Chap12
Multirate Signal Processing

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Chap 12 Multirate signal processing

- 12.1 Sampling rate conversion
- 12.2 Implementation of multirate systems
- 12.3 Filter design for multirate systems
- 12.4 Two-channel filter banks

Sampling rate conversion (Resampling)

Conceptual flow

$x[n]$ **Discrete signal with period T**

↓ Reconstruction, e.g. filtering, polynomial approx.

$x_c(t)$ **Continuous signal**

$$\sum_{k=-\infty}^{\infty} x[n] \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

Ideal signal

↓ Sampling

$$x_0[n] \triangleq x_c(nT_0)$$

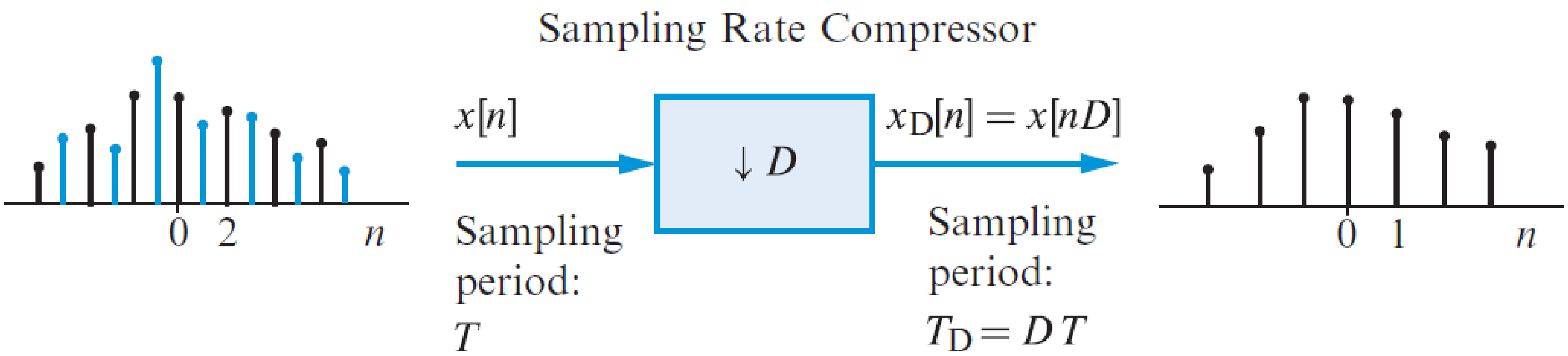
Resampled discrete signal with period T_0

Discrete rate conversion
(D, I: integer)

$$\text{(a) } T_0 = DT, \text{ (b) } T_0 = T/I, \text{ and (c) } T_0 = T(D/I)$$

Downsample **Upsample**

Downsampler: sampling rate compressor



CTFT

$$X_D(e^{j\Omega T_D}) = \frac{1}{DT}\sum_{\ell=-\infty}^{\infty} X_c\left(j\left(\Omega - \ell\frac{2\pi}{DT}\right)\right)$$

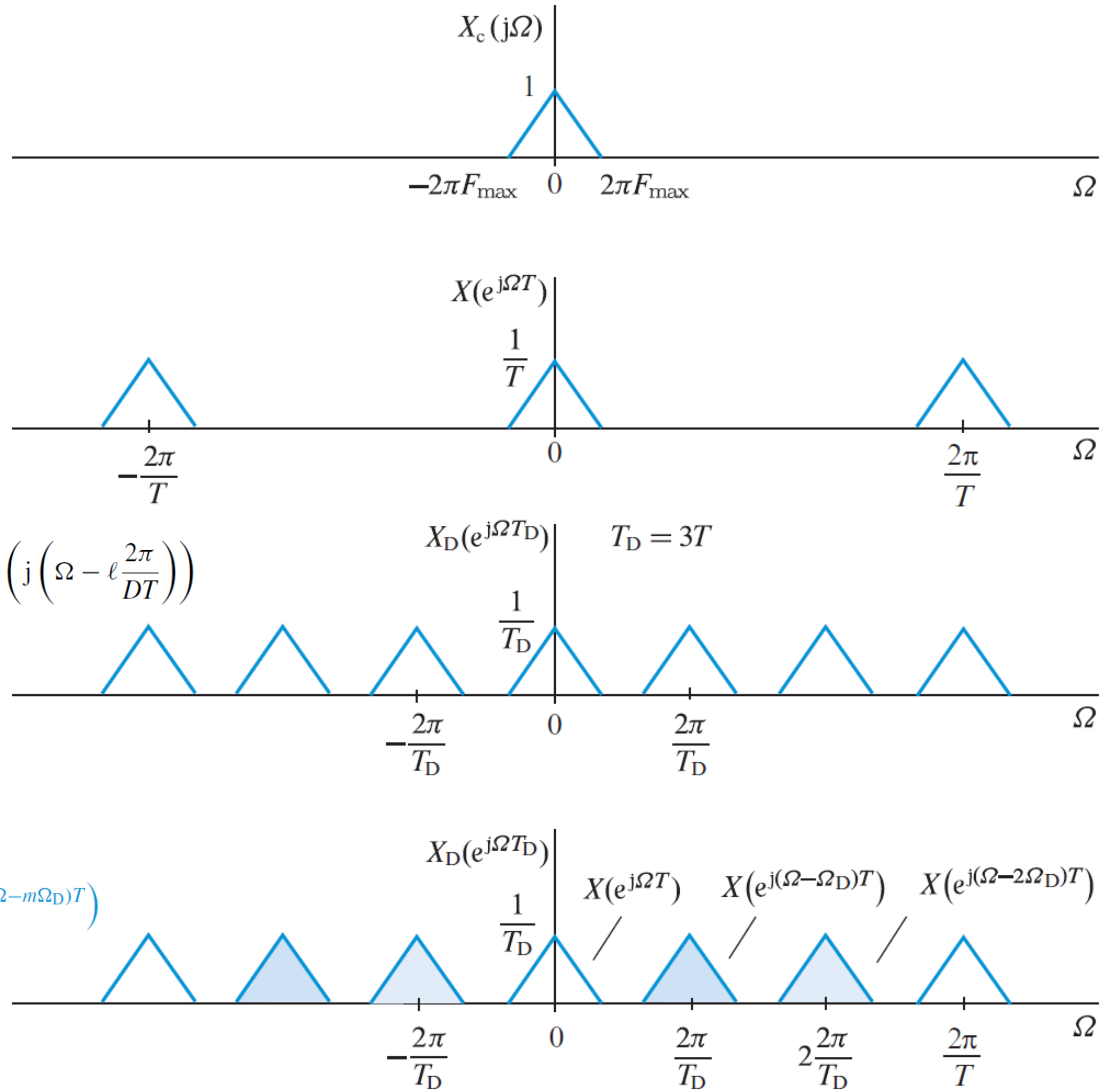
$$X_D(e^{j\Omega T_D}) = \frac{1}{D}\sum_{m=0}^{D-1} X\left(e^{j(\Omega - m\Omega_D)T}\right)$$

DTFT

$$X_D(e^{j\omega}) = \frac{1}{D}\sum_{m=0}^{D-1} X\left(e^{j(\omega - 2\pi m)/D}\right) \qquad \omega \triangleq \Omega T_D$$

Downsampler: CTFT example

D=3

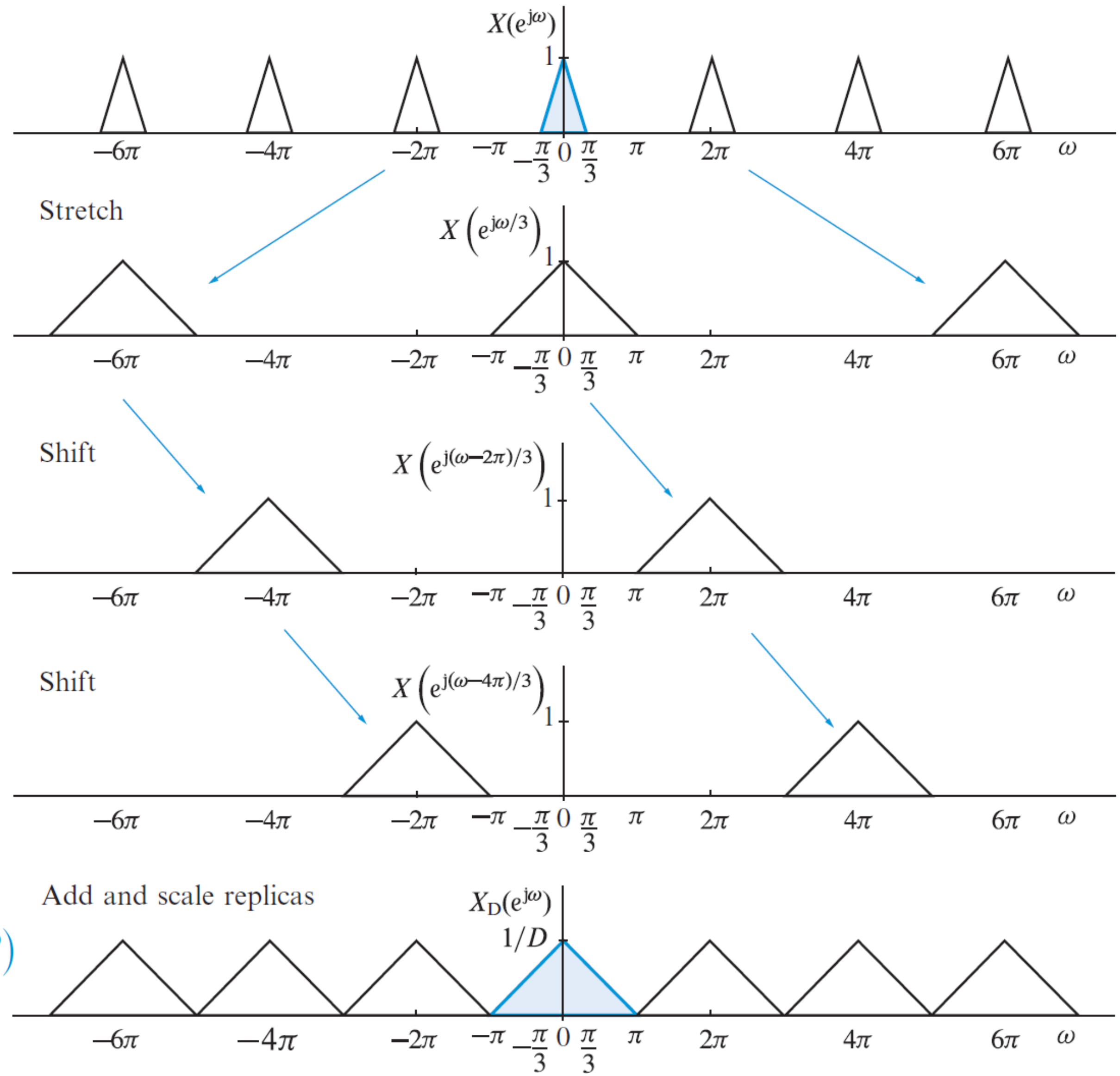


$$X_D(e^{j\Omega T_D}) = \frac{1}{DT} \sum_{\ell=-\infty}^{\infty} X_c\left(j\left(\Omega - \ell\frac{2\pi}{DT}\right)\right)$$

$$X_D(e^{j\Omega T_D}) = \frac{1}{D} \sum_{m=0}^{D-1} X\left(e^{j(\Omega - m\Omega_D)T}\right)$$

Downsampler: DTFT example

D=3



$$X_D(e^{j\omega}) = \frac{1}{D}\sum_{m=0}^{D-1} X\left(e^{j(\omega-2\pi m)/D}\right)$$

Downsampler: D=2 example

General case

$$x[n] = \{\ldots, x[-2], x[-1], x[0], x[1], x[2], \ldots\}$$

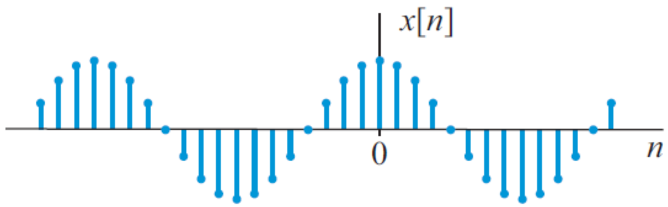
$$\mathcal{D}_2\{x[n]\} = \{\ldots, x[-4], x[-2], x[0], x[2], x[4], \ldots\} = x_\mathrm{D}[n]$$

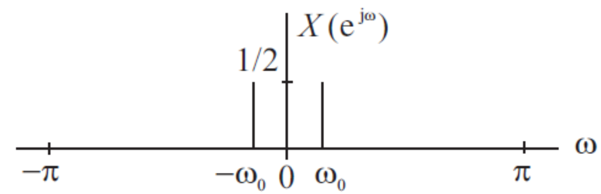
$$\mathcal{D}_2\{x[n+1]\} = \{\ldots, x[-3], x[-1], x[1], x[3], x[5], \ldots\}$$

$$X_\mathrm{D}\left(e^{j\omega}\right) = \frac{1}{2}X\left(e^{j\omega/2}\right) + \frac{1}{2}X\left(e^{j(\omega/2-\pi)}\right)$$

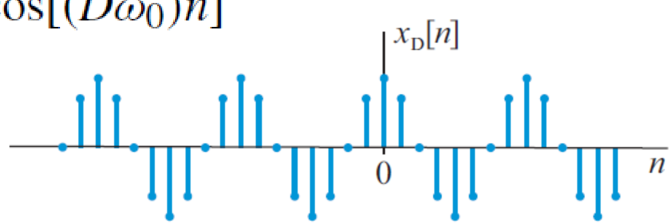
Spectrum expansion

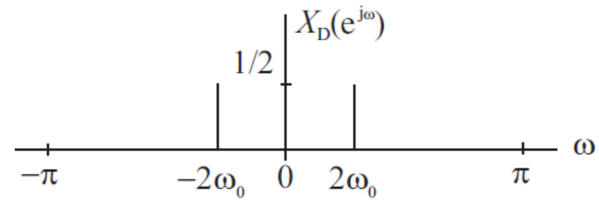
$$x[n] = \cos(\omega_0 n)$$



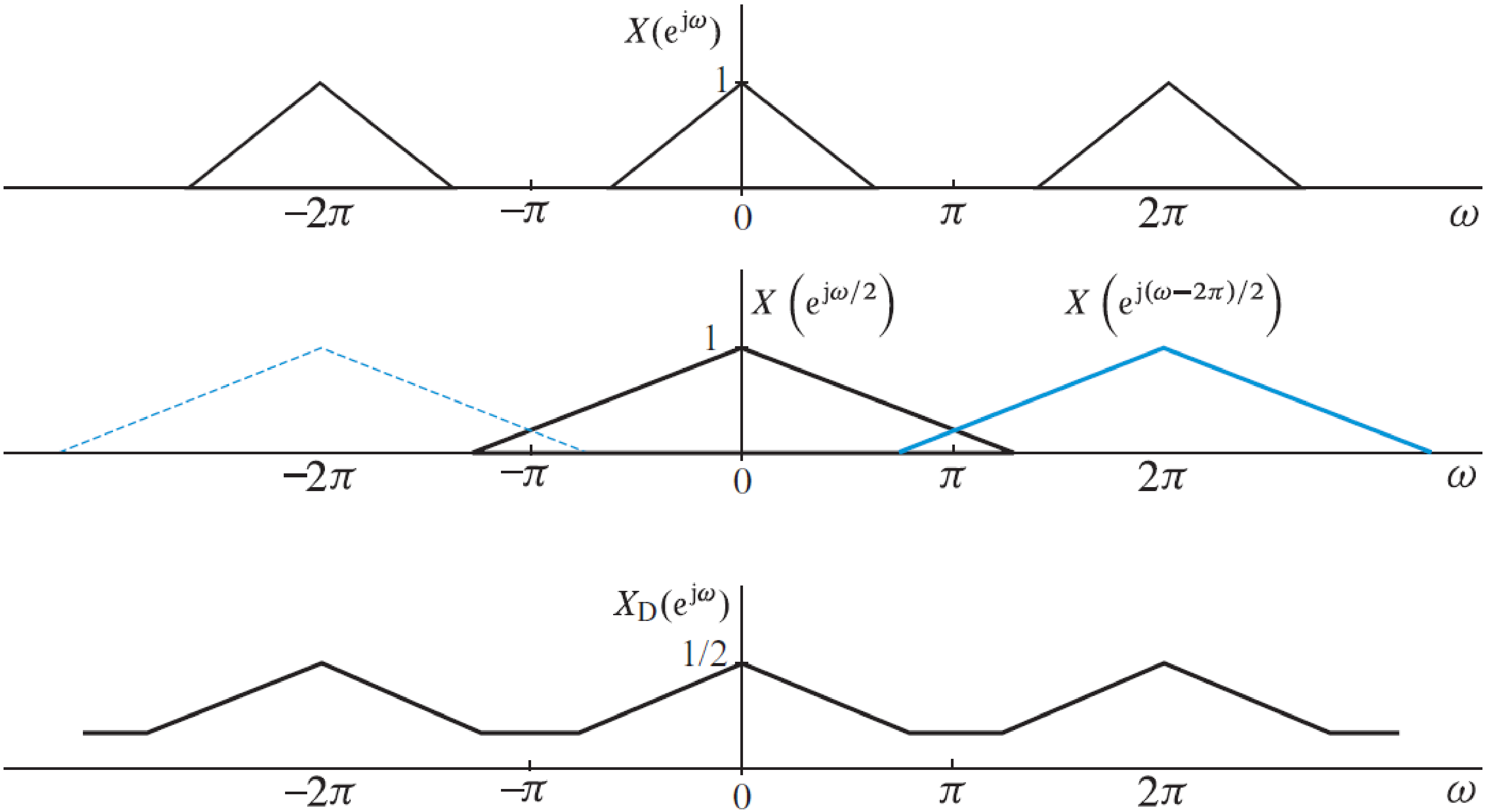


$$x_\mathrm{D}[n] = \cos[\omega_0(nD)] = \cos[(D\omega_0)n]$$

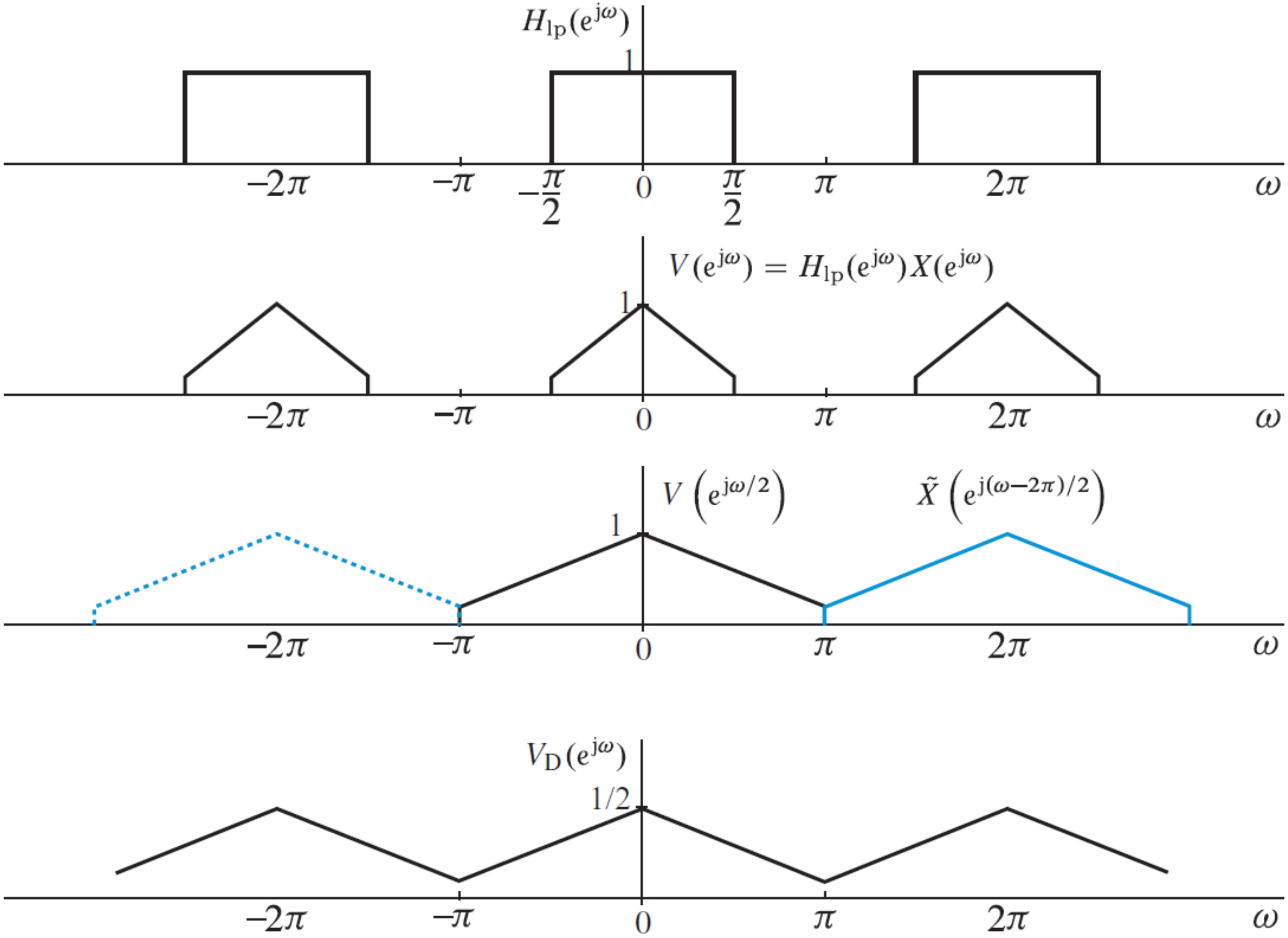




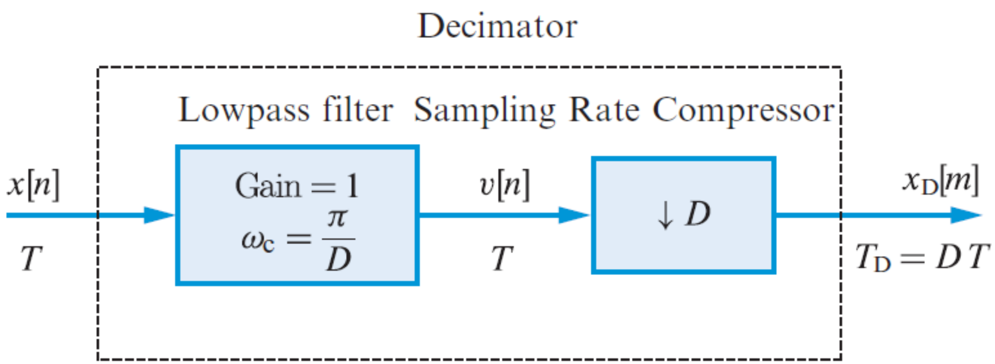
Downsampling with aliasing (D=2)



Downsampling with lowpass prefiltering (Decimation)



Decimator



For FIR filter

$$x_D[m] = v[mD] = \sum_{k=0}^{M} h[k]x[mD-k]$$

Overall computation can be reduced to 1/D

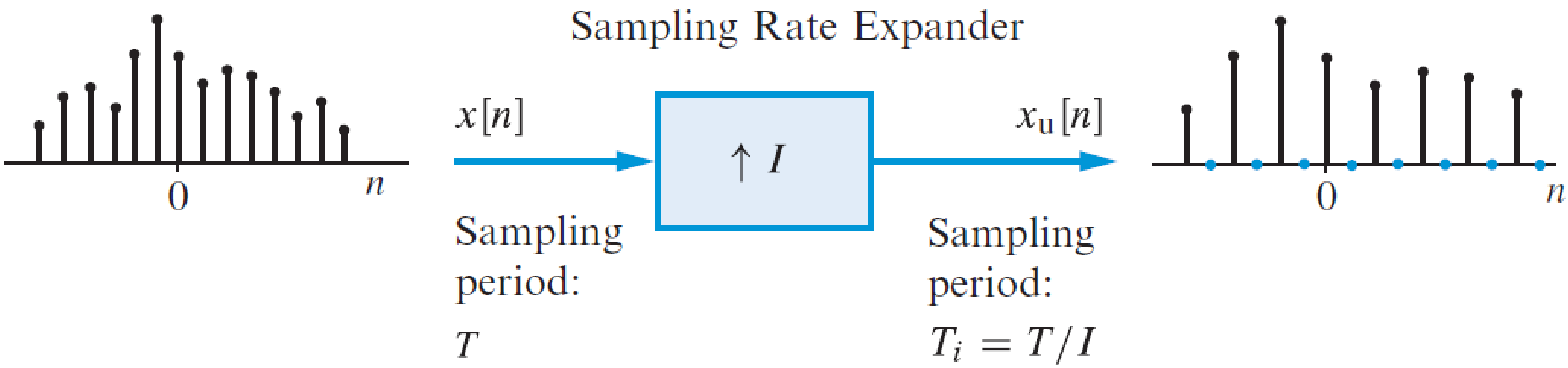
To avoid aliasing

$$\text{If } X(e^{j\omega}) = 0, \quad \omega_H \le |\omega| \le \pi \quad \text{then} \quad \omega_s = \frac{2\pi}{D} \ge 2\omega_H$$

Safe choice

$$\omega_c = \pi/D$$

Upsampler: sampling rate expander



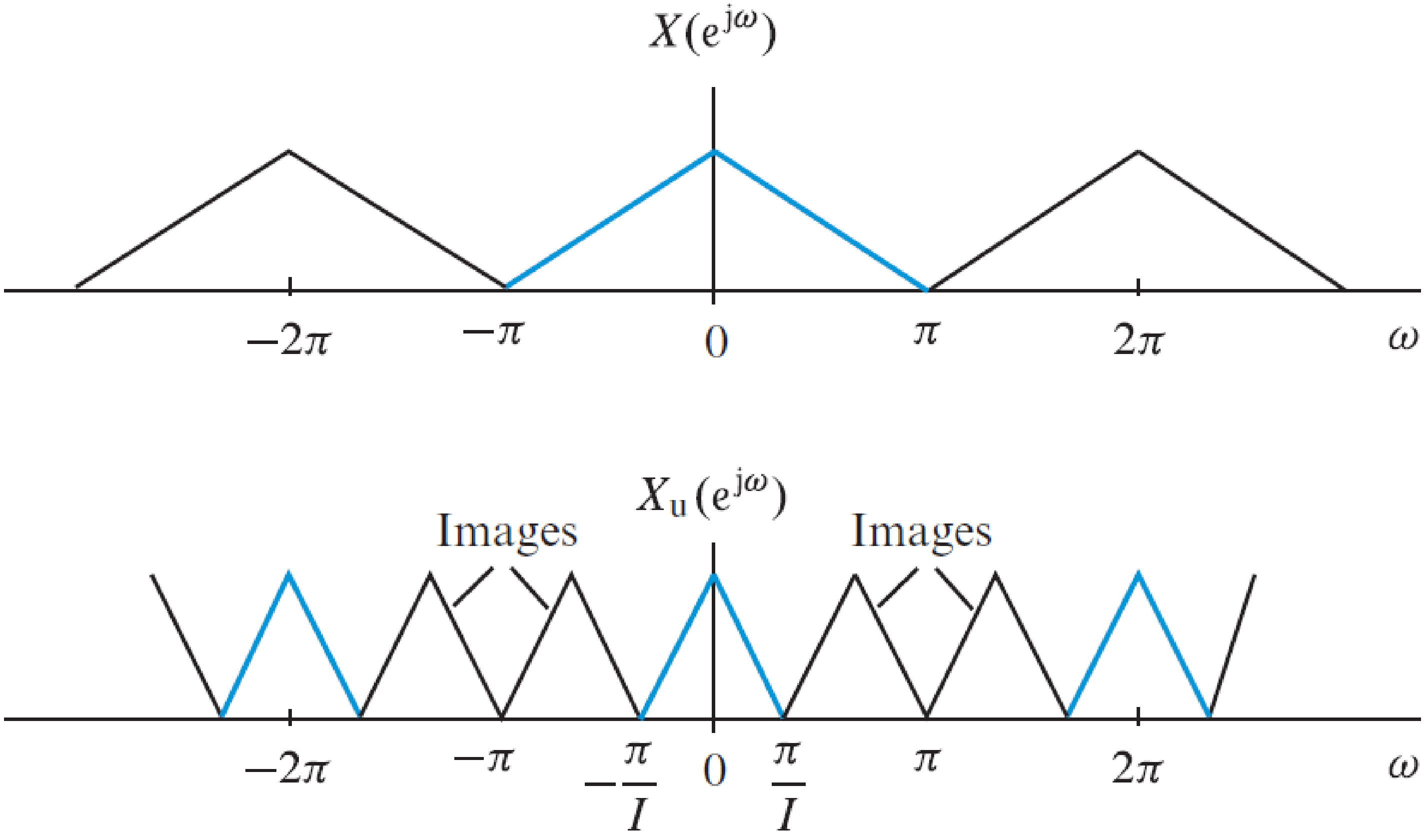
Upsample

$$x_u[n] \triangleq \mathcal{U}_I\{x[n]\} \triangleq x_{\uparrow I}x[n] \triangleq \begin{cases} x[n/I], & n \text{ is a multiple of } I \\ 0, & \text{otherwise} \end{cases}$$

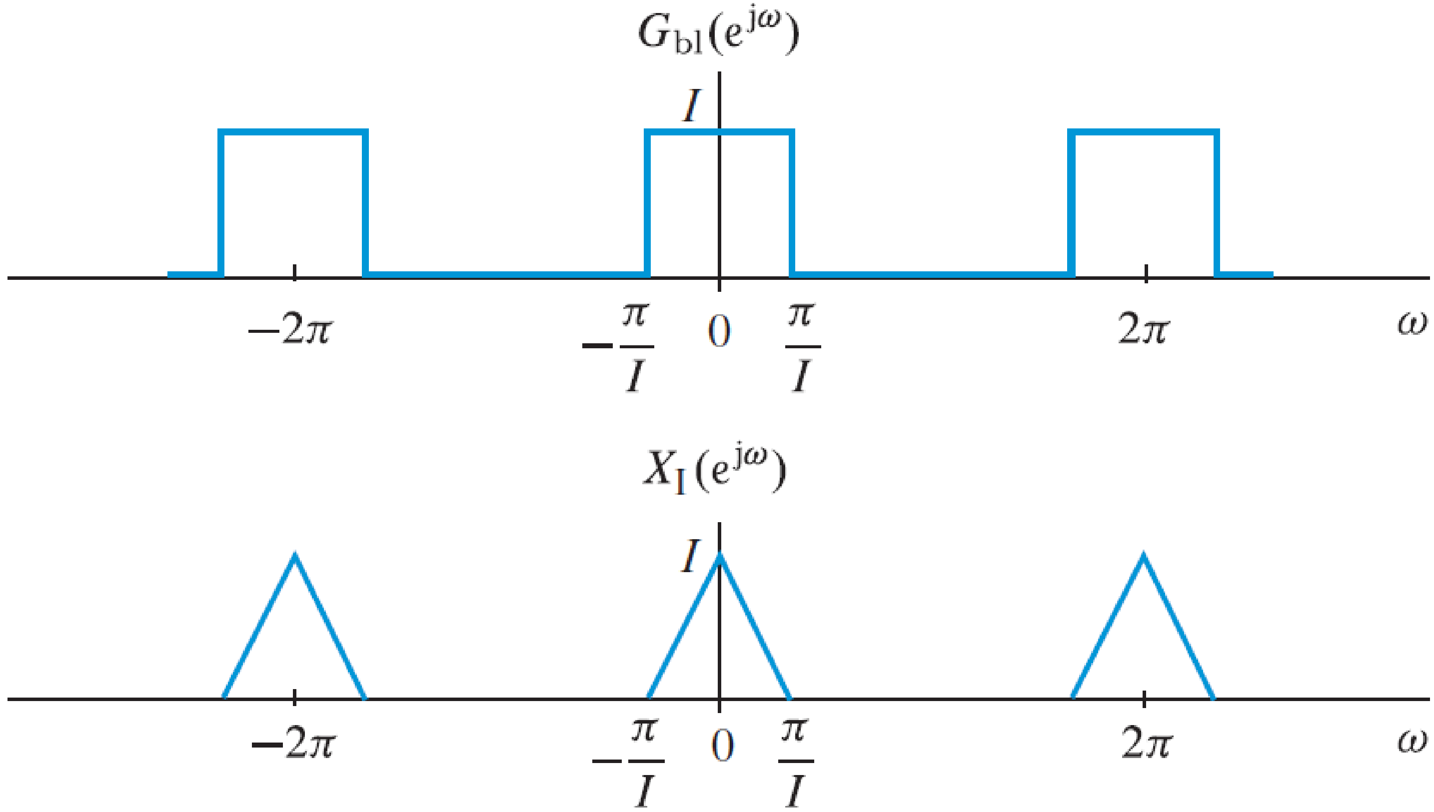
DTFT

$$X_u\left(e^{j\omega}\right) = X\left(e^{j\omega I}\right)$$

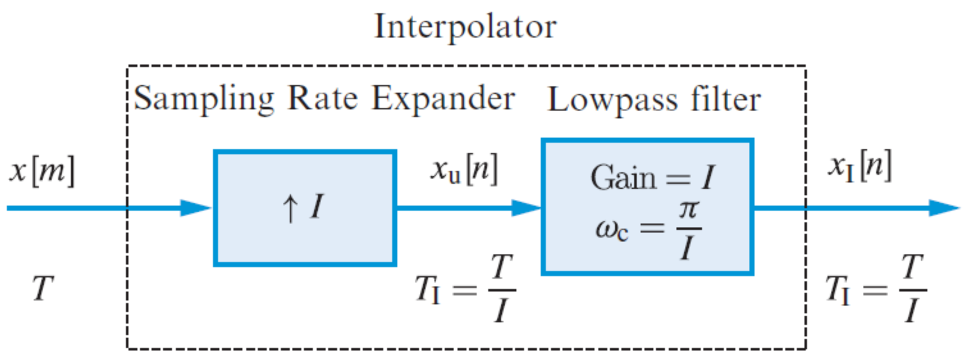
Upsampling with images (I=3)



Upsampling with lowpass postfiltering (Interpolation)



Interpolator



Post-filtering

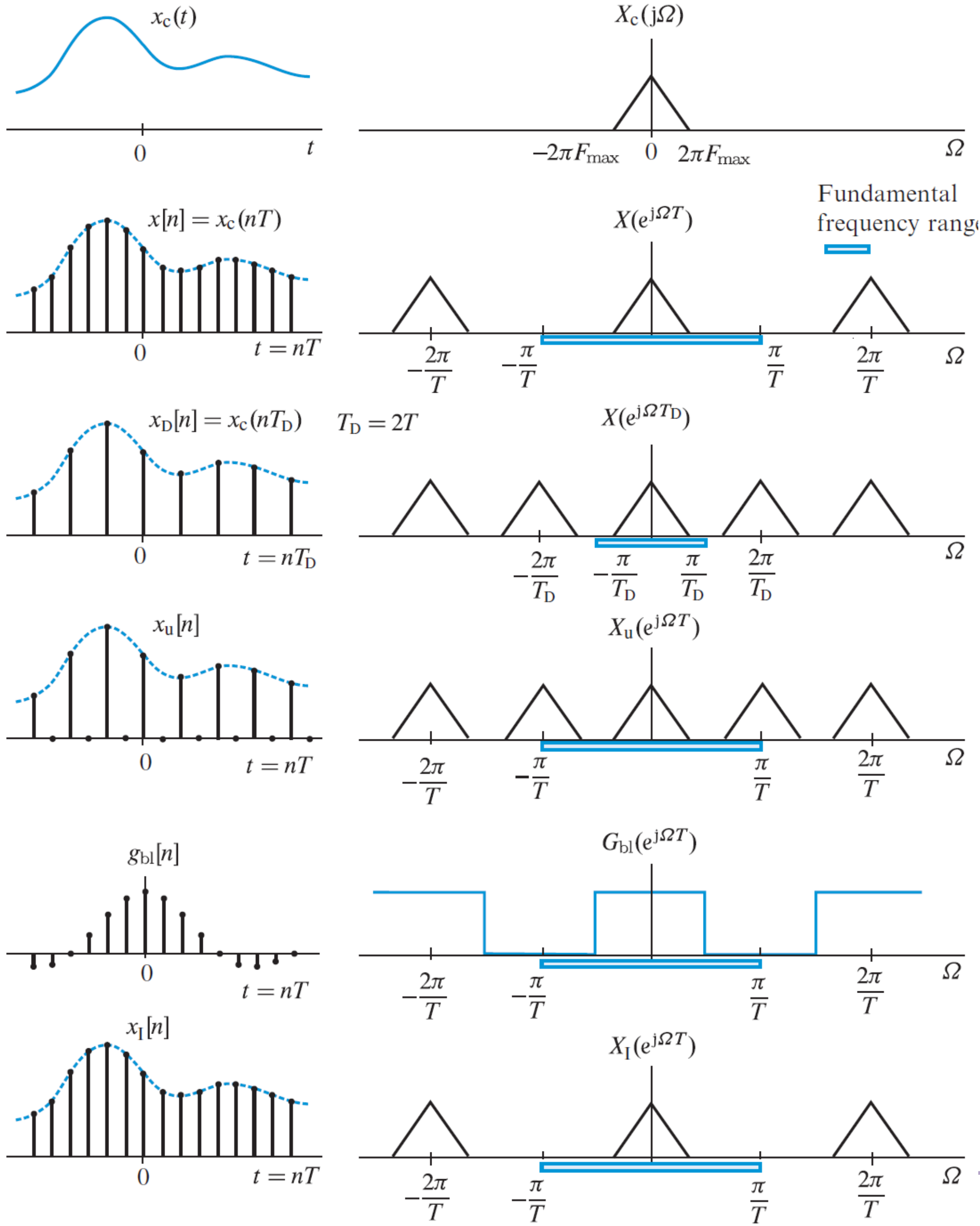
$$X_u(e^{j\omega}) = X(e^{j\omega I})$$

$$X_I(e^{j\omega}) = G_r(e^{j\omega})X_u(e^{j\omega})$$

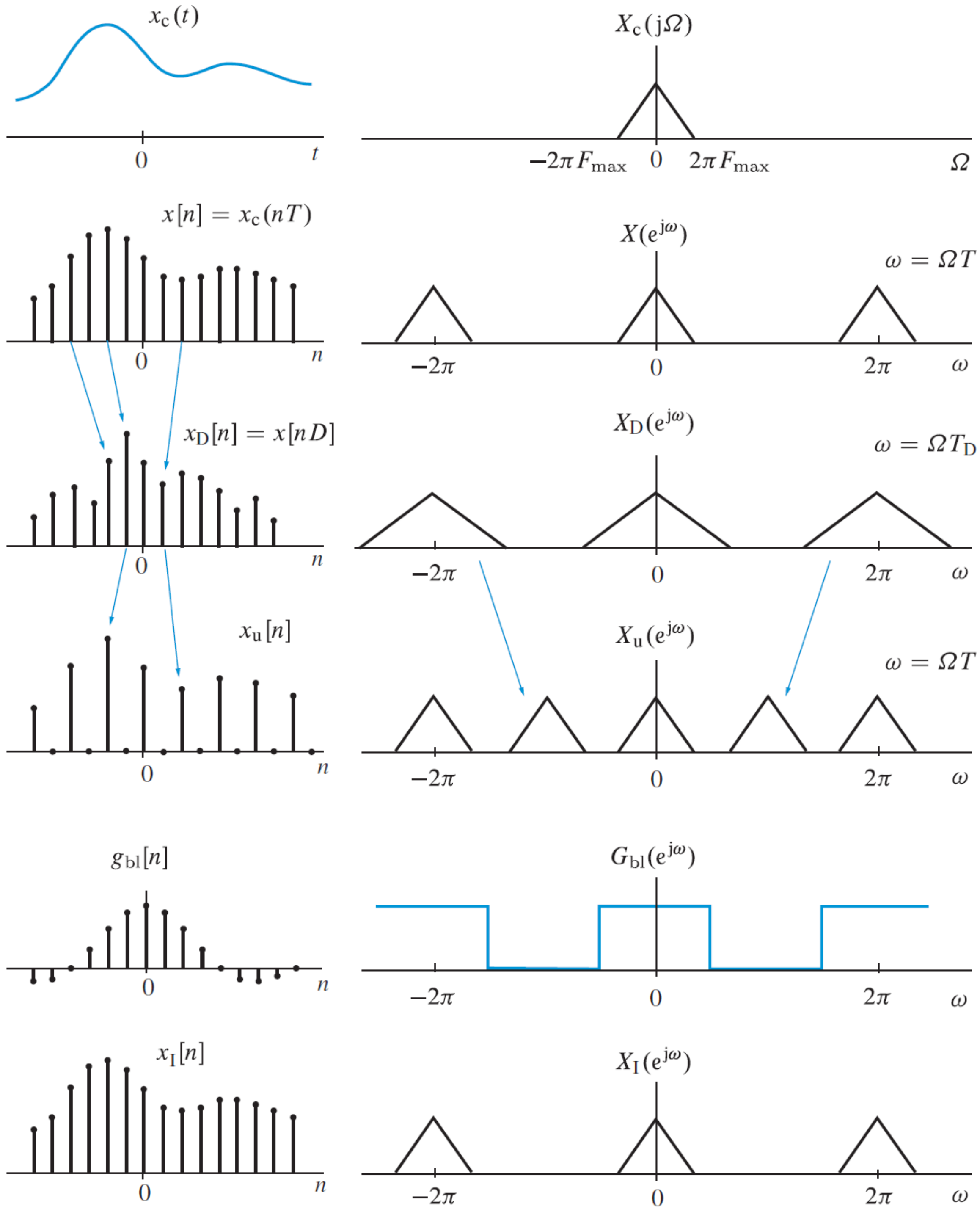
$$x_I[n] = \sum_{k=-\infty}^{\infty} x_u[k]g_r[n-k]$$

Overall computation can be reduced to 1/I for FIR filters

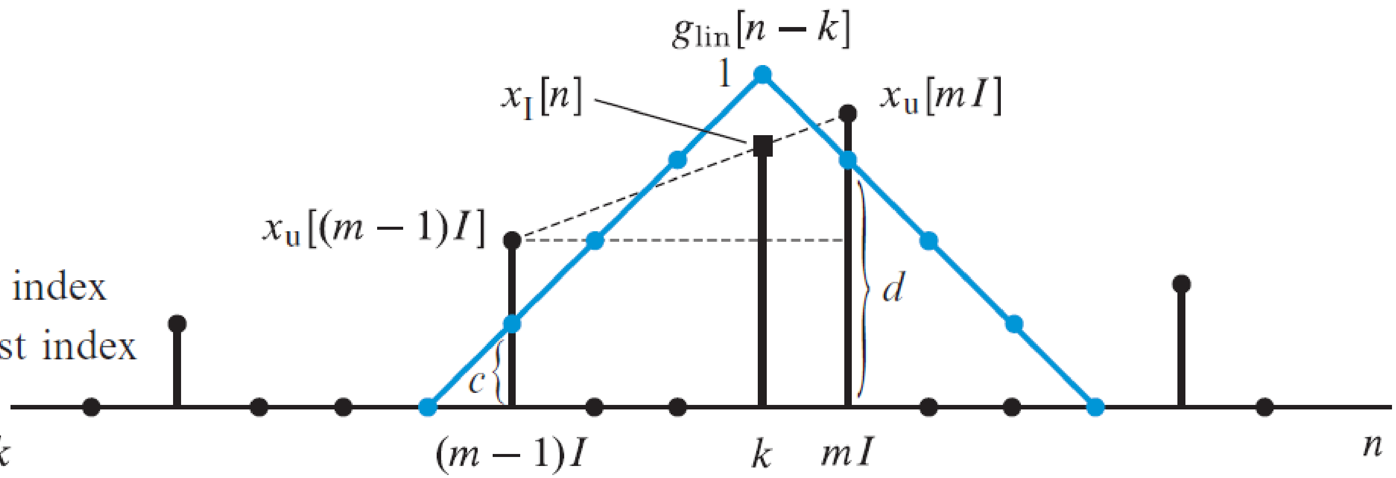
Decimation and interpolation in CTFT



Decimation and interpolation in DTFT



Linear interpolation



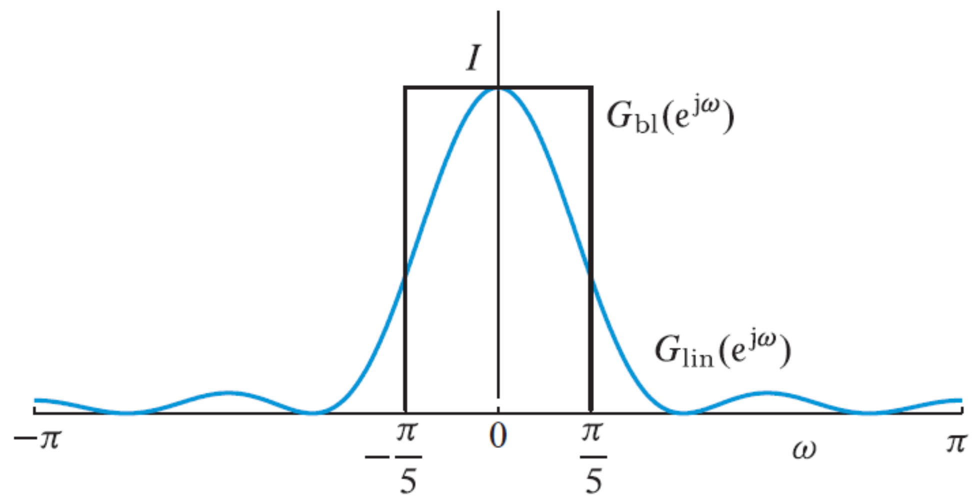
Input: Fast index

Output: Fast index

$n = mI + k$

$$g_{\text{lin}}[n] \triangleq \begin{cases} 1 - \dfrac{|n|}{I}, & -I < n < I \\ 0, & \text{otherwise} \end{cases}$$

$$G_{\text{lin}}(e^{j\omega}) = \frac{1}{I}\left[\frac{\sin(\omega I/2)}{\sin(\omega/2)}\right]^2$$



Fractional delay

Ideal (non-practical)

$$y[n] = y_c(nT) = x_c(nT - t_D) = x[n - \Delta] \qquad \Delta \triangleq t_D/T$$

$$H_{fd}(e^{j\omega}) = e^{-j\omega\Delta}$$

$$h[n] = \mathcal{F}^{-1}[e^{-j\Delta\omega}] = \frac{\sin[\pi(n-\Delta)]}{\pi(n-\Delta)}$$

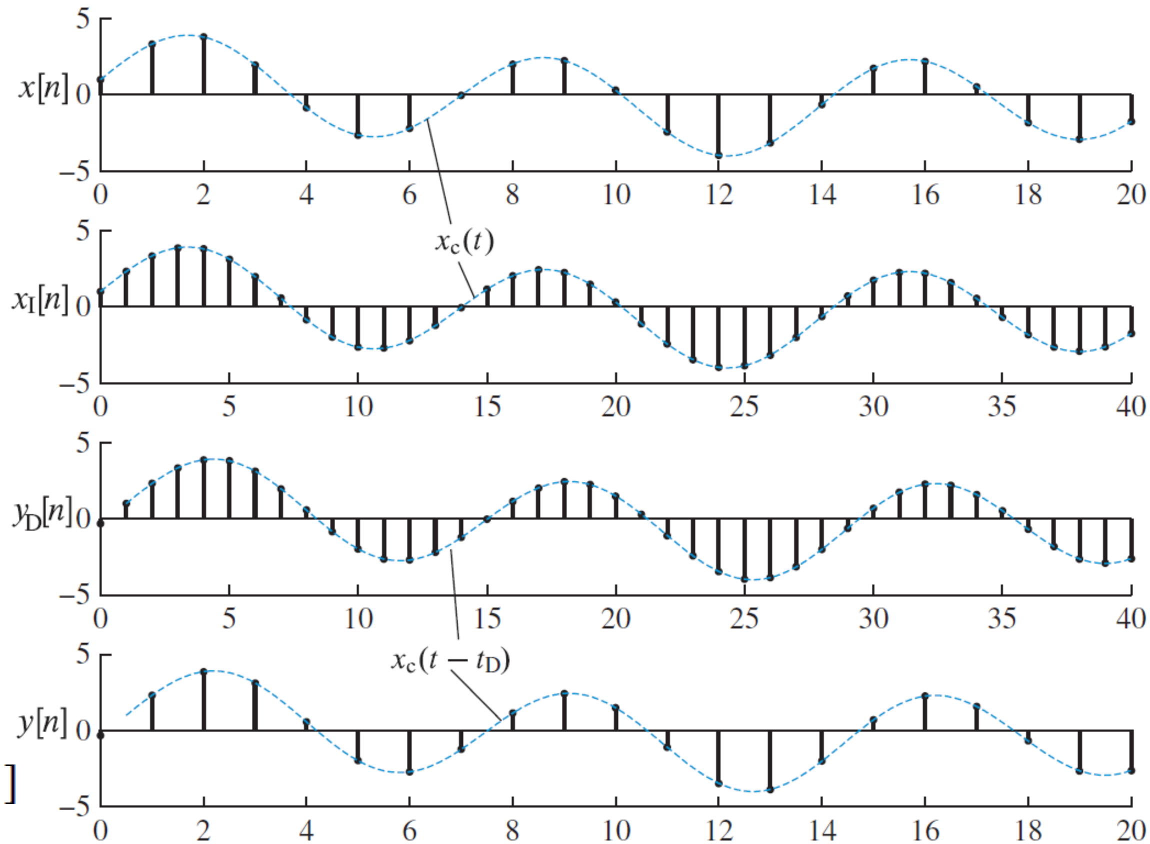
Practical example of $\Delta = 1/2$

$I = 2$

1-sample delay

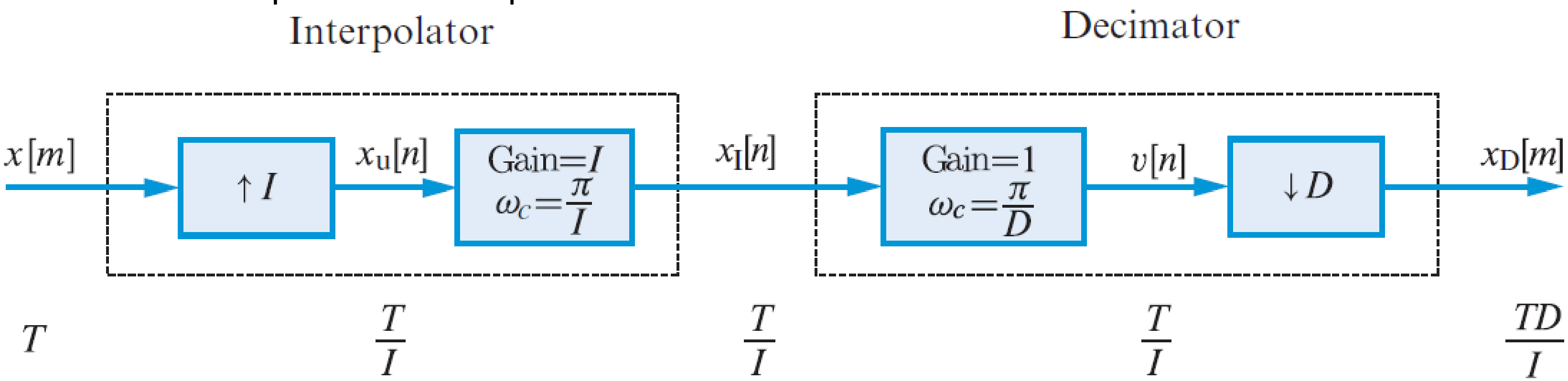
$D = 2$

$y[n] = x[n - \frac{1}{2}]$

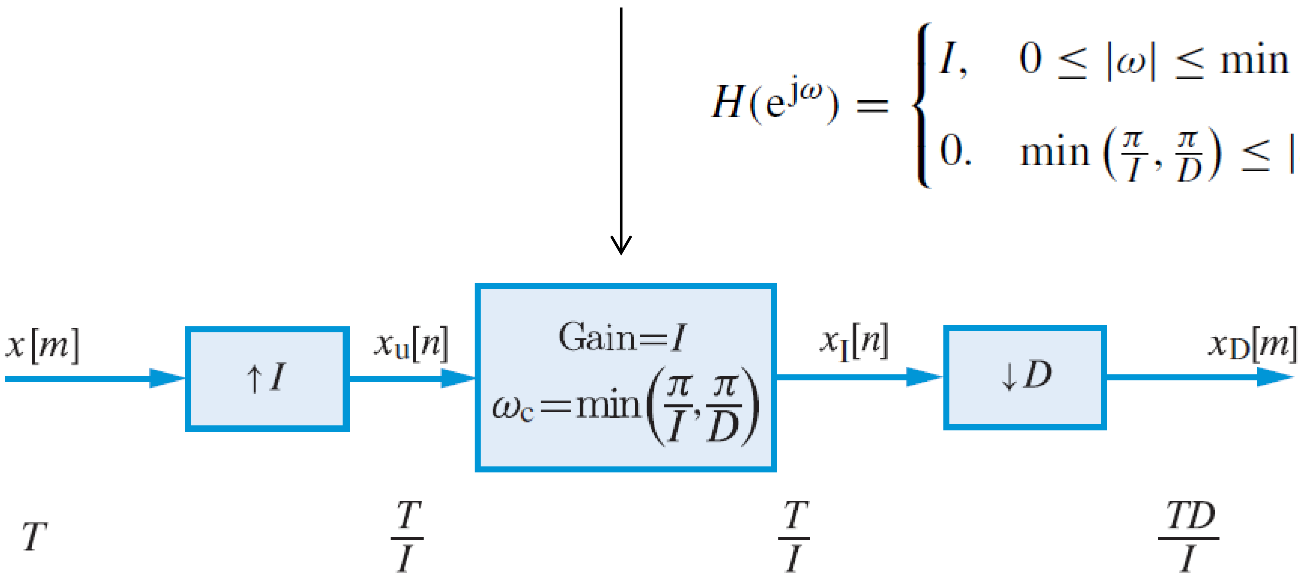


Non-integer sample rate conversion

Information-preserved interpolation first



$$H(e^{j\omega}) = \begin{cases} I, & 0 \le |\omega| \le \min\left(\frac{\pi}{I}, \frac{\pi}{D}\right) \\ 0. & \min\left(\frac{\pi}{I}, \frac{\pi}{D}\right) \le |\omega| \le \pi \end{cases}$$



This approach is only practical when I and D are small integers.

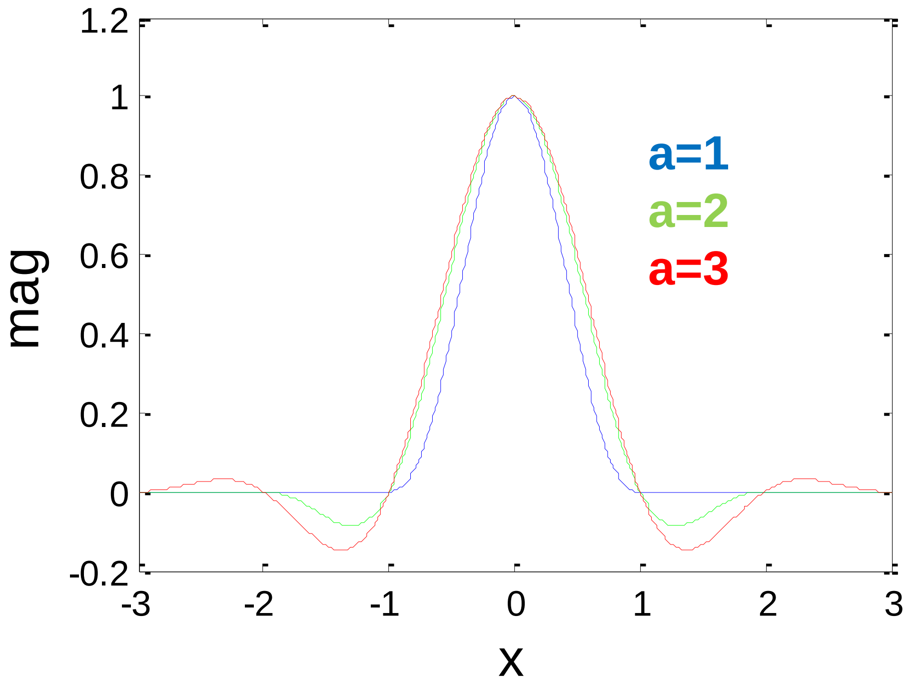
Lanczos resampling

- Ideal sinc function masked by a sinc window
  - Good approximation for sinc function
  - Useful for interpolation, scale-up, scale-down
  - Indexed by parameter **a**

$$L(x) = \begin{cases} sinc(x)sinc(\frac{x}{a}) & \text{if } -a < x < a \\ 0 & \text{otherwise} \end{cases}$$

- For interpolation at α, filter coefficients:

$$h[n] = L(n - \alpha)$$



Multirate identity for downsampling

z transform of downsampling

$$X_D(e^{j\omega}) = \frac{1}{D}\sum_{m=0}^{D-1} X\left(e^{j(\omega-2\pi m)/D}\right)$$

$$y[n] = x[nD] \stackrel{\mathcal{Z}}{\longleftrightarrow} Y(z) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(W_D^k z^{1/D}\right)$$

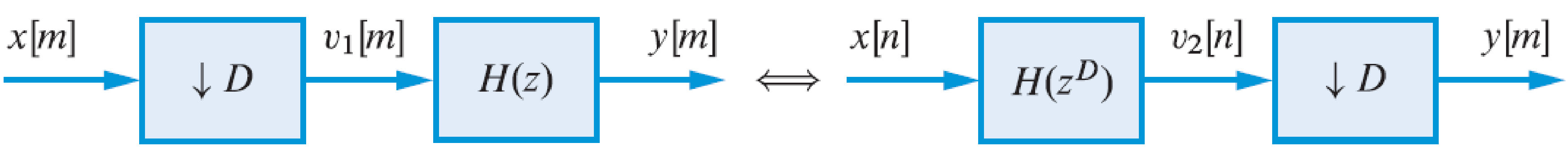
Interchange of filtering with downsampling

$$Y(z) = H(z)V_1(z) = H(z)\frac{1}{D}\sum_{k=0}^{D-1} X\left(z^{1/D} W_D^k\right)$$

$$\Updownarrow$$

$$Y(z) = \frac{1}{D}\sum_{k=0}^{D-1} V_2\left(z^{1/D} W_D^k\right) = H(z)\frac{1}{D}\sum_{k=0}^{D-1} X\left(z^{1/D} W_D^k\right)$$

$$\boxed{W_D^{kD} = 1}$$



Multirate identity for upsampling

z transform of upsampling

$$X_u(e^{j\omega}) = X(e^{j\omega I})$$

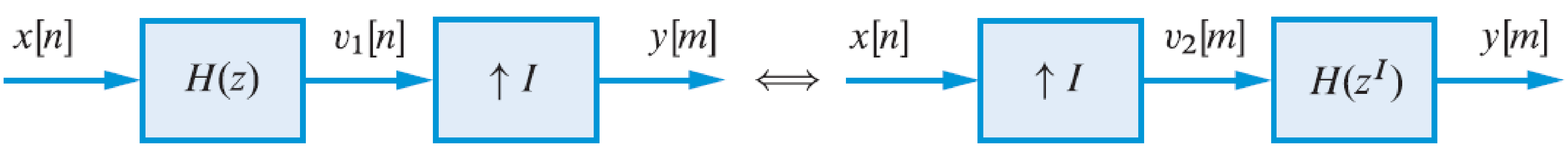
$$y[n] = \begin{cases} x[n/I], & n = 0, \pm I, \pm 2I, \ldots \\ 0. & \text{otherwise} \end{cases} \longleftrightarrow Y(z) = X(z^I)$$

Interchange of filtering with upsampling

$$Y(z) = V_1(z^I) = H(z^I)X(z^I)$$

$$\Updownarrow$$

$$Y(z) = H(z^I)V_2(z) = H(z^I)X(z^I)$$



Polyphase filter structure

Example

$$H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[4]z^{-4} + h[5]z^{-5}$$
$$= \left(h[0] + h[2]z^{-2} + h[4]z^{-4}\right) + z^{-1}\left(h[1] + h[3]z^{-2} + h[5]z^{-4}\right)$$

$$\Updownarrow$$

(M=2)

$$H(z) = P_0\left(z^2\right) + z^{-1}P_1\left(z^2\right)$$

$$P_0(z) \triangleq h[0] + h[2]z^{-1} + h[4]z^{-2}$$
$$P_1(z) \triangleq h[1] + h[3]z^{-1} + h[5]z^{-2}$$

(M=3)

$$H(z) = P_0\left(z^3\right) + z^{-1}P_1\left(z^3\right) + z^{-2}P_2\left(z^3\right)$$

$$P_0(z) \triangleq h[0] + h[3]z^{-1}$$
$$P_1(z) \triangleq h[1] + h[4]z^{-1}$$
$$P_2(z) \triangleq h[2] + h[5]z^{-1}$$

Filter

$$Y(z) = H(z)X(z)$$
$$= P_0(z^3)X(z) + z^{-1}P_1(z^3)X(z) + z^{-2}P_2(z^3)X(z),$$
$$= P_0(z^3)X(z) + z^{-1}\{P_1(z^3)X(z) + z^{-1}[P_2(z^3)X(z)]\}.$$

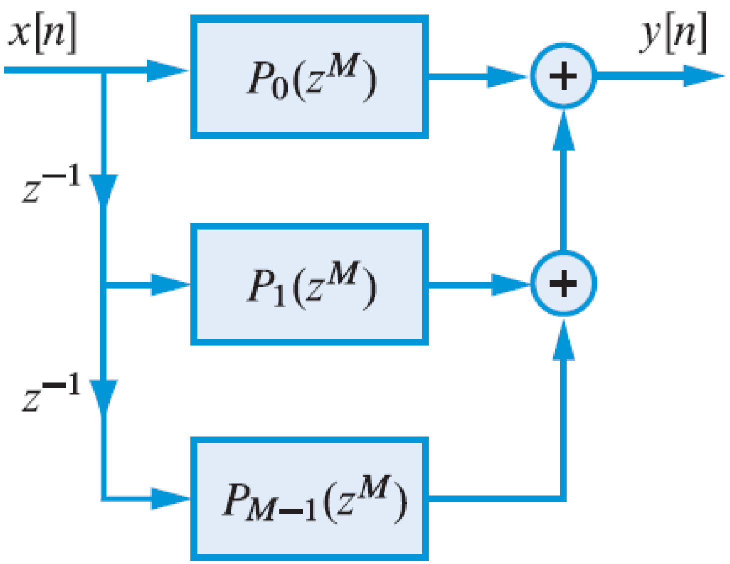
Polyphase filter structure

General case

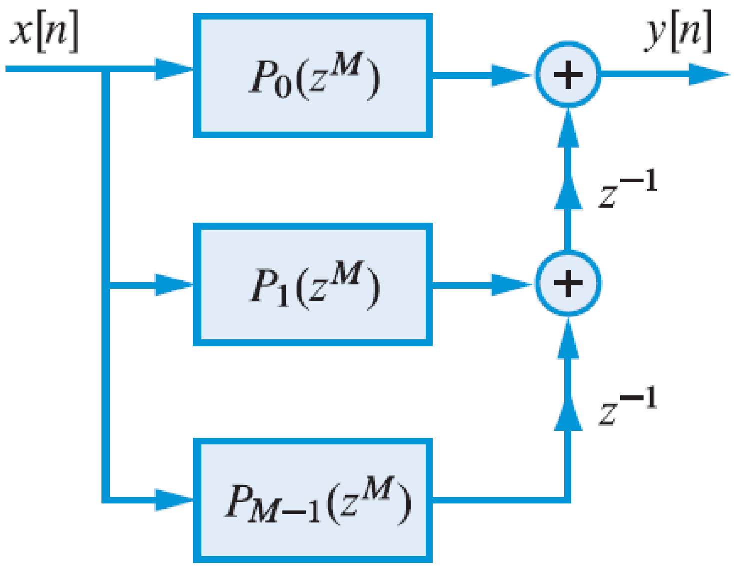
$$H(z) = \sum_{k=0}^{M-1} z^{-k} P_k\left(z^M\right)$$

$$p_k[n] \triangleq h[nM+k], \quad k = 0, 1, \ldots, M-1.$$

Realization



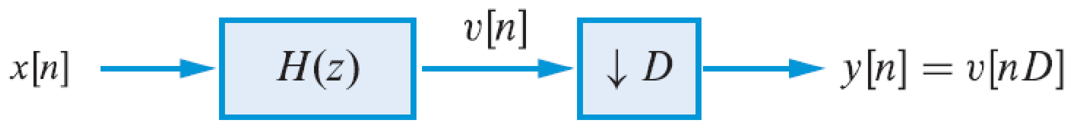
Direct form



Transposed form

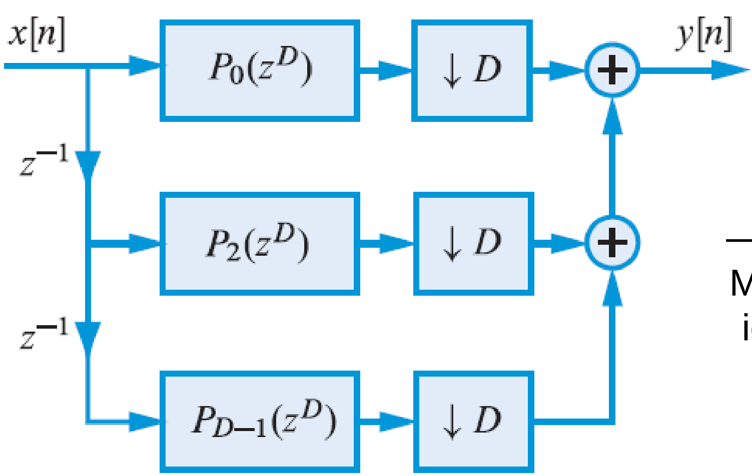
Polyphase decimator

Decimation system

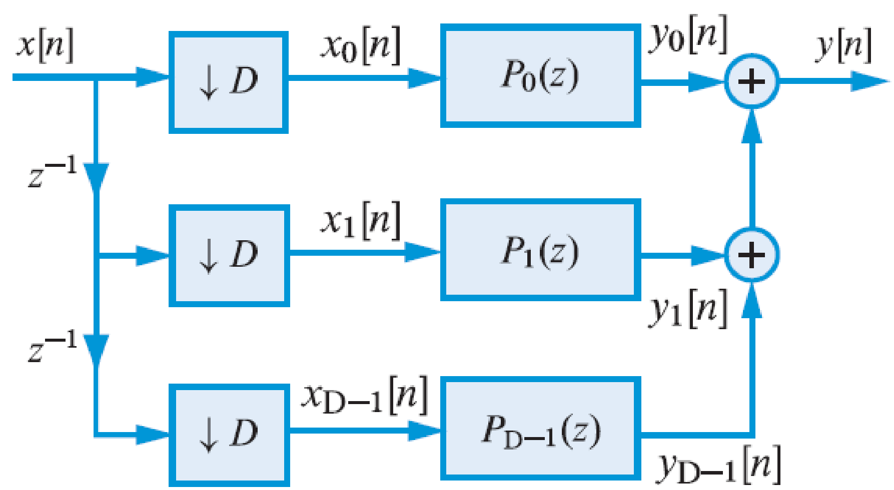


$$H(z) = \sum_{k=0}^{N-1} h[k]z^{-k} = \sum_{k=0}^{D-1} P_k(z^D)z^{-k}$$

Polyphase implementation

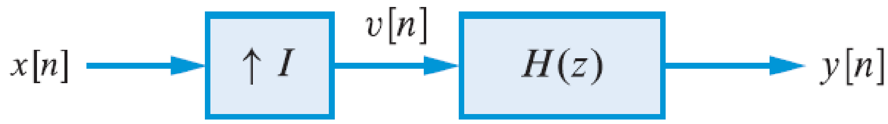


Multirate identity



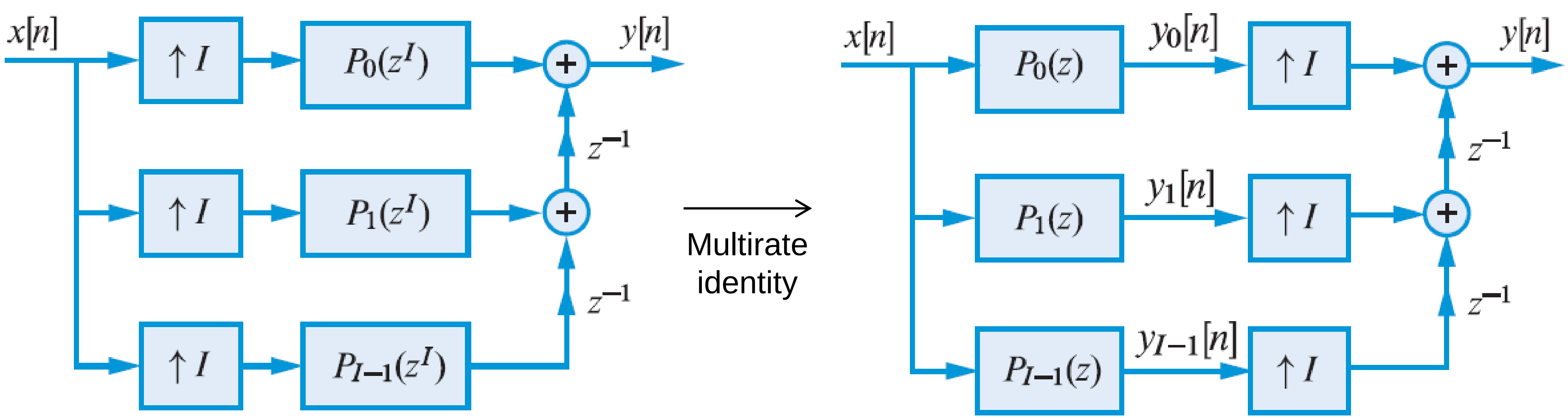
Polyphase interpolator

Interpolation system



$$H(z) = \sum_{k=0}^{N-1} h[k] z^{-k} = \sum_{k=0}^{I-1} P_k(z^I) z^{-k}$$

Polyphase implementation



Half-band filters

Ideal half-band filter
(noncausal zero-phase)

$$h[n] = \frac{\omega_c}{\pi} \frac{\sin \omega_c (n-\alpha)}{\omega_c (n-\alpha)}\Bigg|_{\omega_c=\pi/2} = \begin{cases} 1/2, & n=\alpha \\ 0. & n-\alpha = \pm 2, \pm 4, \ldots \end{cases}$$

General half-band filter
(noncausal zero-phase)

$$h[0] = 1/2, \quad h[2n] = 0. \quad n = \pm 1, \pm 2, \ldots$$

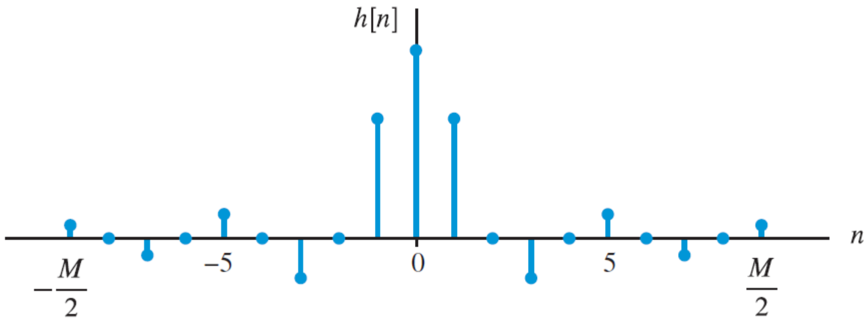
$$h[-n] = h[n] \text{ or } H\left(e^{-j\omega}\right) = H\left(e^{j\omega}\right)$$

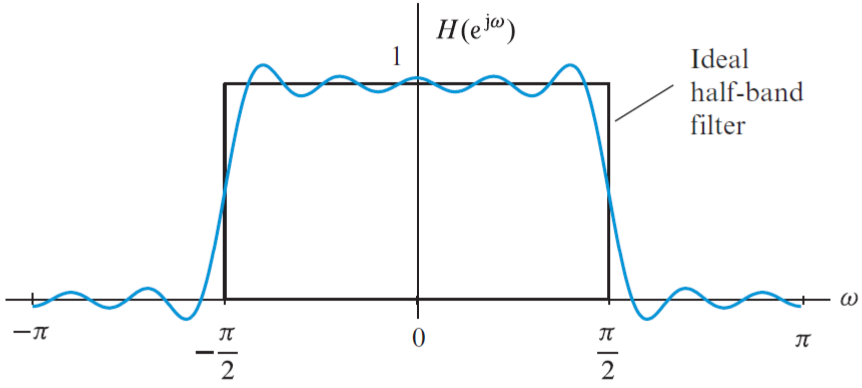
$$H(z) = P_0\left(z^2\right) + z^{-1}P_1\left(z^2\right) = \tfrac{1}{2} + z^{-1}P_1\left(z^2\right)$$

Property

$$H(z) + H(-z) = 1 \qquad H\left(e^{j\omega}\right) + H\left(e^{j(\omega-\pi)}\right) = 1$$

FIR





Half-band FIR design

1. Given the specifications ω_s, A_p, and A_s of the half-band filter, obtain the parameters δ_p, δ_s, and ω_s so that they satisfy the design requirements and the constraints of the half-band filter. That is,

$$\delta \triangleq \min\left(\delta_p, \delta_s\right), \quad \omega_p = \pi - \omega_s. \tag{12.92}$$

2. Design a single band Type II FIR filter $G(z)$ of order $M/2 = 2p - 1$ (odd) with $\tilde{\omega}_p = 2\omega_p$, $\tilde{\omega}_s = \pi$, and $\tilde{\delta} = 2\delta$ using the Parks–McClellan algorithm. Since $G(z)$ is Type II, the frequency response $G(e^{j\omega})$ is equal to zero at $\omega = \pi$.

can be replaced by any filter design method, e.g. windowing

3. Scale the impulse response $g[n]$ by one-half, upsample the result by a factor of two, and set the middle coefficient to $1/2$. The result is an impulse response $h[n]$ with system function $H(z)$ given by

$$H(z) = \frac{1}{2}\left[z^{-M/2} + G(z^2)\right], \tag{12.93}$$

which is a half-band filter with passband cutoff frequency ω_p.

Design example of a half-band filter

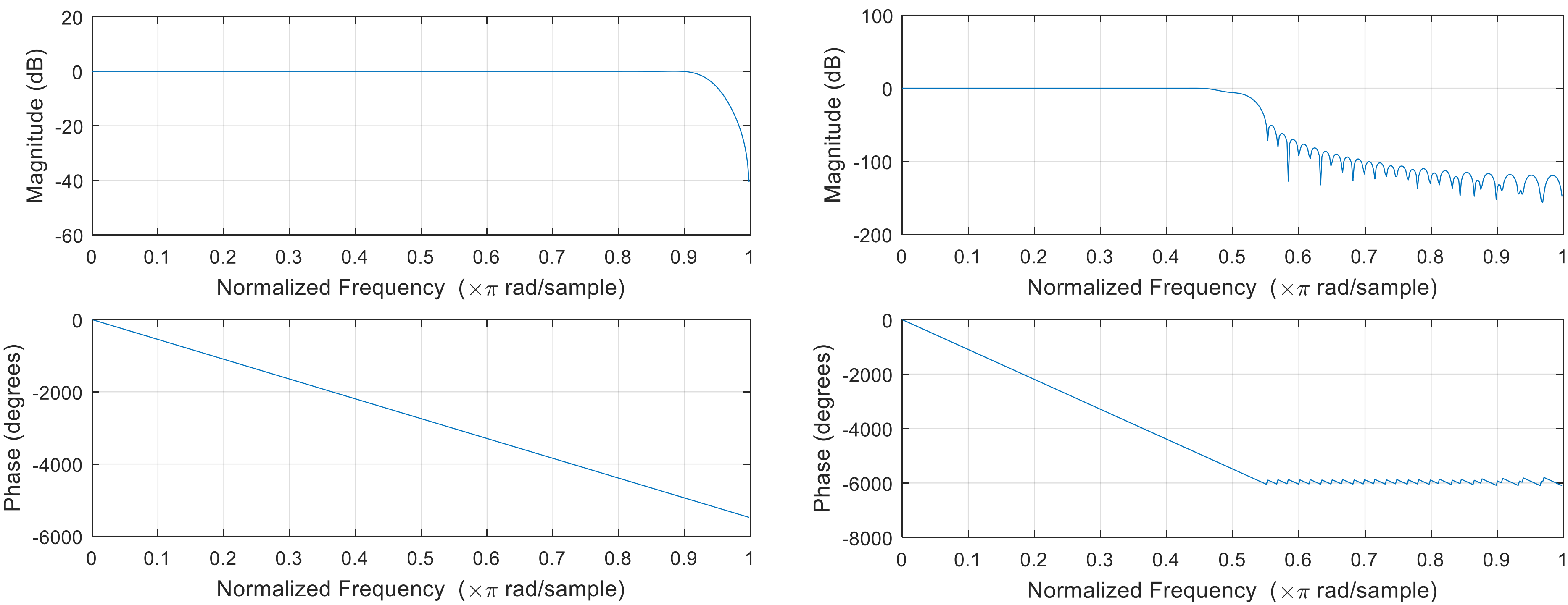
Specification (fixed window)

$\omega_s = 0.55\pi$ → $\omega_p = 0.45\pi$ → $\widetilde{\omega}_p = 0.9\pi$ → $\Delta\widetilde{\omega} = 0.1\pi$

$A_s = 50$ dB → $\tilde{A}_s = 44$ dB → Hann window

$\widetilde{\omega}_c = 0.95\pi$ ← $\Delta\widetilde{\omega} = 0.1\pi$ → $L = 62$ → Hann window

$G(z)$ $H(z)$



Kth-band or Nyquist filters

General Kth-band filter
(noncausal zero-phase)

$$h[n] = \begin{cases} 1/K, & n = 0 \\ 0. & n = \pm K, \pm 2K, \ldots \end{cases}$$

$$H(z) = \frac{1}{K} + \sum_{k=1}^{K-1} z^{-k} P_k(z^K)$$

$$\omega_c = \pi/K$$

$$h[-n] = h[n] \text{ or } H(e^{-j\omega}) = H(e^{j\omega})$$

Property

$$\sum_{k=0}^{K-1} H(zW_K^k) = 1$$

$$\sum_{k=0}^{K-1} H(e^{j(\omega - 2\pi k/K)}) = 1$$

Multistage decimation and interpolation

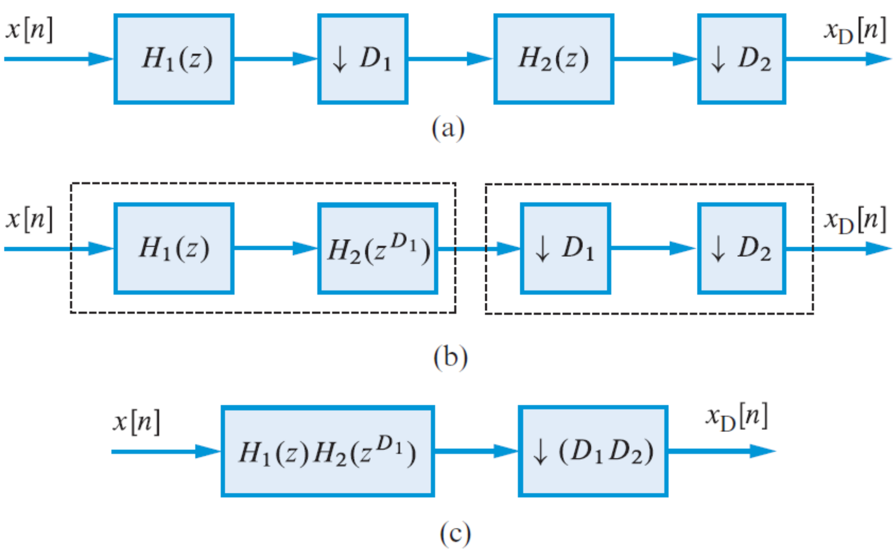
Issue of single-stage systems

Large decimation factors D (or interpolation factors I) require narrow passbands and thus demand smaller transition bands, which results in long FIR filters.

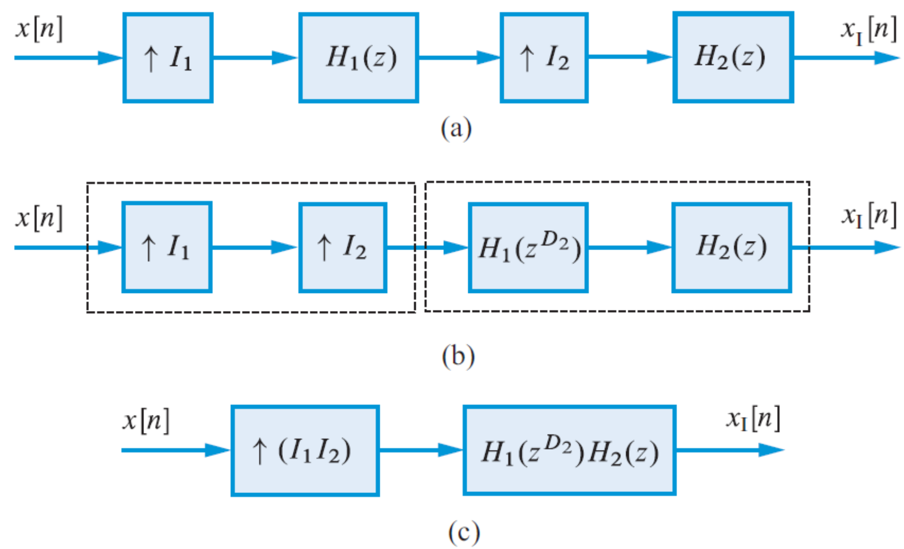
Two-stage systems

Provide shorter filters and require less computation.

Decimation



Interpolation



$$H(z) = H_1(z)H_2(z^{D_1})$$

Example of large decimation factor

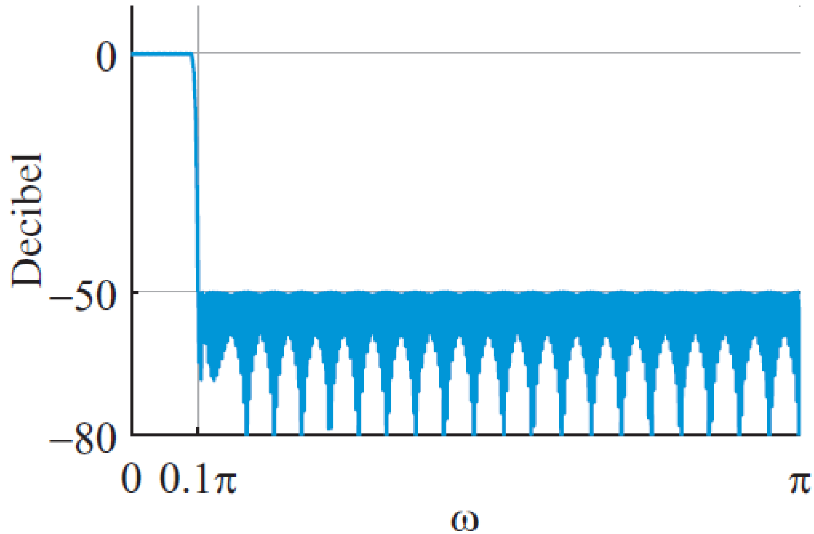
Single-stage decimation

High original sampling rate $F_H = 100 \text{ Hz}$

Reduced sampling rate $F_L = 10 \text{ Hz}$

$D = 10 \qquad \omega_c = \pi / D = 0.1\pi$

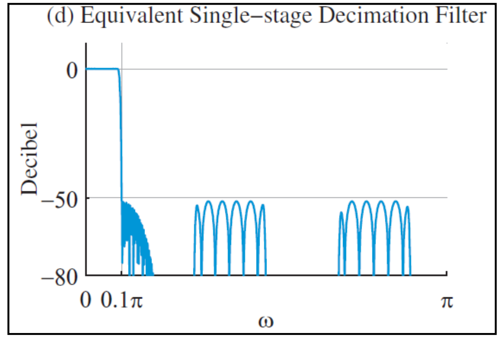
(a) Single–stage Decimation Filter H(z)



$M = 489$

Computation complexity $C_D = (M + 1) F_H / D = 4900 \text{ mults/s}$

Example of large decimation factor



$$C_1 + C_2 = 2080 \text{ mults/s}$$

Two-stage decimation

$$D_1 = 5$$

$$M_1 = 49$$

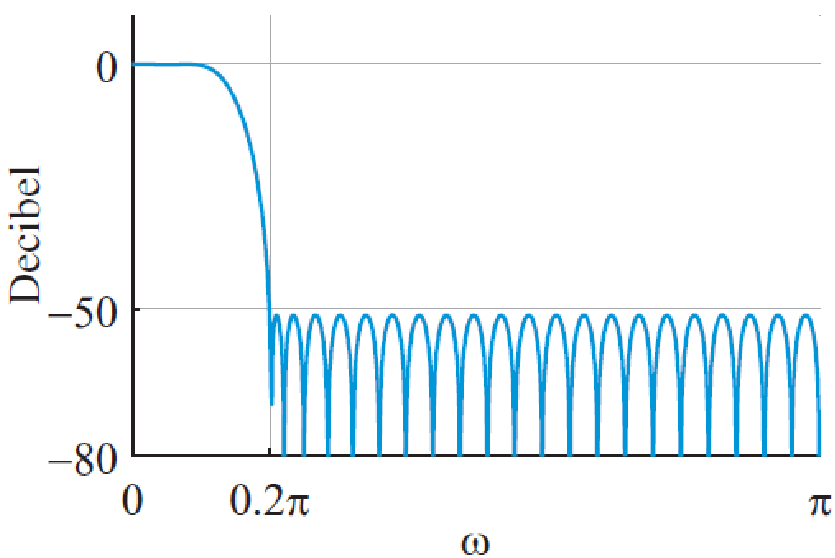
$$C_1 = (M_1 + 1)F_H / D_1 = 1000$$

$$D_2 = 2$$

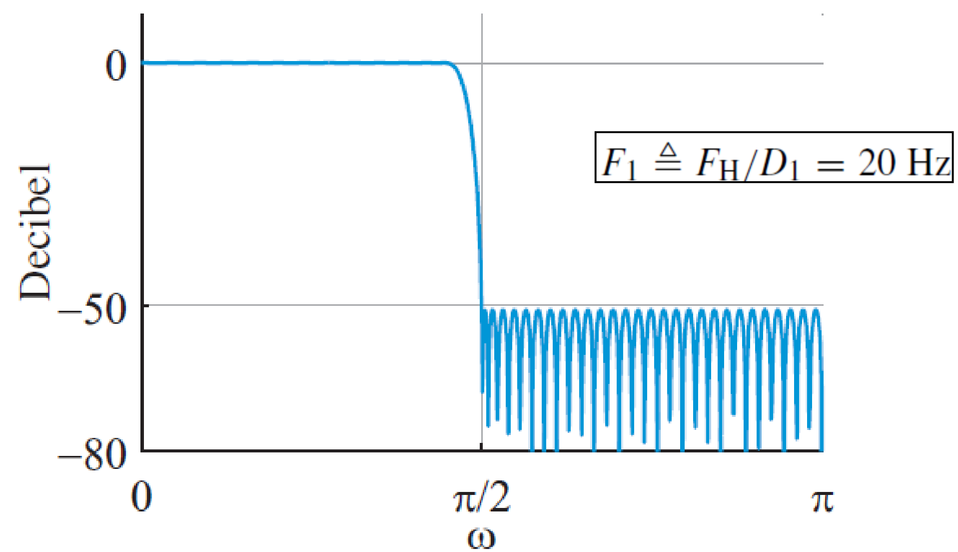
$$M_2 = 107$$

$$C_2 = (M_2 + 1)F_1 / D_2 = 1080$$

(b) Two–stage Decimation Filter $H_1(z)$



(c) Two–stage Decimation Filter $H_2(z)$

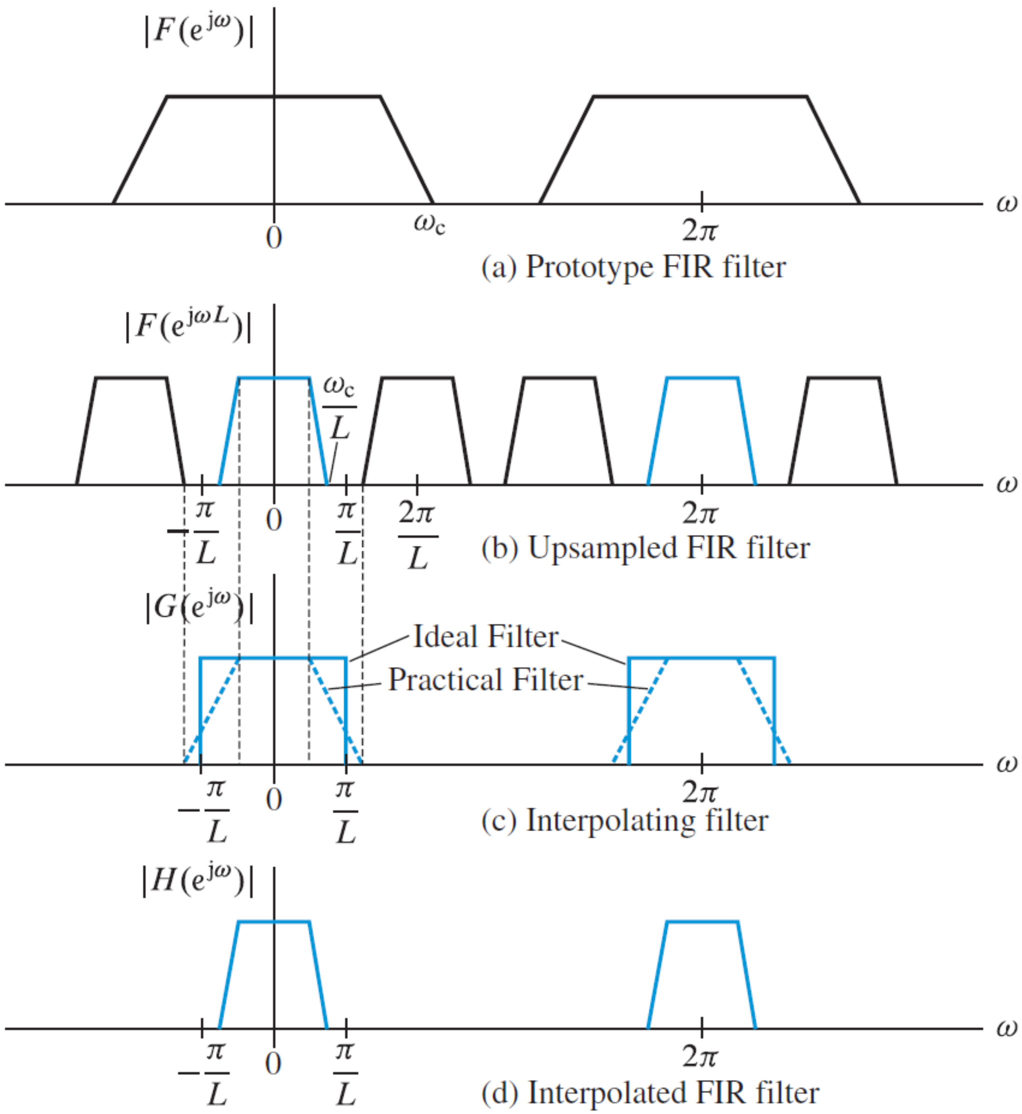


Interpolated FIR (IFIR) filters

$$f_L[n] = \begin{cases} f[n], & n = 0, \pm L, \pm 2L, \ldots \\ 0, & \text{otherwise} \end{cases}$$

$$F_L(e^{j\omega}) = F(e^{j\omega L})$$

$$H(z) = F(z^L)G(z)$$



Example of large decimation factor

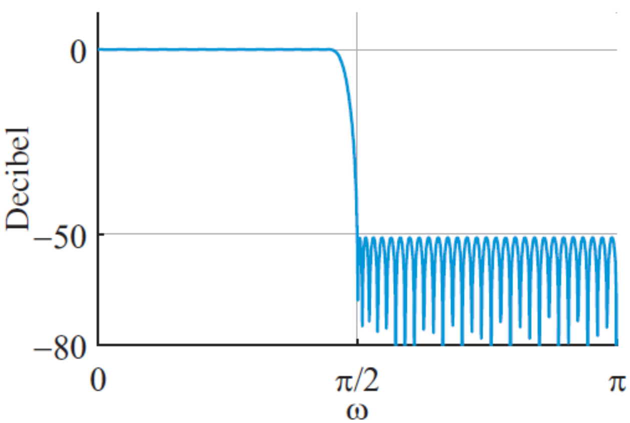
Interpolated FIR

$$L = 5$$

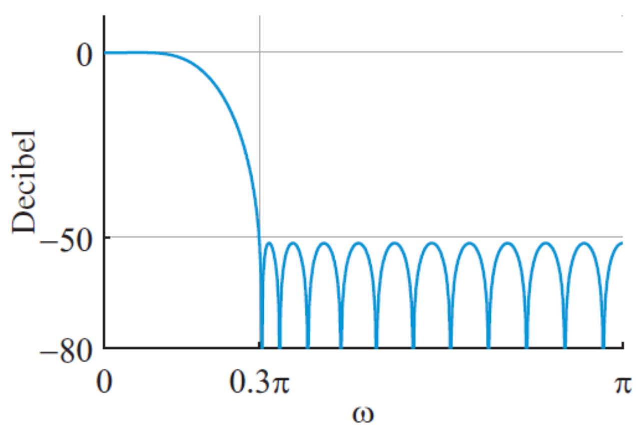
$$\omega_{pF} = \omega_p L = 0.45\pi \qquad \omega_p = 0.09\pi,\ \omega_{sG} = 0.3\pi$$

$$L_F = 108 \qquad L_G = 27 \qquad L_G + L_F = 135$$

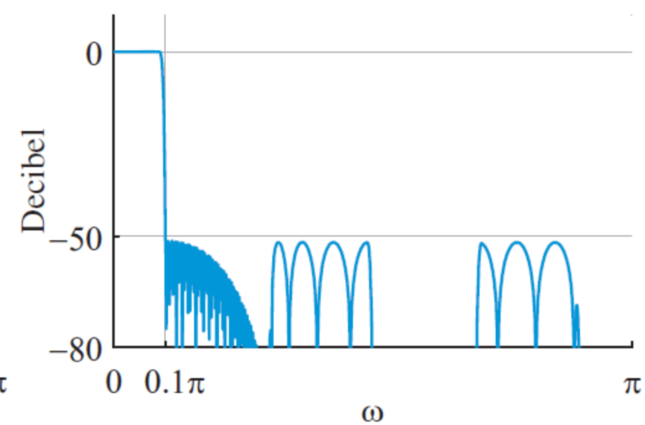
(c) Filter $F(z)$ Response



(b) Filter $G(z)$ Response



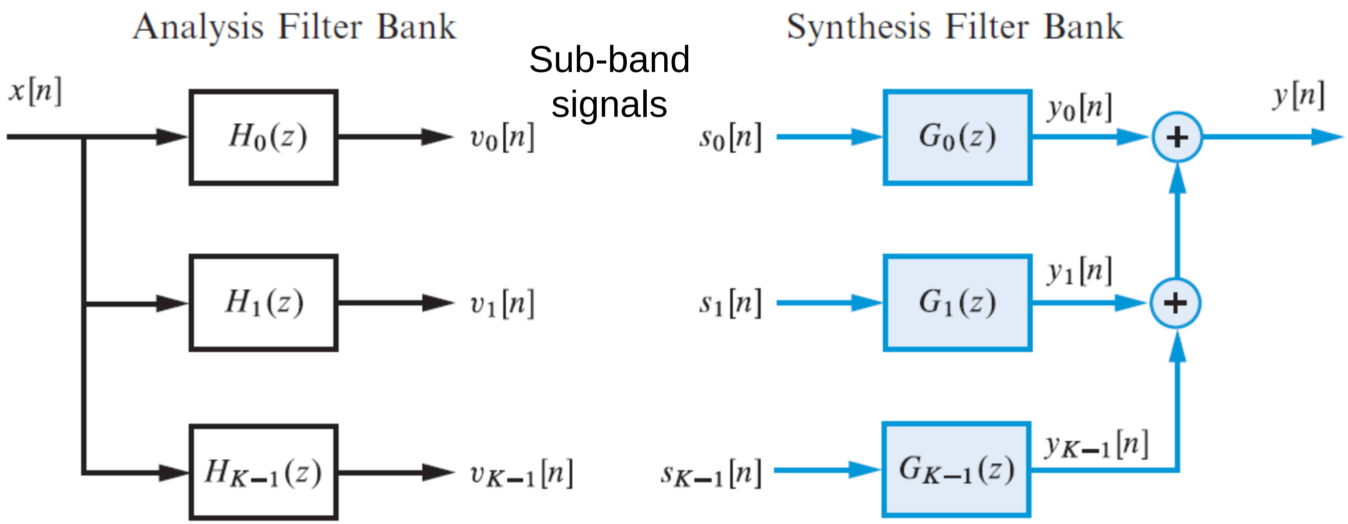
(d) IFIR Filter Response



$C = 1350$ mult/s

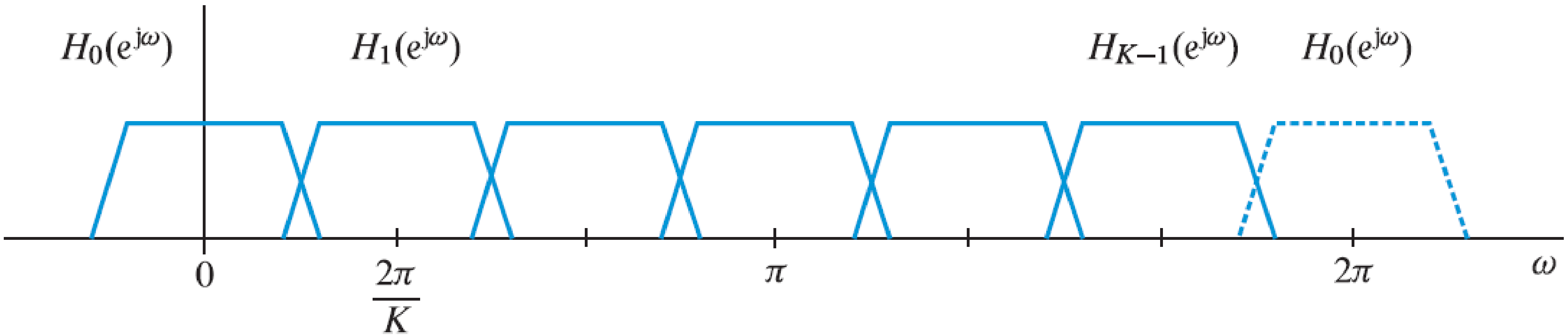
Filter bank

Filter banks

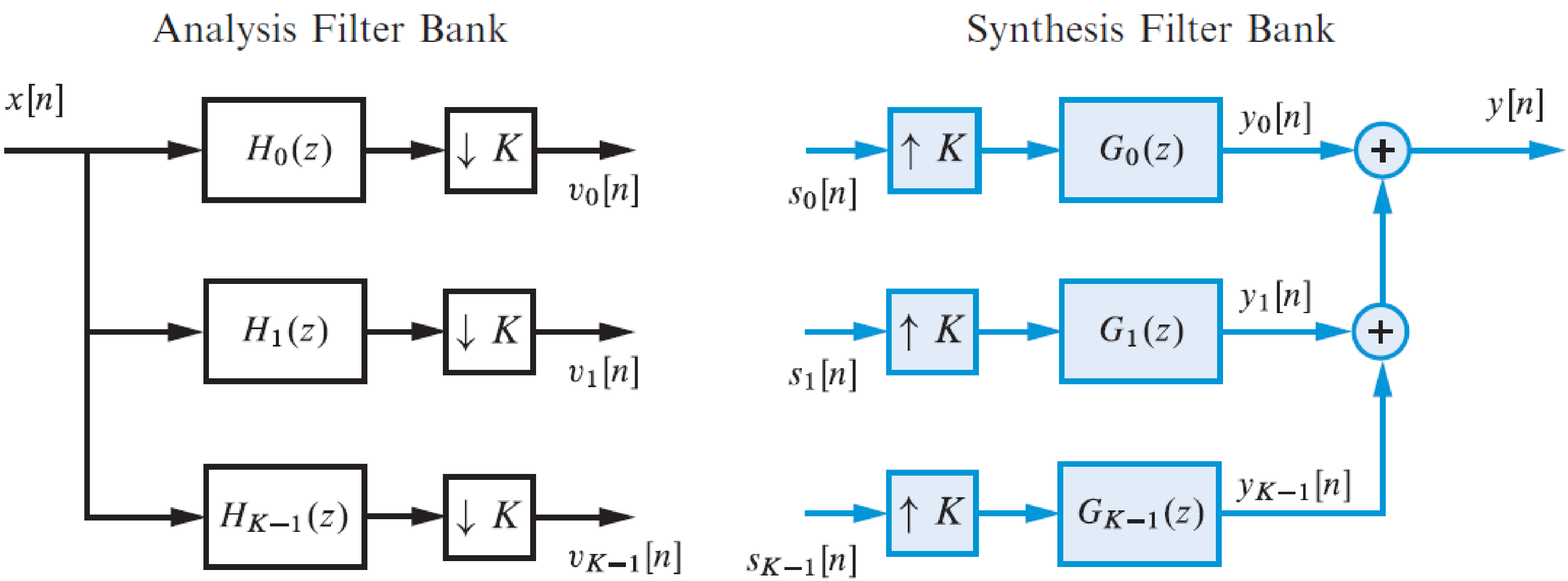


Uniform DFT Filter bank

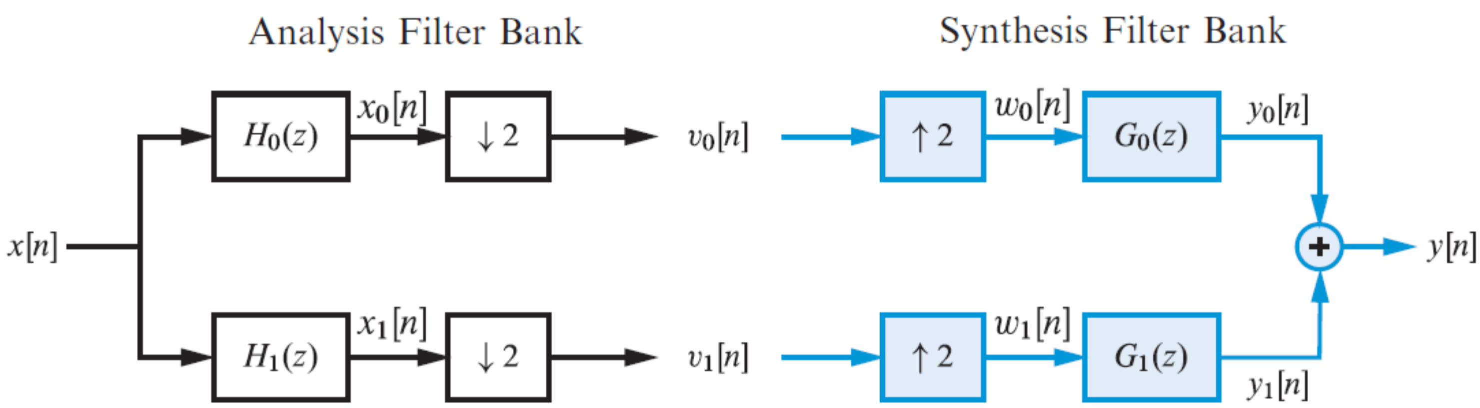
$$H_k(z) = \sum_{n=0}^{\infty} h_k[n] z^{-n} = \sum_{n=0}^{\infty} h[n]\left(zW_K^k\right)^{-n} = H\left(zW_K^k\right)$$



Maximally decimated multirate filter bank



Two-channel filter bank



$$V_0(z) = \frac{1}{2}H_0(z^{1/2})X(z^{1/2}) + \frac{1}{2}H_0(-z^{1/2})X(-z^{1/2}),$$

$$Y_0(z) = V_0(z^2)G_0(z),$$

$$Y_0(z) = \frac{1}{2}[H_0(z)X(z) + H_0(-z)X(-z)]G_0(z).$$

$$Y_1(z) = \frac{1}{2}[H_1(z)X(z) + H_1(-z)X(-z)]G_1(z)$$

Transfer term — Aliasing term

$$Y(z) = \frac{1}{2}[\underline{T(z)X(z)} + \underline{A(z)X(-z)}]$$

$$T(z) \triangleq H_0(z)G_0(z) + H_1(z)G_1(z),$$

$$A(z) \triangleq H_0(-z)G_0(z) + H_1(-z)G_1(z).$$

Perfect reconstruction

Condition

$$A(z) = H_0(-z)G_0(z) + H_1(-z)G_1(z) = 0,$$

$$T(z) = H_0(z)G_0(z) + H_1(z)G_1(z) = Gz^{-n_D},$$

$$\begin{bmatrix} H_0(z) & H_1(z) \\ H_0(-z) & H_1(-z) \end{bmatrix} \begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix} = \begin{bmatrix} Gz^{-n_D} \\ 0 \end{bmatrix}$$

Solution

$$G_0(z) = \frac{2z^{-n_D}}{\Delta_m(z)} H_1(-z), \quad G_1(z) = -\frac{2z^{-n_D}}{\Delta_m(z)} H_0(-z),$$

$$\Delta_m(z) = H_0(z)H_1(-z) - H_0(-z)H_1(z)$$

$\Delta_m(z) \neq 0$

Product filter

$$R(z) \triangleq z^{n_D} H_0(z)G_0(z) = \frac{2}{\Delta_m(z)} H_0(z)H_1(-z)$$

$$R(-z) = \frac{-2}{\Delta_m(z)} H_0(-z)H_1(z) = z^{n_D} H_1(z)G_1(z)$$

$\Delta_m(-z) = -\Delta_m(z)$

$$R(z) + R(-z) = 2$$

(Gain $G = 2$)

Two-channel perfect reconstruction filter bank

Necessary condition

$$R(z) + R(-z) = 2$$

⇩

$$R(z) = 1 + z^{-1}R_1(z^2)$$

R(z) must be a half-band filter

$$R(z) = R_0(z^2) + z^{-1}R_1(z^2)$$

$$R_0(z^2) + z^{-1}R_1(z^2) + R_0(z^2) - z^{-1}R_1(z^2) = 2$$

Proof

Additional conditions

Orthogonal

$$R(z) = H(z)H(z^{-1})$$

R(z) is an autocorrelation sequence

Bi-orthogonal

$$R(z) = H_0(z)G_0(z)$$

R(z) is a correlation sequence

Perfect reconstruction FIR filter bank: Conjugate quadrature filter (CQF)

CQF

$$H_0(z) = H(z),$$
$$H_1(z) = -z^{-M} H\left(-z^{-1}\right).$$

Conjugate:
$H\left(e^{-j\omega}\right) = H^*\left(e^{j\omega}\right)$ for real-valued h

Quadrature:
ω shifted by π $\Rightarrow$ mirrored around $\pi/2$

FIR condition
(M is odd)

$$\Delta_m(z) = H_0(z)H_1(-z) - H_0(-z)H_1(z)$$
$$= z^{-M}\left[H(z)H\left(z^{-1}\right) + H(-z)H\left(-z^{-1}\right)\right]$$

$$\Rightarrow \quad H(z)H\left(z^{-1}\right) + H(-z)H\left(-z^{-1}\right) = 1$$

$$|H(e^{j\omega})|^2 + |H(e^{j(\omega-\pi)})|^2 = 1$$

Power complementary

$$|H_0(e^{j\omega})|^2 + |H_1(e^{j\omega})|^2 = 1$$

Synthesis filters

$$G_0(z) = 2H_1(-z) = -2z^{-M}H\left(z^{-1}\right),$$
$$G_1(z) = -2H_0(-z) = -2H(-z).$$

$$\Rightarrow \quad R(z) = z^{n_D} H_0(z) G_0(z)$$
$$= 2H(z)H\left(z^{-1}\right)$$
$$R(z) + R(-z) = 2$$

Properties of CQF

R(z) as autocorrelation sequence

$$r_h[n] = h[n] * h[-n] \xleftrightarrow{\mathcal{Z}} R_h(z) = H(z)H(z^{-1})$$

$$R_h(e^{j\omega}) = \left|H(e^{j\omega})\right|^2 \geq 0$$

Orthogonal filter

$$\sum_{k=0}^{M} h[k]h[k+2n] = 0. \quad n \neq 0$$

∵Half-band R(z)

CQFs

$$h_0[n] = h[n] \xleftrightarrow{\text{DTFT}} H_0(e^{j\omega}) = H(e^{j\omega}),$$

$$h_1[n] = (-1)^n h[M-n] \xleftrightarrow{\text{DTFT}} H_1(e^{j\omega}) = -H(e^{-j(\omega-\pi)})e^{-j\omega M},$$

$$g_0[n] = h[M-n] \xleftrightarrow{\text{DTFT}} G_0(e^{j\omega}) = 2H(e^{-j\omega})e^{-j\omega M},$$

$$g_1[n] = -(-1)^n h[n] \xleftrightarrow{\text{DTFT}} G_1(e^{j\omega}) = -2H(e^{-j(\omega-\pi)}).$$

Design procedure of a CQF bank

1. Design a lowpass zero-phase half-band FIR filter $R_0(z)$ of order $2M$, where the number M *must* be an *odd* integer (see Section 12.3.1).
2. If the minimum value $\delta_{\min}$ of the real and even function $R_0(e^{j\omega})$ is negative, form a nonnegative function as

$$R_+(e^{j\omega}) = R_0(e^{j\omega}) + |\delta_{\min}| \geq 0. \tag{12.130}$$

   This is equivalent to adding the value $|\delta_{\min}|$ to the sample $r_0[0]$, that is,

$$r_+[n] = r_0[n] + |\delta_{\min}|\,\delta[n]. \tag{12.131}$$

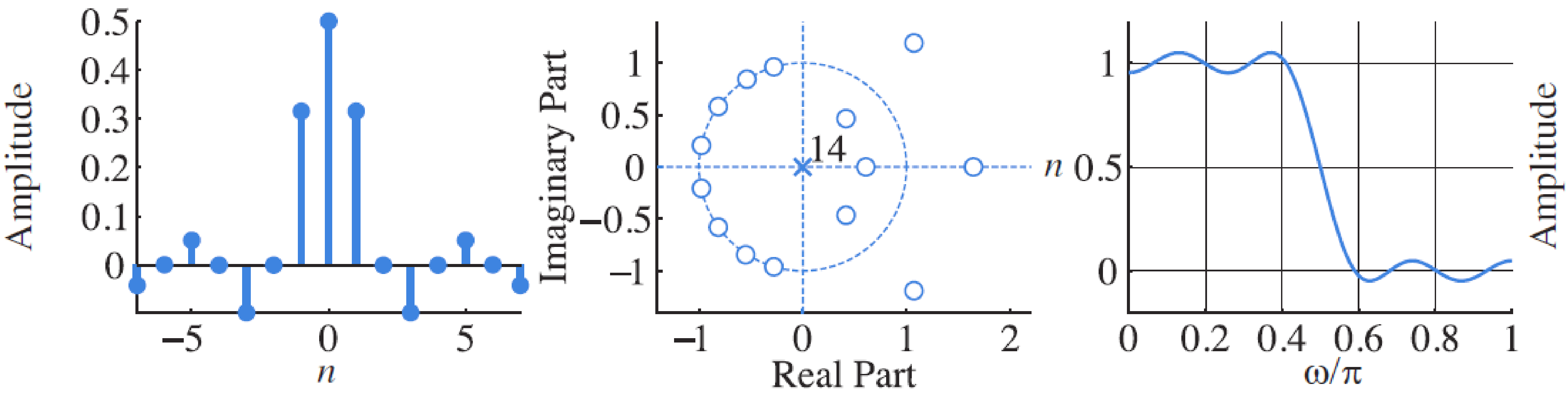
3. Scale $R_+(z)$ so that the frequency response is equal to $1/2$ at $\omega = \pi/2$,

$$R(z) = \frac{1/2}{1/2 + |\delta_{\min}|} R_+(z). \tag{12.132}$$

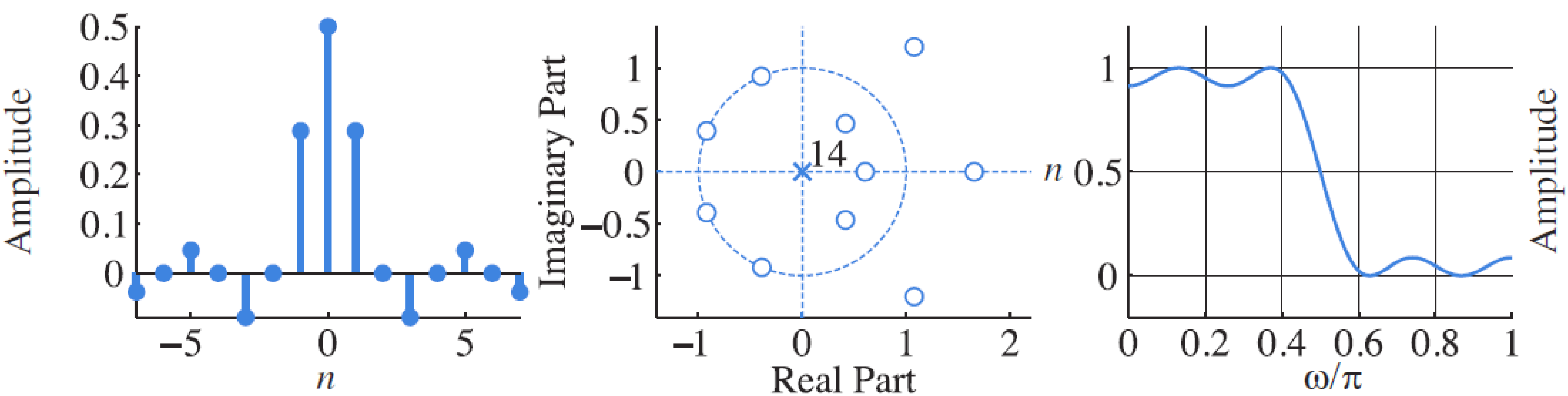
4. Determine the minimum-phase filter $H(z)$ by solving the spectral factorization problem $R(z) = H(z)H(z^{-1})$ (see Section 5.8).
5. Specify the remaining filters of the bank using (12.115b) and (12.120).

Example of a CQF bank (1/2)

half-band FIR filter $R_0(z)$ $M = 7$ Parks–McClellan algorithm

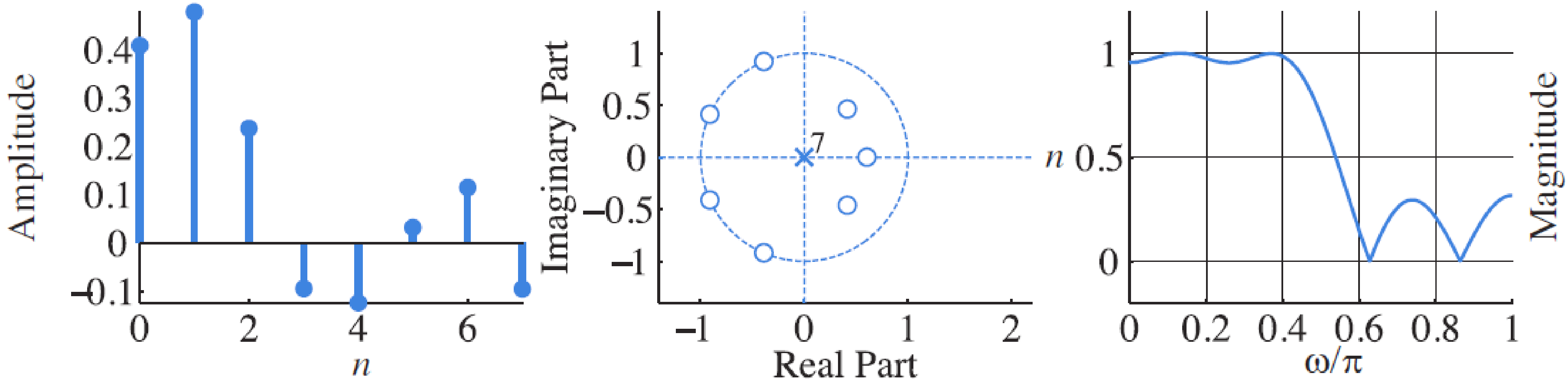


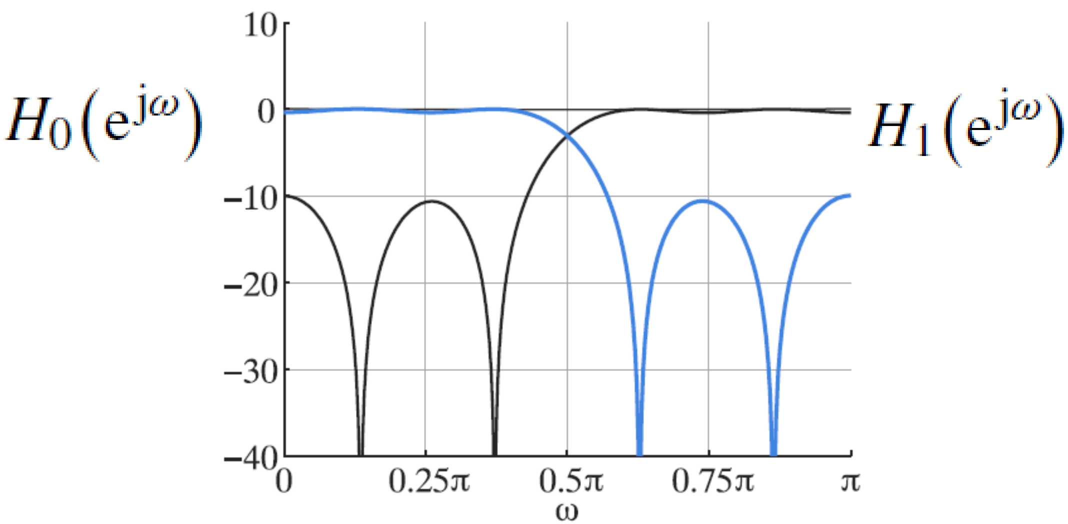
$R_+(z)$



Example of a CQF bank (2/2)

$H(z)$ Minimum-phase (**not linear-phase**)





Perfect reconstruction FIR filter bank: Quadrature mirror filter (QMF)

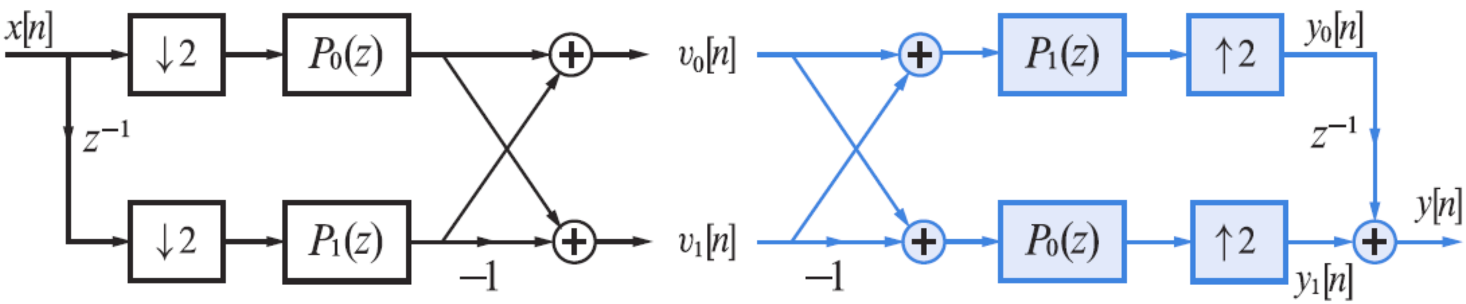
Quadrature:
ω shifted by π ⇒ mirrored around $\pi/2$

QMF

$$H_0(z) = H(z), \quad H_1(z) = H(-z),$$
$$G_0(z) = H(z), \quad G_1(z) = -H(-z).$$

Poly-phase implementation

$$H(z) = P_0(z^2) + z^{-1} P_1(z^2)$$



Aliasing-free

$$A(z) = H_0(-z)G_0(z) + H_1(-z)G_1(z)$$
$$= H(-z)H(z) - H(z)H(-z) = 0.$$

Design H(z) directly for QMF

Transfer function

$$T(z) = H_0(z)G_0(z) + H_1(z)G_1(z)$$
$$= H^2(z) - H^2(-z).$$

$$T(z) = \Delta_m(z) = 2z^{-1}P_0(z^2)P_1(z^2)$$

PR is not practical

$$P_0(z) = b_0 z^{-n_0}$$
$$P_1(z) = b_1 z^{-n_1}$$
$$\Rightarrow \quad H(z) = b_0 z^{-2n_0} + b_1 z^{-(2n_1+1)}$$

Linear-phase H(z)
(M is odd)

$$H\left(\mathrm{e}^{\mathrm{j}\omega}\right) = A\left(\mathrm{e}^{\mathrm{j}\omega}\right)\mathrm{e}^{-\mathrm{j}\omega M/2}$$

$$\Rightarrow \quad T\left(\mathrm{e}^{\mathrm{j}\omega}\right) = \mathrm{e}^{-\mathrm{j}\omega M}\left[\left|H\left(\mathrm{e}^{\mathrm{j}\omega}\right)\right|^2 + \left|H\left(\mathrm{e}^{\mathrm{j}(\omega-\pi)}\right)\right|^2\right]$$

Design criterion by Johnston

$$J = \alpha \int_{\omega_s}^{\pi} \left|H\left(\mathrm{e}^{\mathrm{j}\omega}\right)\right|^2 \mathrm{d}\omega + (1-\alpha)\int_0^{\pi}\left(1 - \left|T\left(\mathrm{e}^{\mathrm{j}\omega}\right)\right|^2\right)\mathrm{d}\omega$$

Example of a QMF bank