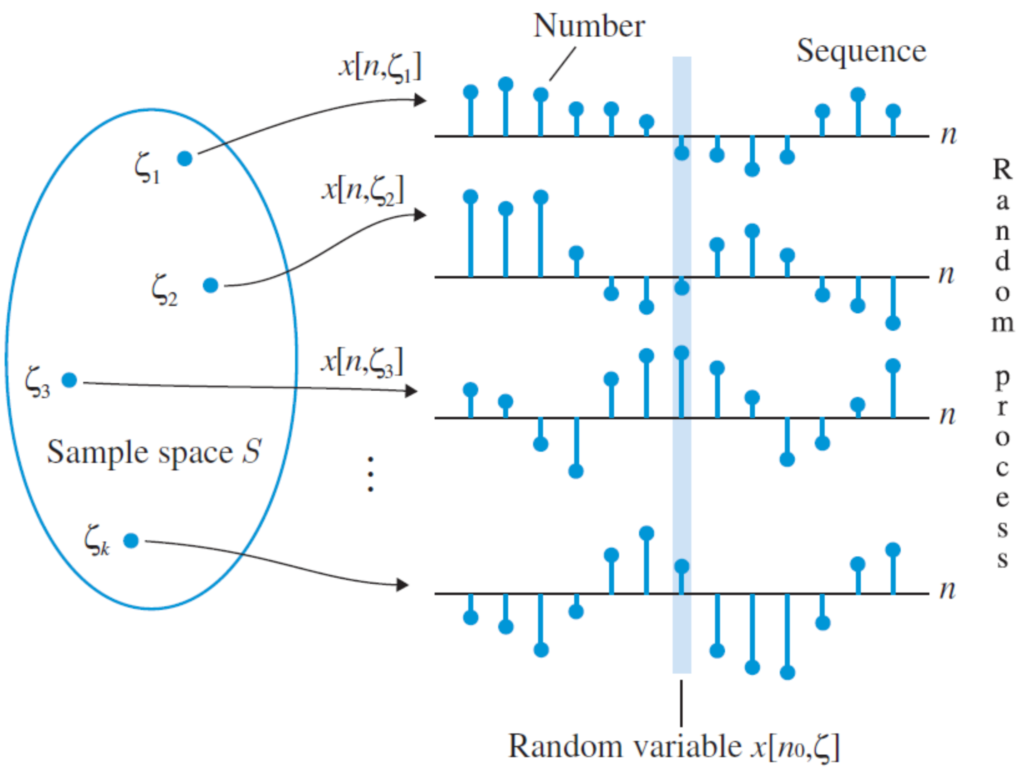
$$\text{E}(Y^2) = \boldsymbol{a}^{\text{T}} \boldsymbol{R_x} \boldsymbol{a} \geq 0$$

R_x is nonnegative definite

Random process

A collection or ensemble of functions (or sequences) with probability assigned to each

$$X[n, \zeta]$$



One realization is a sequence

Figure 13.9 The concept of a random (stochastic) process as a mapping from the sample space of a random experiment to an ensemble of sequences.

Stationary random processes

Strictly stationary the sets of random variables $x[n_1], \ldots, x[n_p]$ and $x[n_1+k], \ldots, x[n_p+k]$ have the same joint probability distribution for any set of points, any number p, and any shift k.

p=1 $f(x[n]) = f(x[n+k])$ (ensemble average is static over time)

$$\mathrm{E}(x[n]) = m_x \quad \text{and} \quad \mathrm{var}(x[n]) = \sigma_x^2, \text{ for all } n$$

p=2 $f(x[n], x[m])$, depend only upon the *lag* (time difference) $\ell \triangleq n - m$

$$c_{xx}[n,m] \triangleq \mathrm{cov}(x[n], x[m]) = c_{xx}[\ell]. \text{ for all } m, n$$

Autocovariance sequence (ACVS) (ensemble covariance is static over time)

Wide-sense stationary (WSS) A random process which satisfies the condition of p=1 and p=2.

$$r_{xx}[m+\ell, m] \triangleq \mathrm{E}(x[m+\ell]x[m]) = r_{xx}[\ell] = c_{xx}[\ell] + m_x^2$$

Autocorrelation sequence (ACRS)

Examples

Sinusoidal random process

$$x[n, \zeta_k] = A(\zeta_k) \cos[\omega(\zeta_k)n + \phi(\zeta_k)]$$

not stationary

WSS sinusoidal random process

$$x[n] = A\cos(\omega n + \phi), \quad \phi \sim (0, 2\pi)$$

$$\mathrm{E}(x[n]) = A\mathrm{E}[\cos(\omega n + \phi)] = \frac{A}{2\pi}\int_0^{2\pi} \cos(\omega n + \phi)\mathrm{d}\phi = 0$$

$$\begin{aligned} r_x[n, m] &= \mathrm{E}[A\cos(\omega n + \phi)A\cos(\omega m + \phi)] \\ &= \frac{A^2}{2\pi}\int_0^{2\pi} \frac{1}{2}\{\cos[\omega(n-m)] + \cos[\omega(n+m) + 2\phi]\}\,\mathrm{d}\phi \\ &= \frac{A^2}{2}\cos\omega(n-m). \end{aligned}$$

$$r_x(\ell) = \frac{1}{2}A^2\cos\omega\ell$$

Properties of correlation and covariance sequences

Jointly WSS

$$r_{xy}[\ell] = c_{xy}[\ell] + m_x m_y$$

cross-correlation sequence

$$c_{xy}[n,m] \triangleq \mathrm{cov}(x[n], y[m]),$$

$$\boxed{\ell = n - m}$$

$$r_{xy}[n,m] \triangleq \mathrm{E}(x[n]y[m]) = c_{xy}[n,m] + \mathrm{E}(x[n])\mathrm{E}(y[m])$$

Symmetry

$$r_{xx}[\ell] = r_{xx}[-\ell]$$

$$c_{xx}[\ell] = c_{xx}[-\ell]$$

even symmetry

$$r_{xy}[\ell] = r_{yx}[-\ell]$$

$$c_{xy}[\ell] = c_{yx}[-\ell]$$

not even

Correlation matrix

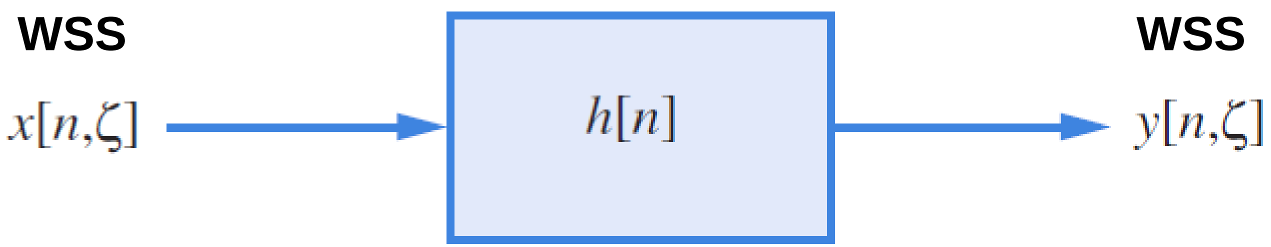
Toeplitz
(so as covariance matrix)

$$\boldsymbol{R}_x = \begin{bmatrix} r_x[0] & r_x[1] & \dots & r_x[p-1] \\ r_x[1] & r_x[0] & \dots & r_x[p-2] \\ \vdots & \vdots & \ddots & \vdots \\ r_x[p-1] & r_x[p-2] & \dots & r_x[0] \end{bmatrix}$$

$$r_{ij} = \mathrm{E}(x[n-i]x[n-j]) = r_x[j-i]$$

$$\boldsymbol{a}^{\mathrm{T}}\boldsymbol{R}_x\boldsymbol{a} \geq 0$$

Response of LTI systems to random processes



$$y[n,\zeta] = \sum_{k=-\infty}^{\infty} h[k]x[n-k,\zeta] \xrightarrow{\text{simplified}} y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

Expectation

$$\mathrm{E}(y[n]) = \sum_{k=-\infty}^{\infty} h[k]\mathrm{E}(x[n-k]) \longrightarrow \mathrm{E}(y[n]) = m_x \sum_{k=-\infty}^{\infty} h[k] \triangleq m_y$$

Cross-correlation

$$\mathrm{E}(x[m]y[n]) = \sum_{k=-\infty}^{\infty} h[k]\mathrm{E}(x[m]x[n-k]) \qquad \boxed{\ell = m-n}$$

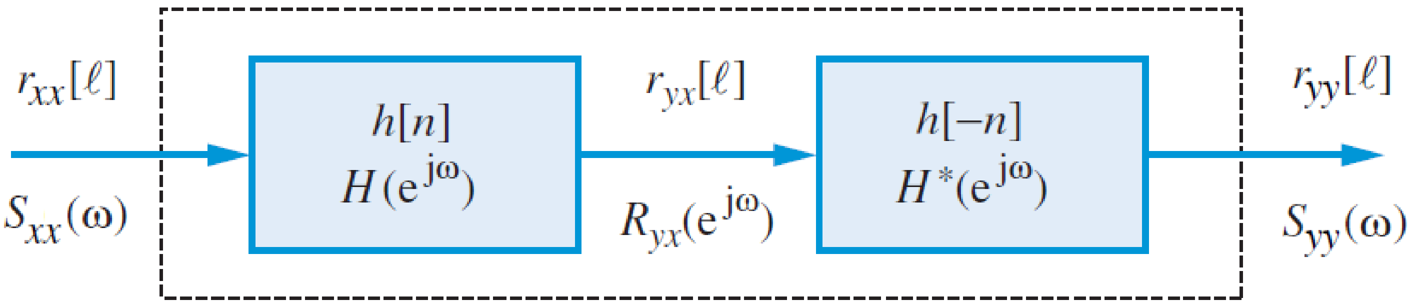
$$r_{xy}[\ell] = \sum_{k=-\infty}^{\infty} h[k]r_{xx}[\ell+k] = \sum_{i=-\infty}^{\infty} h[-i]r_{xx}[\ell-i] = h[-\ell] * r_{xx}[\ell]$$

Response of LTI systems to random processes

Cross-correlation $$r_{yx}[\ell] = \sum_{k=-\infty}^{\infty} h[k] r_{xx}[\ell - k] = h[\ell] * r_{xx}[\ell]$$

ACRS $$r_{yy}[\ell] = \sum_{m=-\infty}^{\infty} r_{hh}[m] r_{xx}[\ell - m] = r_{hh}[\ell] * r_{xx}[\ell]$$

$$r_{yy}[\ell] = h[\ell] * h[-\ell] * r_{xx}[\ell]$$



$$S_{yy}(\omega) = \left|H(e^{j\omega})\right|^2 \; S_{xx}(\omega)$$

Power spectral density (PSD)

[Assume x[n] and h[n] are all real-valued]

$$S_{xx}(\omega) \triangleq \sum_{\ell=-\infty}^{\infty} r_{xx}[\ell]\mathrm{e}^{-\mathrm{j}\ell\omega}$$

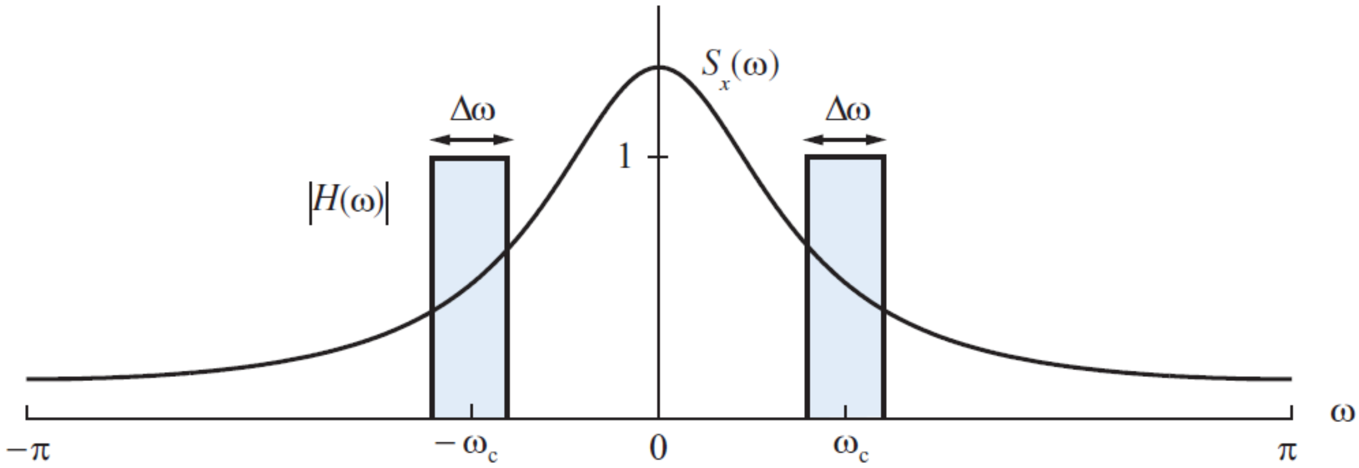
ACRS and PSD are DTFT pairs.

$$S_{xx}(\omega) = \mathbf{E}_\zeta\left[X\left(e^{j\omega},\zeta\right)X^*(e^{j\omega},\zeta)\right]$$

$$= \mathbf{E}_\zeta\left[\left|X(e^{j\omega},\zeta)\right|^2\right] \geq 0$$

Average power on frequency ω

$$S_{xx}(-\omega) = S_{xx}(\omega). \quad \text{(real and even)} \quad \Leftarrow \quad r_x[\ell] = r_x[-\ell]$$



Properties of PSD

LTI response

$$S_{yy}(\omega) = \left|H(e^{j\omega})\right|^2 S_{xx}(\omega)$$

$$E(y^2[n]) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|H(e^{j\omega})\right|^2 S_{xx}(\omega)d\omega$$

Cross PSD

$$S_{yx}(\omega) = H(e^{j\omega})S_{xx}(\omega)$$

Remark

$$r_{xx}[\ell] = c_{xx}[\ell] + m_x^2$$

$$\downarrow$$

$$S_{xx}(\omega) = C_{xx}(e^{j\omega}) + 2\pi m_x^2 \delta(\omega)$$

$$\downarrow$$

Cause problems for analysis, e.g. large leakage power due to windowing

$$\downarrow$$

Prefer to remove the mean value.
(default if not mentioned in the following)