Chap10
Design of FIR Filters

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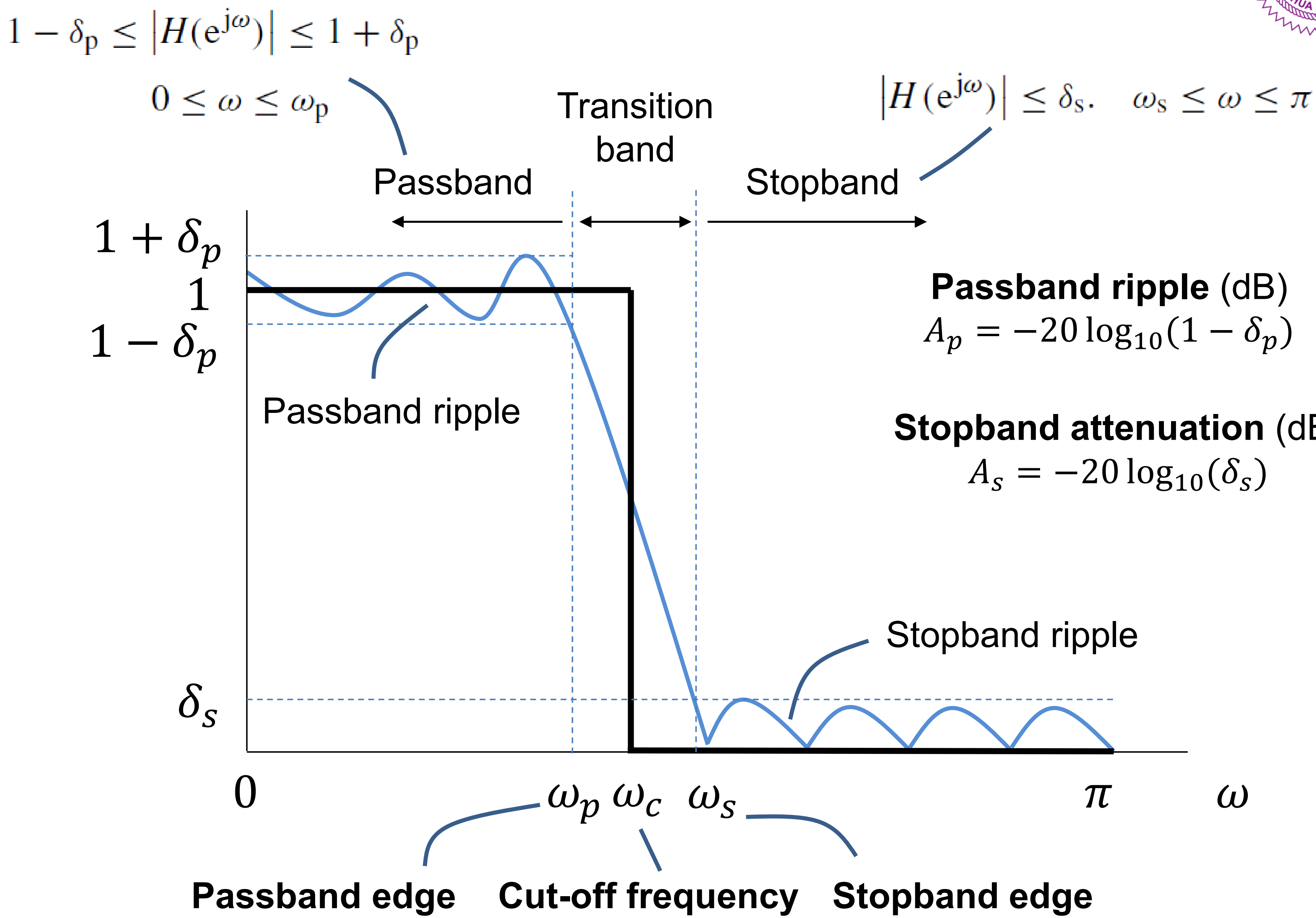
Chap 10 Design of FIR filters

- 10.1 The filter design problem
- 10.2 FIR filters with linear phase
- 10.3 Design of FIR filters by windowing
- 10.4 Design of FIR filters by frequency sampling

Practical filter design problem

- **Specification**
  - Stopband/passband ripple, cut-off frequency, transition band width (10.1)
  - Linear phase (10.2)
- **Approximation**
  - Windowing (10.3) on **ideal** filters
  - Frequency sampling (10.4) on **DTFT** of target filters
  - Chebyshev minimax (10.5-10.6)
- Quantization
- Verification
- Implementation
  - SFG structures (chap 9)

Magnitude/Amplitude Specifications



Passband ripple (dB)

$$A_p = -20\log_{10}(1-\delta_p)$$

Stopband attenuation (dB)

$$A_s = -20\log_{10}(\delta_s)$$

(Recap) Magnitude and phase responses cannot be specified independently

Assume

$$R(z) = H(z)H^*(1/z^*), \qquad \text{complex } h[n]$$
$$= H(z)H(1/z). \qquad \text{real } h[n]$$
$$\Downarrow$$
$$R(z)|_{z=e^{j\omega}} = |H(e^{j\omega})|^2 = H(e^{j\omega})H^*(e^{j\omega})$$

Consider

$$H_1(z) = \left(1 - az^{-1}\right)\left(1 - bz^{-1}\right),$$
$$H_2(z) = \left(1 - az^{-1}\right)(1 - bz),$$
$$H_3(z) = (1 - az)\left(1 - bz^{-1}\right),$$
$$H_4(z) = (1 - az)(1 - bz).$$

Have the same magnitude response since their R(z) are identical

$$R(z) = H(z)H(1/z) = \left(1 - az^{-1}\right)\left(1 - bz^{-1}\right)(1 - az)(1 - bz)$$

Constraints for causal and stable filters

Theorem 1 (Paley–Wiener): If $h[n]$ has finite energy and $h[n] = 0$ for $n < 0$, then

$$\int_{-\pi}^{\pi} \left| \ln \left| H(e^{j\omega}) \right| \right| d\omega < \infty. \quad (10.11)$$

Conversely, if $|H(e^{j\omega})|$ is square integrable and the integral (10.11) is finite, then we can obtain a phase response $\angle H(e^{j\omega})$ so that the filter $H(e^{j\omega}) = |H(e^{j\omega})| \times e^{j\angle H(e^{j\omega})}$ is causal; the solution $\angle H(e^{j\omega})$ is unique if $H(z)$ is minimum phase. A proof of this theorem and its implications are discussed in Papoulis (1977).

Frequency response cannot be zero over any finite band
⇒ Any stable ideal filter must be non-causal

Given magnitude response, we cannot assign phase response arbitrarily
⇒ 1. **Impose linear phase constraint** $\angle H(e^{j\omega}) = -\alpha\omega$
⇒ 2. Simply disregard the phase response

Constraints for real, causal and stable filters

$$h[n] = h_{\mathrm{e}}[n] + h_{\mathrm{o}}[n]$$

$$h_{\mathrm{e}}[n] = \tfrac{1}{2}(h[n] + h[-n]),$$

$$h_{\mathrm{o}}[n] = \tfrac{1}{2}(h[n] - h[-n]).$$

$\downarrow$ h[n] is causal

$$h[n] = 2h_{\mathrm{e}}[n]u[n] - h_{\mathrm{e}}[0]\delta[n]$$

$\downarrow$ h[n] is absolutely summable

$$H(\mathrm{e}^{\mathrm{j}\omega}) = H_{\mathrm{R}}(\mathrm{e}^{\mathrm{j}\omega}) + \mathrm{j}H_{\mathrm{I}}(\mathrm{e}^{\mathrm{j}\omega})$$

$H_{\mathrm{R}}(\mathrm{e}^{\mathrm{j}\omega})$ is the DTFT of $h_{\mathrm{e}}[n]$

$\downarrow$

$$H_R(e^{j\omega}) \Leftrightarrow h_e[n] \Leftrightarrow h[n]$$

Optimality criteria for filter design

- **Minimum mean-squared-error (MMSE) approximation**
  - Interval of interest $\mathcal{B}$: usually union of passbands and stopbands

$$E_2 \triangleq \left[\frac{1}{2\pi} \int_{\mathcal{B}} \left| H_\mathrm{d}\left(\mathrm{e}^{\mathrm{j}\omega}\right) - H\left(\mathrm{e}^{\mathrm{j}\omega}\right) \right|^2 \mathrm{d}\omega \right]^{1/2}$$

- **Minimax approximation**
  - Chebyshev minimax (10.5-10.6)

$$E_\infty \triangleq \max_{\omega \in \mathcal{B}} \left| H_\mathrm{d}\left(\mathrm{e}^{\mathrm{j}\omega}\right) - H\left(\mathrm{e}^{\mathrm{j}\omega}\right) \right|$$

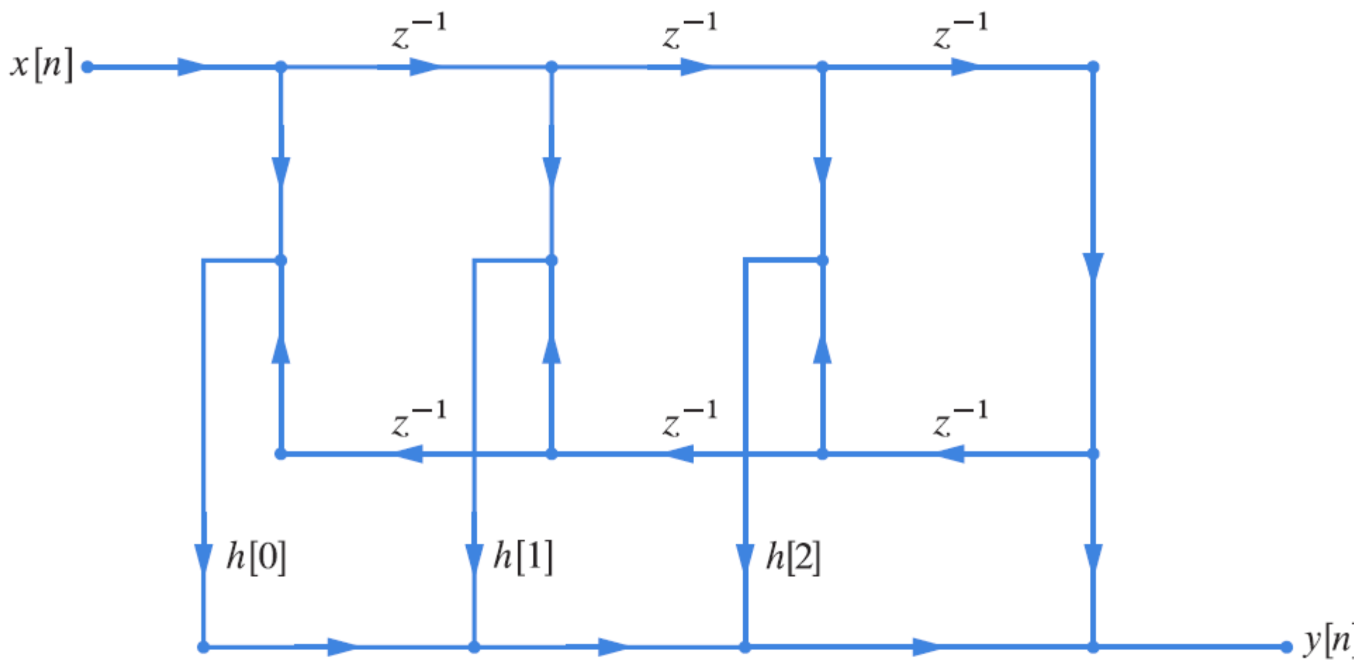
- **Maximally-flat approximation**
  - Butterworth approximation

$$E(\omega) \triangleq A_\mathrm{d}(\omega) - A(\omega) = \frac{A_\mathrm{d}^{(m)}(\omega_0) - A^{(m)}(\omega_0)}{m!}(\omega - \omega_0)^m + \cdots$$

(Recap) Direct form for linear-phase FIR

Type I:	M even,	symmetric
Type II:	M odd,	symmetric
Type III:	M even,	anti-symmetric
Type IV:	M odd,	anti-symmetric

$$h[n] = \pm h[M-n], \quad 0 \le n \le M$$



Causal filters with linear phase

Ideal lowpass filter
(with delay α)

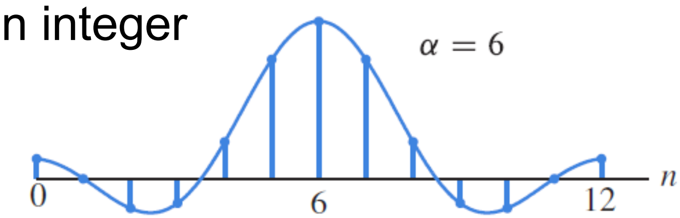
$$H_{\mathrm{lp}}(e^{j\omega}) = \begin{cases} e^{-j\alpha\omega}, & |\omega| < \omega_c \\ 0. & \omega_c < |\omega| \le \pi \end{cases}$$

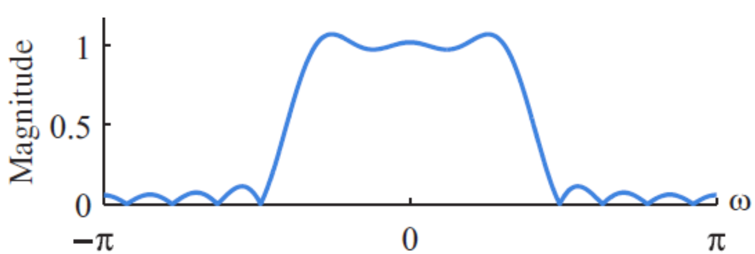
$$h_{\mathrm{lp}}[n] = \frac{\sin \omega_c (n-\alpha)}{\pi (n-\alpha)}$$

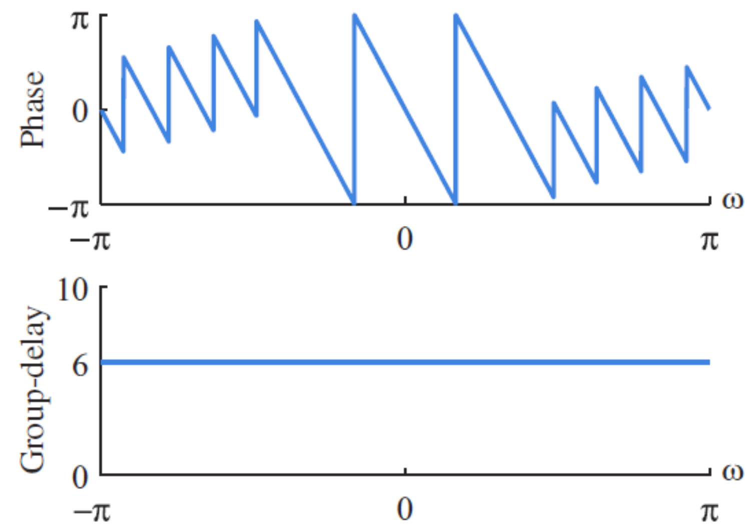
Causal FIR filter

$$h[n] = h_{\mathrm{lp}}[n] \text{ for } 0 \le n \le M$$

$2\alpha = M =$ even integer
(linear phase)

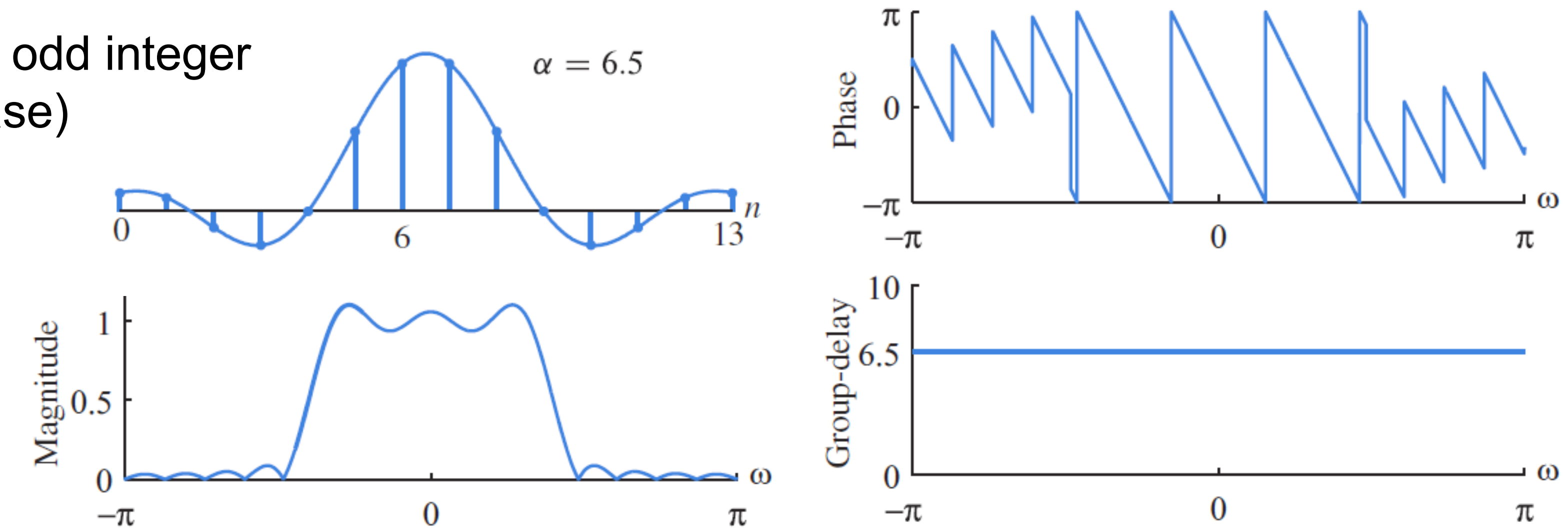




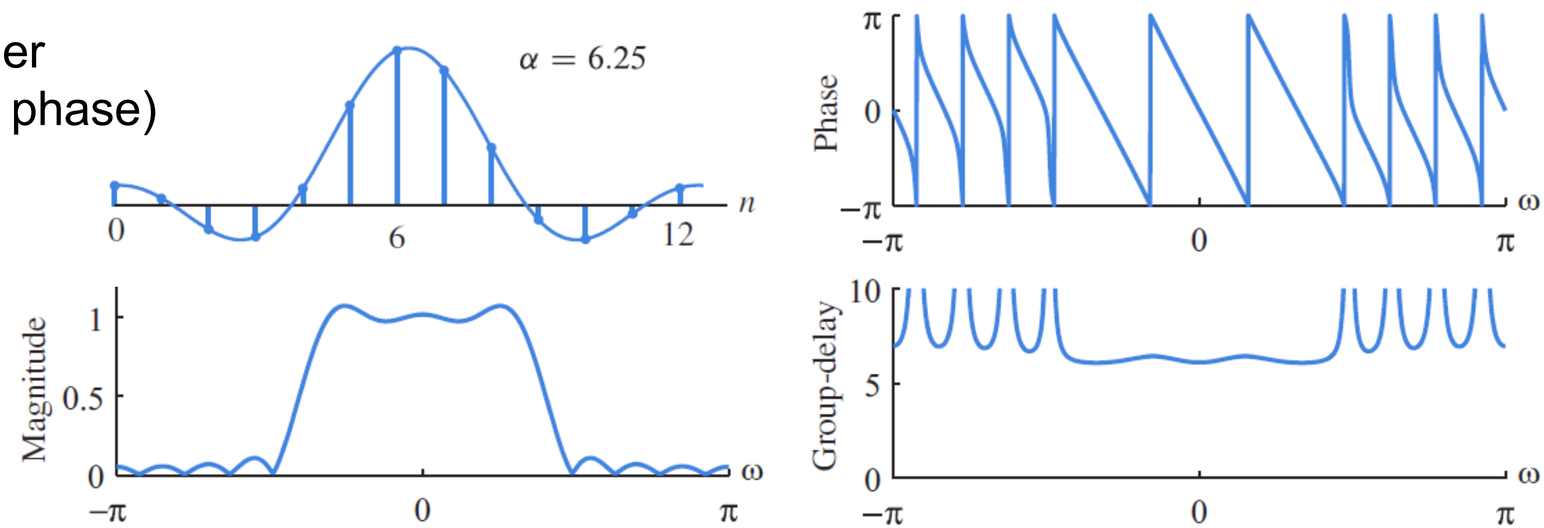


Causal filters with linear phase

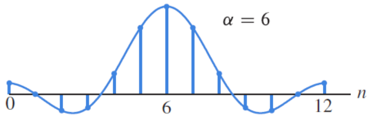
$2\alpha = M =$ odd integer
(linear phase)



$2\alpha \neq$ integer
(non-linear phase)



Note: We cannot have causal IIR filters with linear phase ($M = \infty$).



Type-I linear-phase filter

Formulation $h[n] = h[M-n]. \quad 0 \le n \le M$ Even order M; odd tap L=M+1.

Example
(M=4)

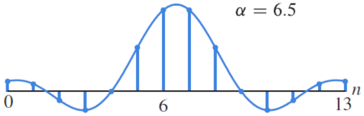
$$
\begin{aligned}
H(e^{j\omega}) &= h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[3]e^{-j3\omega} + h[4]e^{-j4\omega} \\
&= \left(h[0]e^{j2\omega} + h[1]e^{j\omega} + h[2] + h[3]e^{-j\omega} + h[4]e^{-j2\omega}\right) e^{-j2\omega} \\
&= \left(h[0]e^{j2\omega} + h[1]e^{j\omega} + h[2] + h[1]e^{-j\omega} + h[0]e^{-j2\omega}\right) e^{-j2\omega} \\
&= (h[2] + 2h[1]\cos\omega + 2h[0]\cos 2\omega)e^{-j2\omega} \\
&\triangleq (a[0] + a[1]\cos\omega + a[2]\cos 2\omega)e^{-j2\omega}.
\end{aligned}
$$

General case

$$H(e^{j\omega}) = \left(\sum_{k=0}^{M/2} a[k]\cos\omega k\right) e^{-j\omega M/2} \triangleq A(e^{j\omega})e^{-j\omega M/2}$$

$$a[0] = h[M/2], \quad a[k] = 2h[(M/2)-k].$$

Amplitude response is even and real.



Type-II linear-phase filter

Formulation $h[n] = h[M-n]. \quad 0 \le n \le M$ Odd order M; even tap L=M+1.

Example
(M=5)

$$H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[2]e^{-j3\omega} + h[1]e^{-j4\omega} + h[0]e^{-j5\omega}$$

$$= \{2h[2]\cos(\omega/2) + 2h[1]\cos(3\omega/2) + 2h[0]\cos(5\omega/2)\}\, e^{-j(5/2)\omega}$$

$$\triangleq \{b[1]\cos(\omega/2) + b[2]\cos(3\omega/2) + b[3]\cos(5\omega/2)\}\, e^{-j(5/2)\omega}.$$

$$\downarrow \quad 2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$$

$$A(e^{j\omega}) = b[1]\cos(\omega/2) + b[2]\cos(3\omega/2) + b[3]\cos(5\omega/2)$$

$$= \cos\left(\frac{\omega}{2}\right)\{(b[1]-b[2]+b[3]) + 2(b[2]-b[3])\cos\omega + 2b[3]\cos 2\omega\}.$$

General case

$$H(e^{j\omega}) = \left(\sum_{k=1}^{(M+1)/2} b[k]\cos\left[\omega\left(k-\frac{1}{2}\right)\right]\right)e^{-j\omega M/2} \triangleq A(e^{j\omega})e^{-j\omega M/2} \qquad b[k] = 2h[(M+1)/2-k]$$

$$\downarrow \quad 2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$$

$$A(e^{j\omega}) = \cos\left(\frac{\omega}{2}\right)\sum_{k=0}^{(M-1)/2} \tilde{b}[k]\cos\omega k$$

$\omega = \pi,\ A(e^{j\omega}) = 0$

$\Rightarrow$ cannot serve as highpass filter

Type-III/-IV linear-phase filter

Formulation $h[n] = -h[M-n]. \quad 0 \le n \le M$

$$c[k] = 2h[M/2 - k]$$

Type-III
(even M; odd L)

$$H(e^{j\omega}) = \left(\sum_{k=1}^{M/2} c[k] \sin \omega k\right) je^{-j\omega M/2} \triangleq jA(e^{j\omega})e^{-j\omega M/2}$$

$\downarrow$ $2 \sin\alpha \cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$

$$A(e^{j\omega}) = \sin\omega \sum_{k=0}^{M/2} \tilde{c}[k] \cos \omega k$$

Type-IV
(odd M; even L)

$$H(e^{j\omega}) = \left(\sum_{k=1}^{(M+1)/2} d[k] \sin\left[\omega\left(k - \frac{1}{2}\right)\right]\right) je^{-j\omega M/2} \triangleq jA(e^{j\omega})e^{-j\omega M/2}$$

$\downarrow$

$$d[k] = 2h[(M+1)/2 - k]$$

$$A(e^{j\omega}) = \sin\left(\frac{\omega}{2}\right) \sum_{k=0}^{(M-1)/2} \tilde{d}[k] \cos \omega k$$

$A(e^{j\omega}) = 0$ at $\omega = 0$

$\Rightarrow$ good for differentiators and Hilbert transformers

Amplitude response function

Type	$h[k]$	M	$A(e^{j\omega})$	$A(e^{j\omega})$	$\Psi(e^{j\omega})$
I	even	even	$\sum_{k=0}^{M/2} a[k]\cos\omega k$	even–no restriction	$-\frac{\omega M}{2}$
II	even	odd	$\sum_{k=1}^{\frac{M+1}{2}} b[k]\cos\left[\omega\left(k-\frac{1}{2}\right)\right]$	even $A(e^{j\pi})=0$	$-\frac{\omega M}{2}$
III	odd	even	$\sum_{k=1}^{M/2} c[k]\sin\omega k$	odd $A(e^{j0})=0$ $A(e^{j\pi})=0$	$\frac{\pi}{2}-\frac{\omega M}{2}$
IV	odd	odd	$\sum_{k=1}^{\frac{M+1}{2}} d[k]\sin\left[\omega\left(k-\frac{1}{2}\right)\right]$	odd $A(e^{j0})=0$	$\frac{\pi}{2}-\frac{\omega M}{2}$

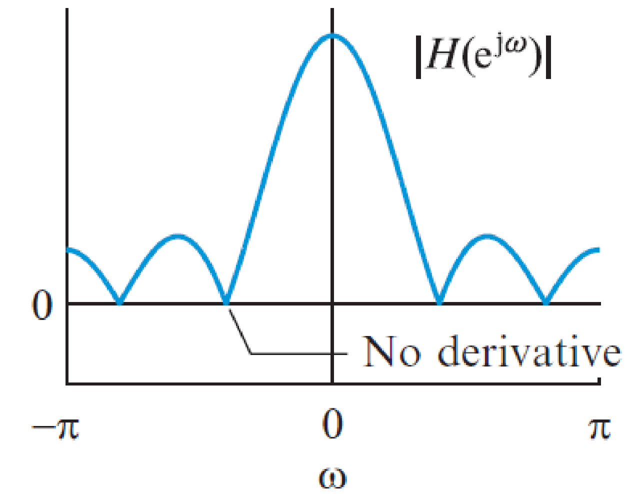
$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} \triangleq A(e^{j\omega})e^{j\Psi(e^{j\omega})}$$

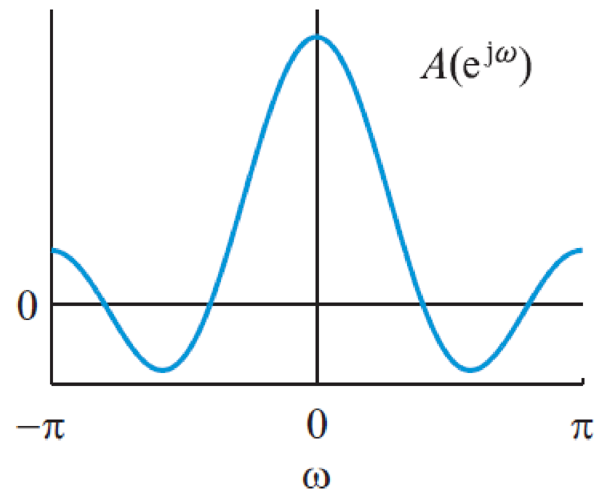
$$\Psi(e^{j\omega}) = -\alpha\omega + \beta$$

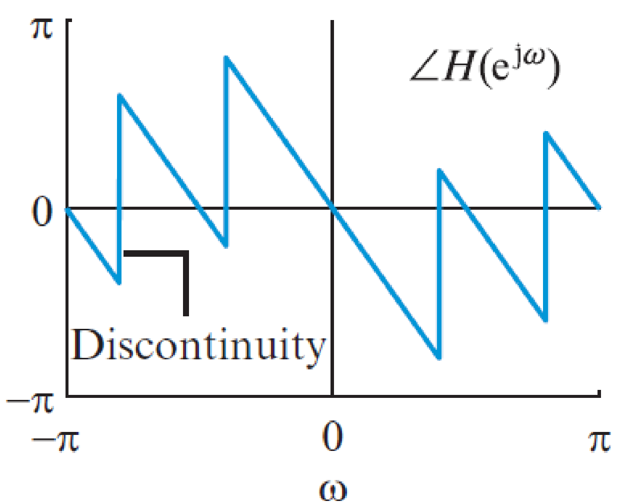
Amplitude response

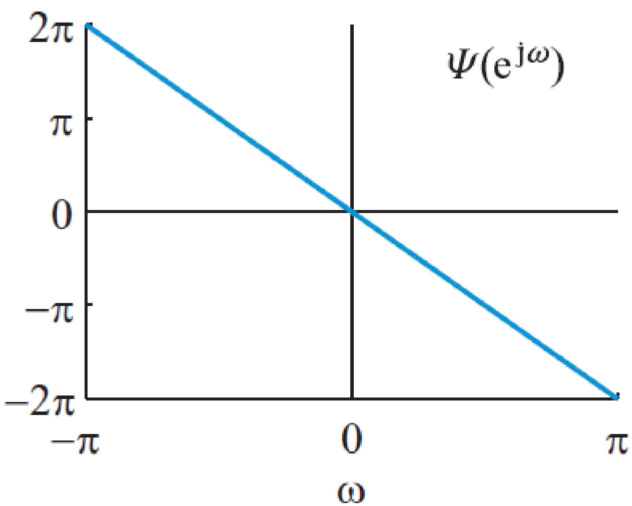
Continuous phase

Magnitude vs. Amplitude response









$$\tau_{gd}(\omega) = -\frac{d\Psi\left(e^{j\omega}\right)}{d\omega} = \alpha$$

Constant group delay

Unified representation

Type	M	$Q(e^{j\omega})$	$P(e^{j\omega})$	$H(e^{j\omega}) = 0$	Uses
I	even	1	$\sum_{k=0}^{M/2} \tilde{a}[k] \cos \omega k$		LP, HP, BP, BS, multiband filters
II	odd	$\cos(\omega/2)$	$\sum_{k=0}^{\frac{M-1}{2}} \tilde{b}[k] \cos \omega k$	$\omega = \pi$	LP, BP
III	even	$\sin \omega$	$\sum_{k=0}^{M/2} \tilde{c}[k] \cos \omega k$	$\omega = 0, \pi$	differentiators, Hilbert transformers
IV	odd	$\sin(\omega/2)$	$\sum_{k=0}^{\frac{M-1}{2}} \tilde{d}[k] \cos \omega k$	$\omega = 0$	differentiators, Hilbert transformers

$$A(e^{j\omega}) = Q(e^{j\omega}) P(e^{j\omega})$$

Fixed function of ω

Dependent on filter coefficients

Zero locations of type-I filters

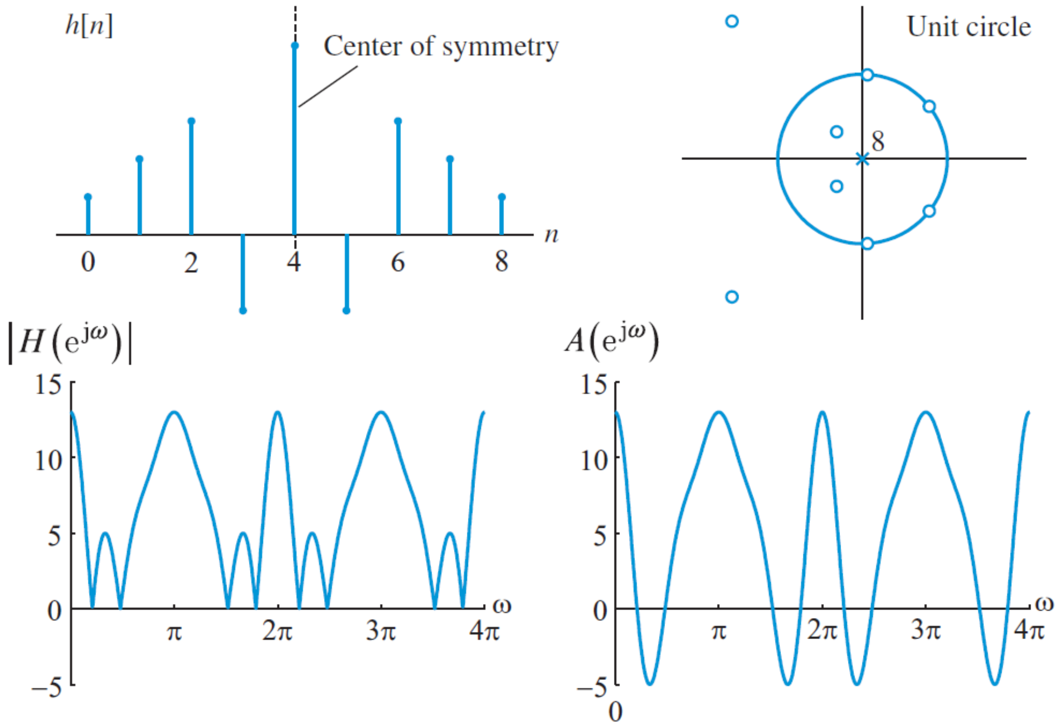
Mirror-image polynomial
(h is real)

$$H(z)=\sum_{n=0}^{M}h[n]z^{-n}=\sum_{n=0}^{M}h[M-n]z^{-n}$$

$$=\sum_{k=M}^{0}h[k]z^{k}z^{-M}=z^{-M}H(z^{-1}).$$

$\Rightarrow$ zeros appear in conjugate reciprocal $(1-re^{j\theta}z^{-1})(1-re^{-j\theta}z^{-1})(1-r^{-1}e^{j\theta}z^{-1})(1-r^{-1}e^{-j\theta}z^{-1})$

Example



MMSE FIR design by rectangular window

Desired response

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d[n] e^{-j\omega n}$$

MSE

$$\varepsilon^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| H_d(e^{j\omega}) - H(e^{j\omega}) \right|^2 d\omega$$

$$\varepsilon^2 = \sum_{n=0}^{M} (h_d[n] - h[n])^2 + \sum_{n=-\infty}^{-1} h_d^2[n] + \sum_{n=M+1}^{\infty} h_d^2[n]$$

MMSE solution

$$h[n] = \begin{cases} h_d[n], & 0 \le n \le M \\ 0. & \text{otherwise} \end{cases}$$

Rectangular window

$$w[n] = \begin{cases} 1, & 0 \le n \le M \\ 0. & \text{otherwise} \end{cases}$$

$$W(e^{j\omega}) = \sum_{n=0}^{M} e^{-j\omega n} = \frac{1 - e^{-j\omega(M+1)}}{1 - e^{-j\omega}} = \frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$$

Frequency domain effects

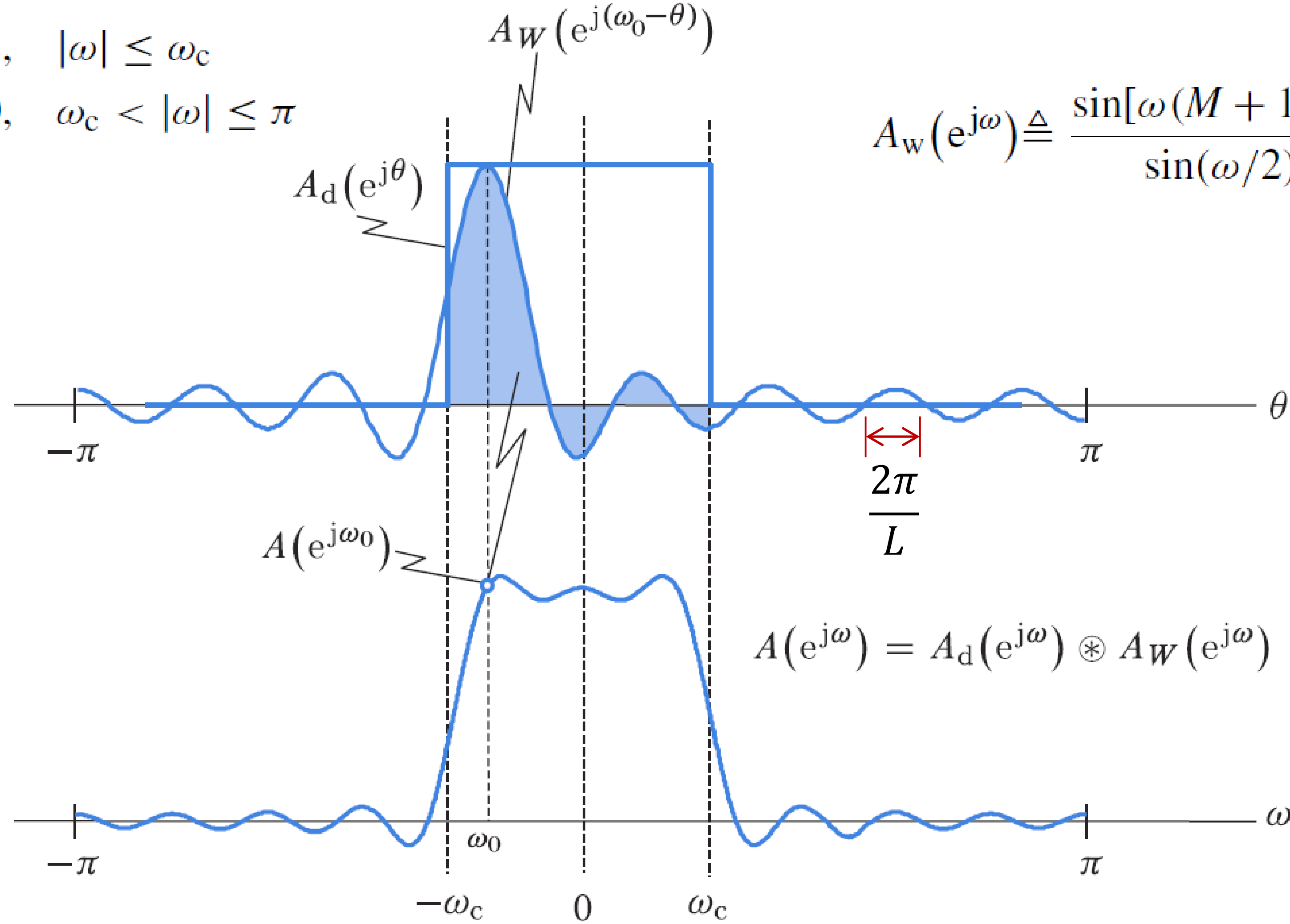
Windowing effect

$$h[n] = h_d[n]w[n]$$

$$H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\theta})W(e^{j(\omega-\theta)})d\theta$$

$$A_d(e^{j\omega}) \triangleq \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

$$A_w(e^{j\omega}) \triangleq \frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)}$$



$$A(e^{j\omega}) = A_d(e^{j\omega}) \circledast A_W(e^{j\omega})$$

Computation of ripples

Amplitude response

$$A(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} A_d(e^{j\theta})A_w(e^{j(\omega-\theta)})d\theta = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} A_w(e^{j(\omega-\theta)})d\theta$$

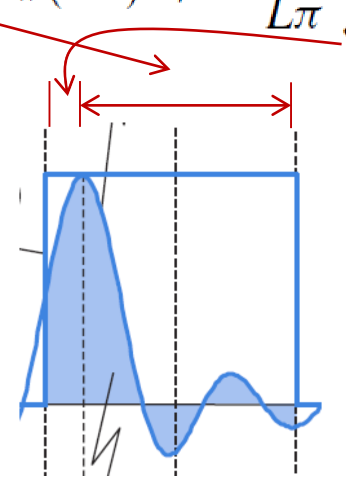
$\boldsymbol{\omega < \omega_c}$
$\boldsymbol{(\omega \cong \omega_c)}$

$$A(e^{j\omega}) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega} A_w(e^{j(\omega-\theta)})d\theta + \frac{1}{2\pi}\int_{\omega}^{\omega_c} A_w(e^{j(\omega-\theta)})d\theta$$

$\phi = \omega - \theta$

$\phi = (\omega - \theta)L/2$

$$A(e^{j\omega}) = \frac{1}{2\pi}\int_{0}^{\omega+\omega_c} A_w(e^{j\phi})d\phi + \frac{1}{L\pi}\int_{0}^{(\omega_c-\omega)L/2} A_w(e^{j(2\phi/L)})d\phi$$



$$\mathrm{Si}(\xi) \triangleq \int_0^{\xi} \frac{\sin\phi}{\phi}d\phi$$

Sine integral function

$$A(e^{j\omega}) \approx \frac{1}{2} + \frac{1}{\pi}\int_0^{(\omega_c-\omega)L/2} \frac{\sin\phi}{\phi}d\phi = \frac{1}{2} + \frac{1}{\pi}\mathrm{Si}[(\omega_c-\omega)L/2]$$

Evaluation of ripples

$\omega > \omega_c$
$(\omega \cong \omega_c)$

$$A(e^{j\omega}) \approx \frac{1}{2} - \frac{1}{\pi}\mathrm{Si}[(\omega - \omega_c)L/2]$$

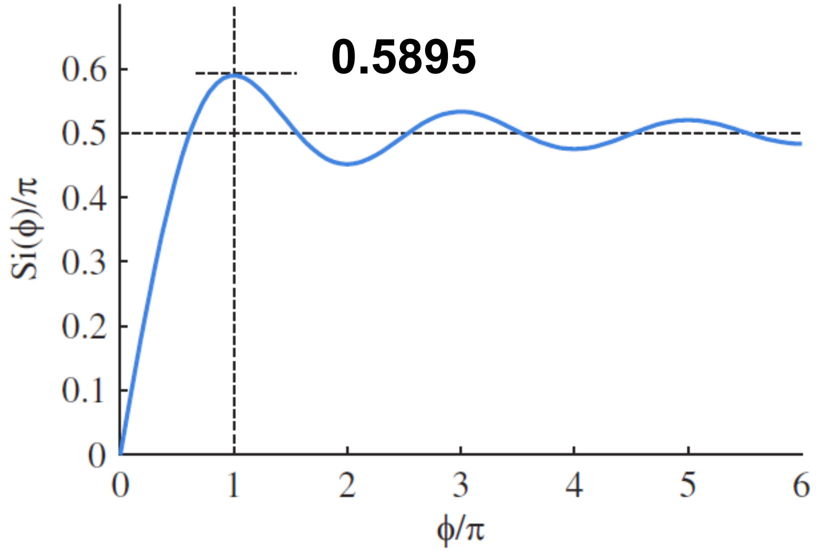
Ripples

$\omega = \omega_c$ $\quad A(e^{j\omega}) = 0.5,$

$\omega = \omega_c - 2\pi/L$ $\quad A(e^{j\omega}) = 0.5 + \mathrm{Si}(\pi)/\pi \approx 1.0895$

$\omega = \omega_c + 2\pi/L$ $\quad A(e^{j\omega}) = 0.5 - \mathrm{Si}(\pi)/\pi \approx -0.0895$

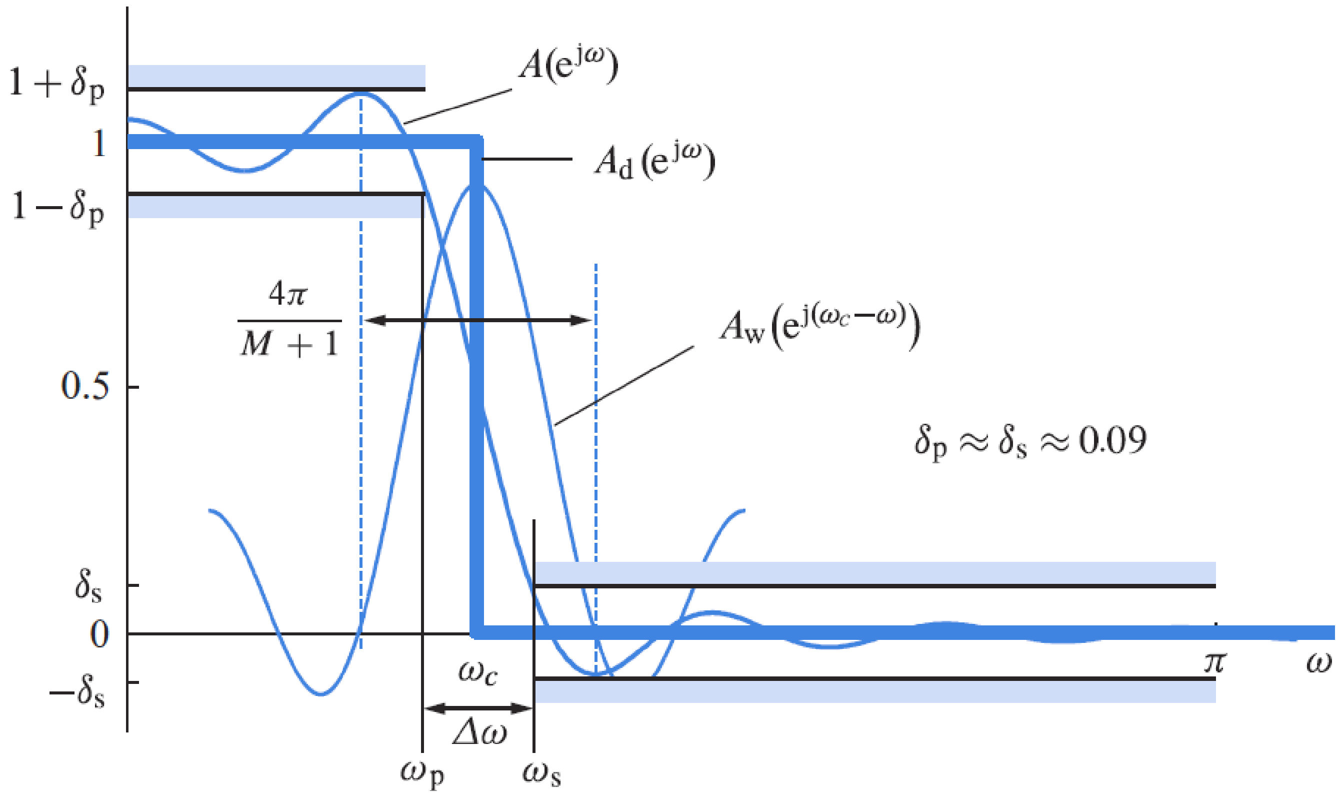
$\Rightarrow \quad \delta_p \approx \delta_s \approx 0.0895 \qquad A_s$=21 dB **(Irrespective of M)**



Evaluation of transition band

Transition bandwidth

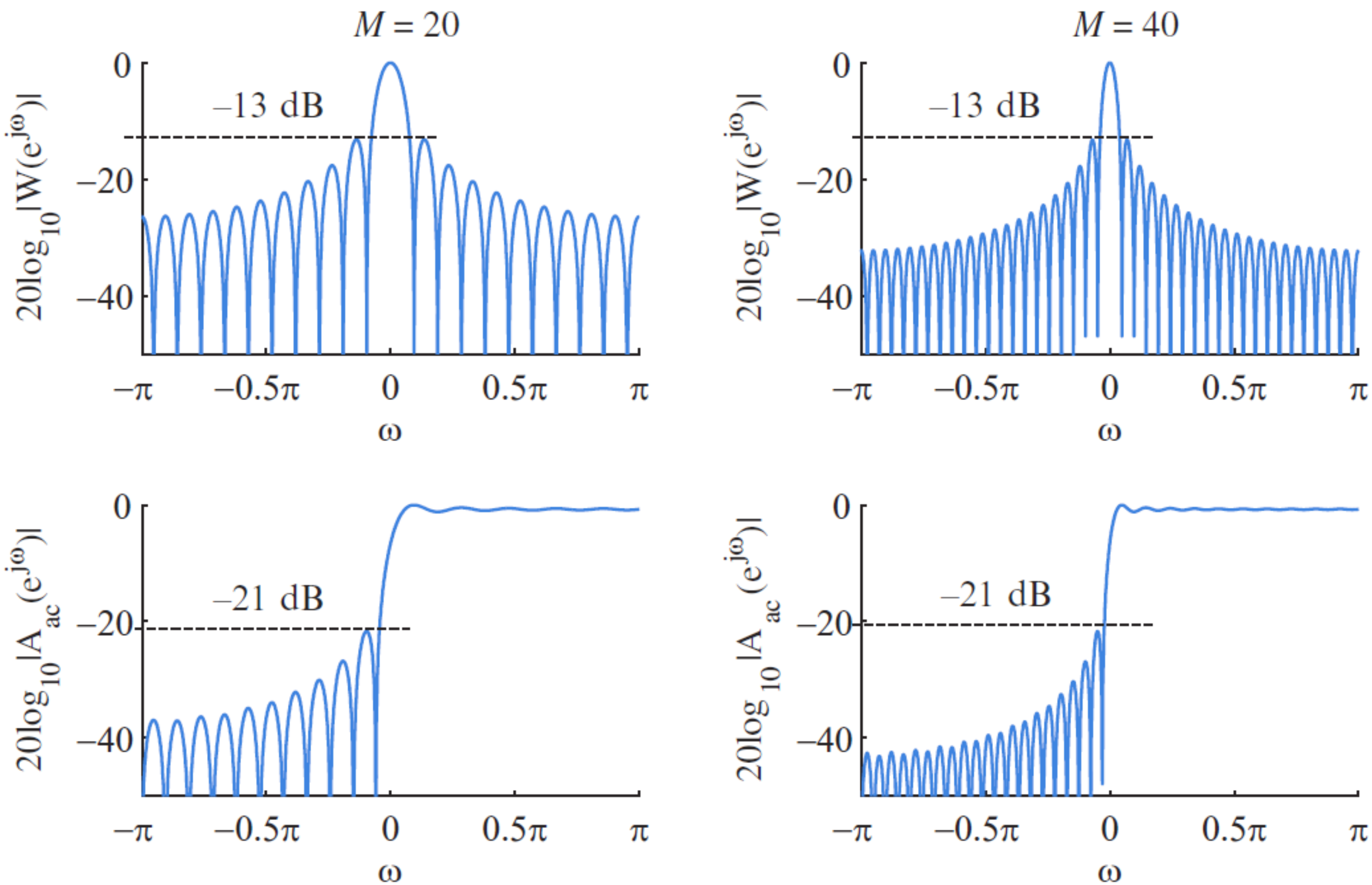
$$\Delta\omega \triangleq \omega_s - \omega_p \approx \frac{1.8\pi}{L} \leq \frac{4\pi}{M+1} \cdot \frac{1}{2}$$



Accumulated amplitude response

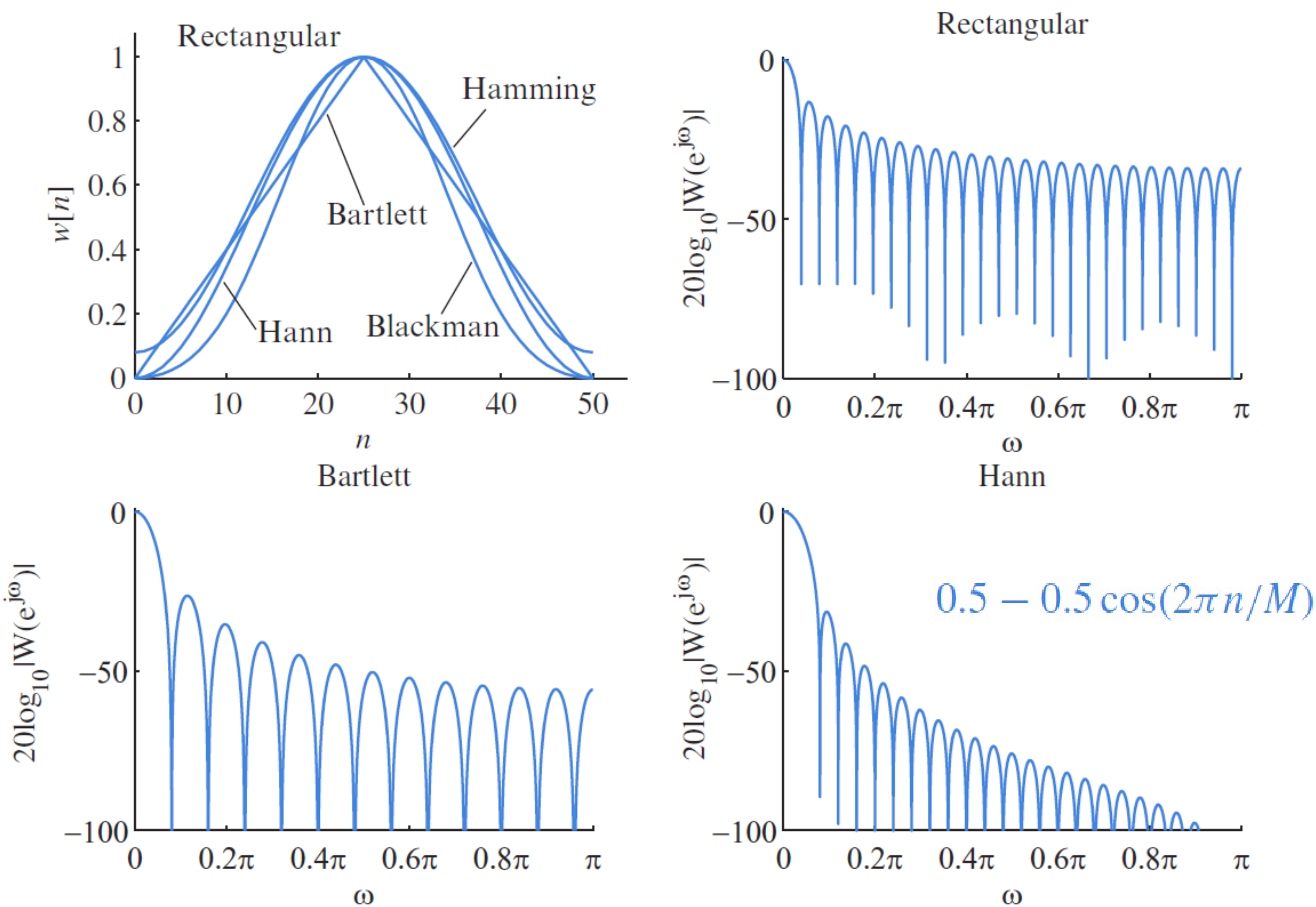
$$A_{ac}(e^{j\omega}) \triangleq \int_{-\pi}^{\omega} A_w(e^{j\theta})d\theta$$

(Approximation for passband and stopband ripples for large M)

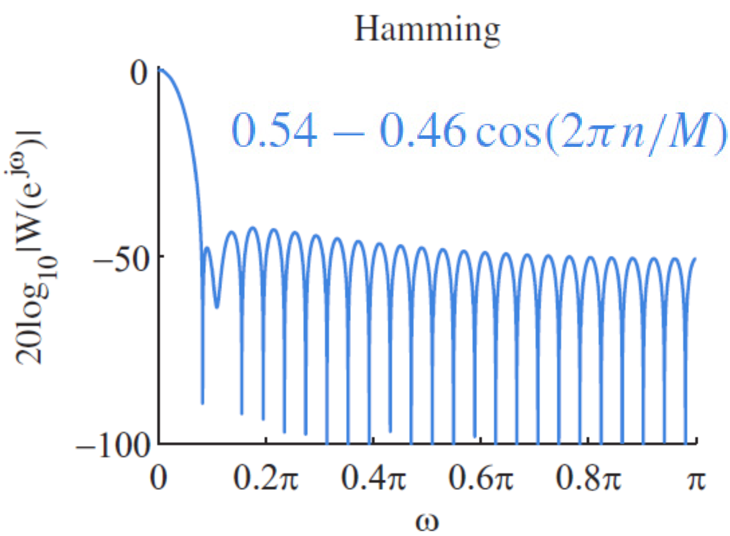


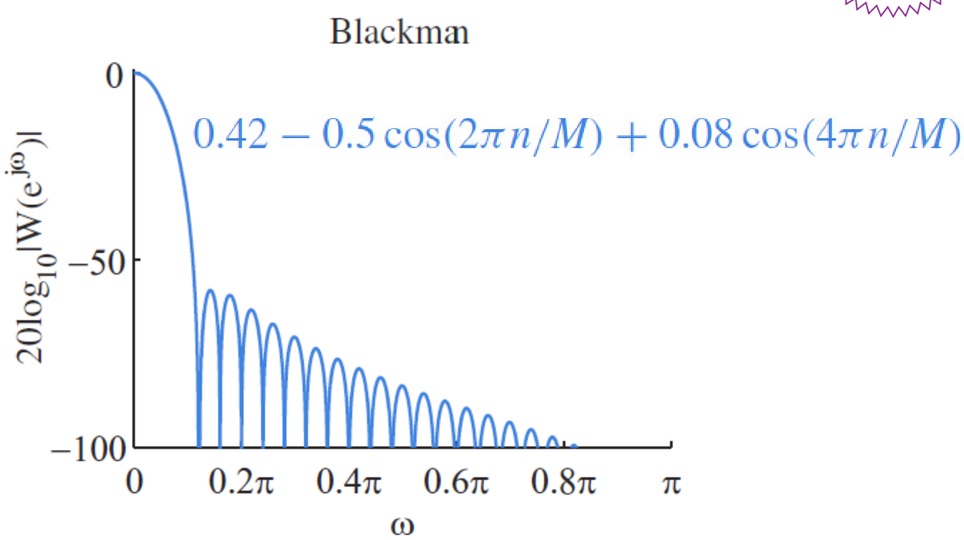
FIR design by non-rectangular window

Window shape determines ripples, and transition bandwidth can be reduced by large M (but not MMSE any more)



FIR design by non-rectangular window





Amplitude function integral

$$\Lambda_{\mathrm{w}}(\phi) = \frac{1}{L}\int_0^{\phi} A_{\mathrm{w}}\left(e^{j2\theta/L}\right)d\theta$$

$$A\left(e^{j\omega}\right) \approx \begin{cases} 0.5 + \frac{1}{\pi}\Lambda_{\mathrm{w}}[0.5(\omega_c - \omega)L], & \omega < \omega_c \\ 0.5 - \frac{1}{\pi}\Lambda_{\mathrm{w}}[0.5(\omega - \omega_c)L], & \omega > \omega_c \end{cases}$$

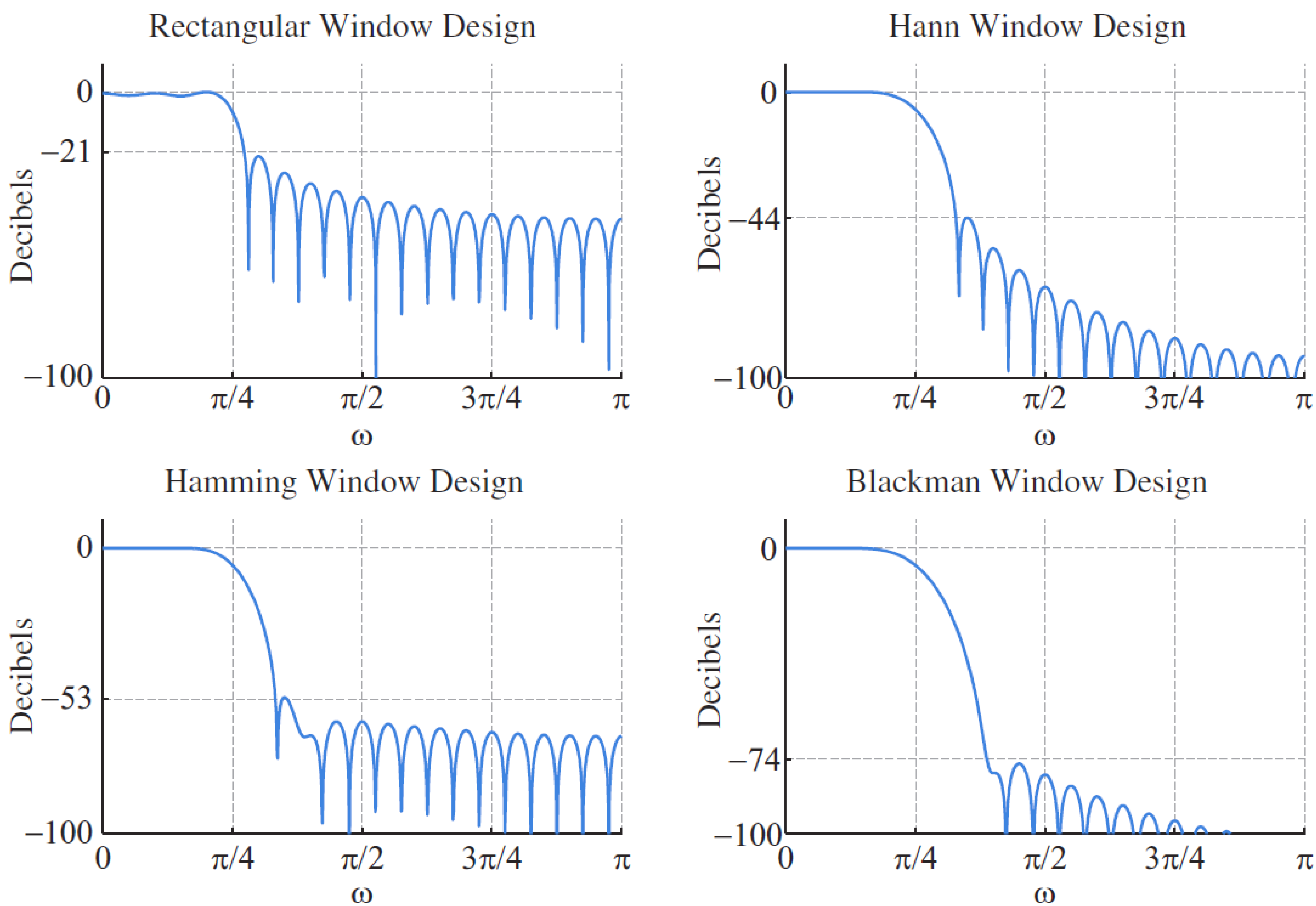
(Window shape determines ripples)

Table 10.3 Properties of commonly used windows ($L = M + 1$).

Window name	Side lobe level (dB)	Approx. $\Delta\omega$	Exact $\Delta\omega$	$\delta_p \approx \delta_s$	A_s (dB)
Rectangular	−13	$4\pi/L$	$1.8\pi/L$	0.09	21
Bartlett	−25	$8\pi/L$	$6.1\pi/L$	0.05	26
Hann	−31	$8\pi/L$	$6.2\pi/L$	0.0063	44
Hamming	−41	$8\pi/L$	$6.6\pi/L$	0.0022	53
Blackman	−57	$12\pi/L$	$11\pi/L$	0.0002	74

Trade-off: To obtain smaller ripples for the same transition bandwidth, you need to use a smoother window and a longer-tap filter.

FIR design examples (M=40)



LP FIR filter design using fixed windows

1. Given the design specifications $\{\omega_p, \omega_s, A_p, A_s\}$, determine the ripples δ_p and δ_s and set $\delta = \min\{\delta_p, \delta_s\}$.
2. Since the transition band is symmetric about ω_c (see Figure 10.8), determine the cutoff frequency of the ideal lowpass prototype by $\omega_c = (\omega_p + \omega_s)/2$.
3. Determine the design parameters $A = -20\log_{10}\delta$ and $\Delta\omega = \omega_s - \omega_p$.
4. From Table 10.3, choose the window function that provides the smallest stopband attenuation greater than A. For this window function, determine the required value of $M = L - 1$ by selecting the corresponding value of $\Delta\omega$ from the column labeled "exact $\Delta\omega$". If M is odd, we may increase it by one to have a flexible type-I filter.
5. Determine the impulse response of the ideal lowpass filter by

$$h_d[n] = \frac{\sin[\omega_c(n - M/2)]}{\pi(n - M/2)}. \tag{10.80}$$

6. Compute the impulse response $h[n] = h_d[n]w[n]$ using the chosen window.
7. Check whether the designed filter satisfies the prescribed specifications; if not, increase the order M and go back to step 5.

Design example of a lowpass linear-phase filter

Specification (fixed window)

$\omega_p = 0.25\pi, \quad \omega_s = 0.35\pi, \quad A_s = 50\,\text{dB}.$

$\omega_c = (\omega_p + \omega_s)/2 = 0.3\pi$

$\delta_s = 0.0032$

$\Delta\omega = \omega_s - \omega_p = 0.1\pi$

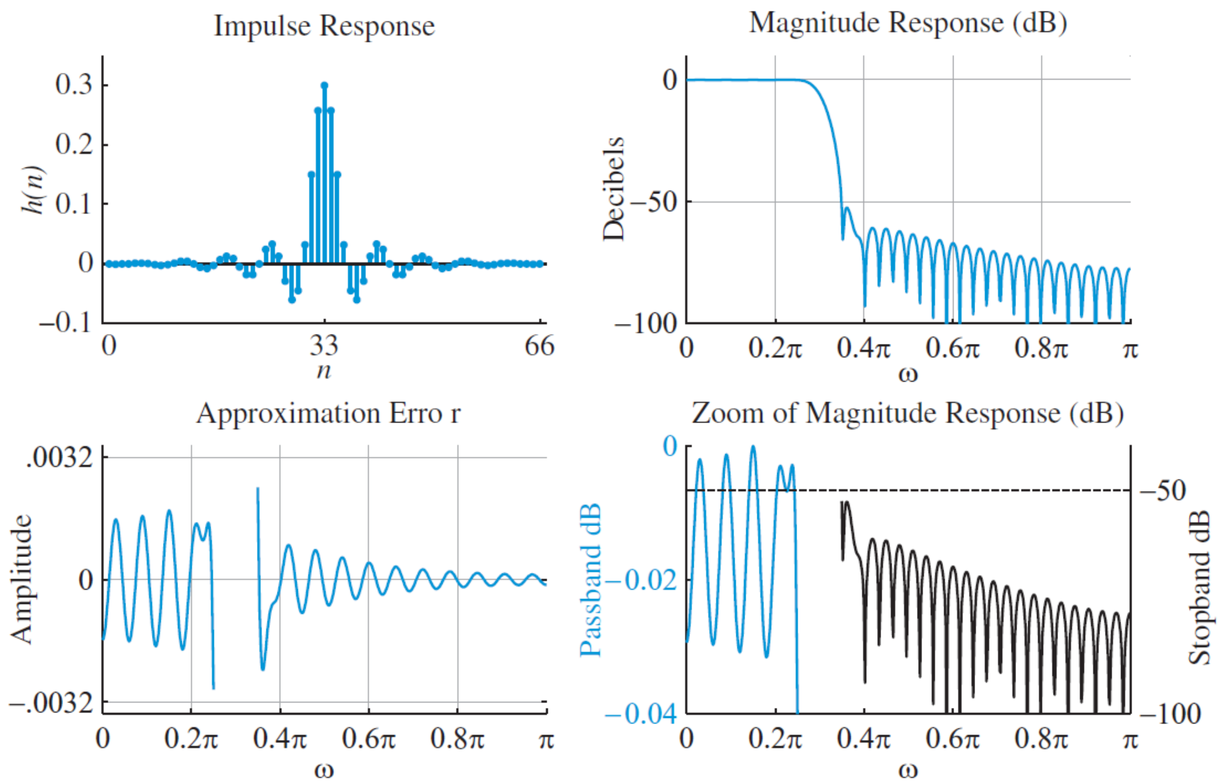
Hamming window

$\Delta\omega \approx 6.6\pi/L$

$L = 66$

Type-I

$L = 67$



Kaiser window: adjustable ripples

Window

$$w[n] = \begin{cases} \dfrac{I_0\left[\beta\sqrt{1-[(n-\alpha)/\alpha]^2}\right]}{I_0(\beta)}, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

$$I_0(x) = 1 + \sum_{m=1}^{\infty}\left[\frac{(x/2)^m}{m!}\right]$$ (zeroth-order modified Bessel function)

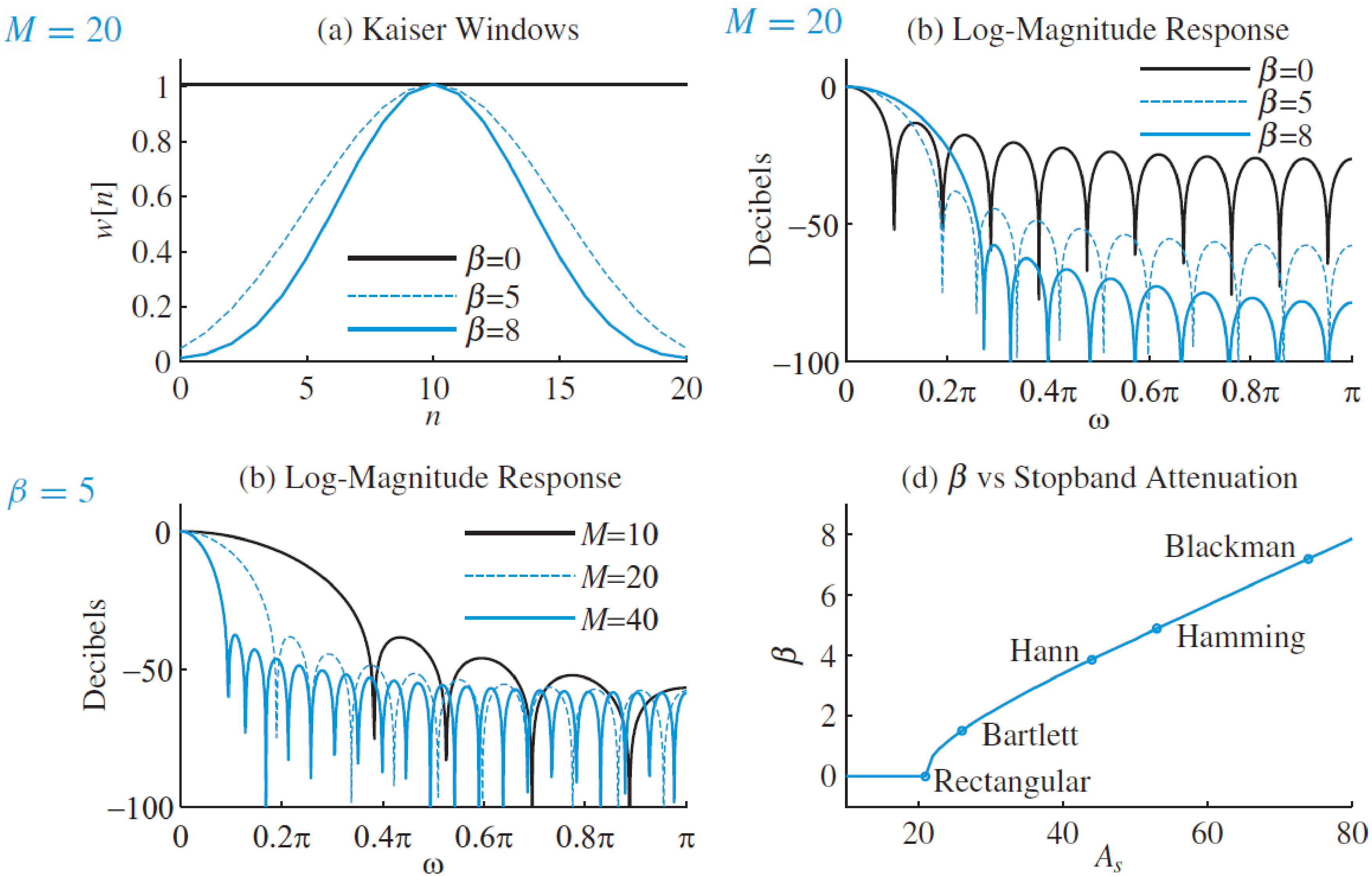
Adjustable A=A_s

$$\beta = \begin{cases} 0, & A < 21 \\ 0.5842(A-21)^{0.4} + 0.07886(A-21), & 21 \le A \le 50 \\ 0.1102(A-8.7), & A > 50 \end{cases}$$

Transition band

$$M = \frac{A-8}{2.285\Delta\omega}$$

Kaiser window: adjustable ripples



LP FIR filter design using Kaiser window

1. Given the design specifications $\{\omega_\mathrm{p}, \omega_\mathrm{s}, A_\mathrm{p}, A_\mathrm{s}\}$, determine the ripples δ_p and δ_s and set $\delta = \min\{\delta_\mathrm{p}, \delta_\mathrm{s}\}$.
2. Because the transition band is symmetric about ω_c, determine the cutoff frequency of the ideal lowpass prototype by $\omega_\mathrm{c} = (\omega_p + \omega_s)/2$.
3. Determine the design parameters $A = -20\log_{10}\delta$ and $\Delta\omega = \omega_\mathrm{s} - \omega_\mathrm{p}$.
4. Determine the required values of β and M from (10.84) and (10.85), respectively. If M is odd, we may increase it by one to have a flexible type-I filter.
5. Determine the impulse response of the ideal lowpass filter using (10.80).
6. Compute the impulse response $h[n] = h_\mathrm{d}[n]w[n]$ using the Kaiser window.
7. Check whether the designed filter satisfies the prescribed specifications; if not, increase the order M and go back to step 5.

Design example of a lowpass linear-phase filter

Specification (Kaiser window)

$\omega_p = 0.25\pi, \quad \omega_s = 0.35\pi, \quad A_s = 50\,\text{dB}.$

$\omega_c = (\omega_p + \omega_s)/2 = 0.3\pi$

$\delta_s = 0.0032$

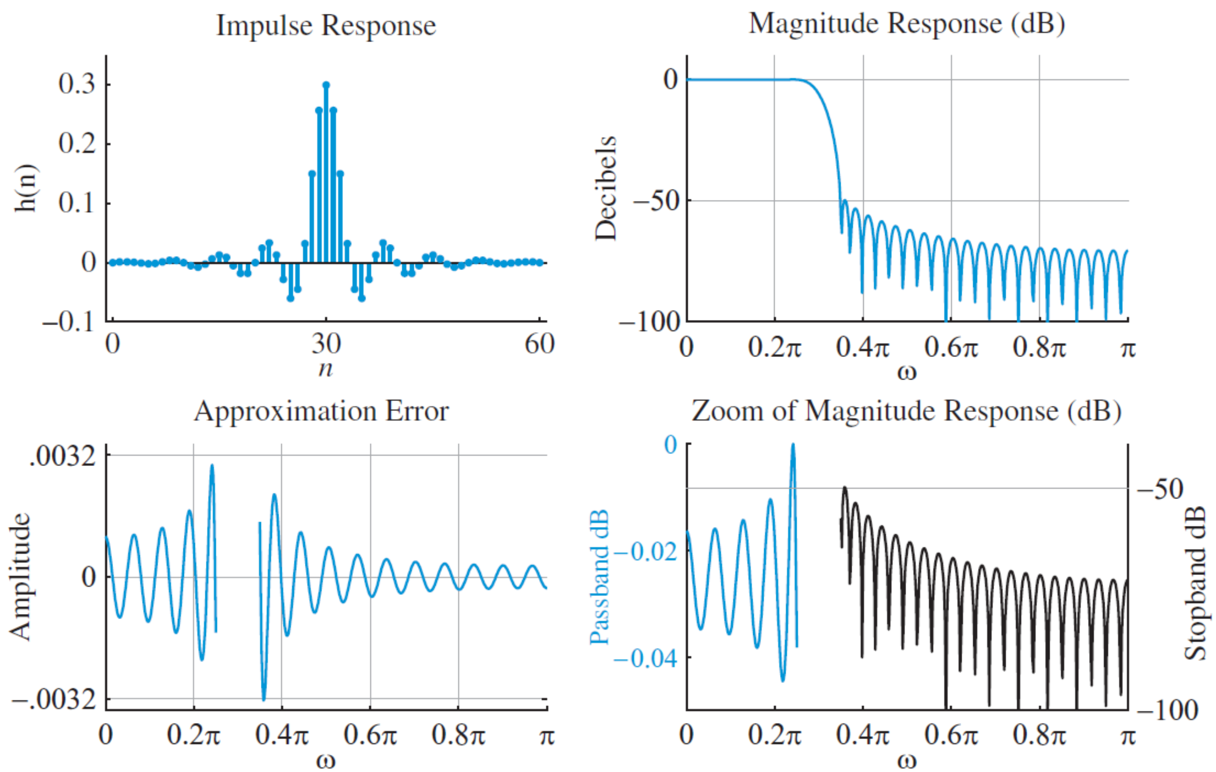
$\Delta\omega = \omega_s - \omega_p = 0.1\pi$

$\beta = 4.528$

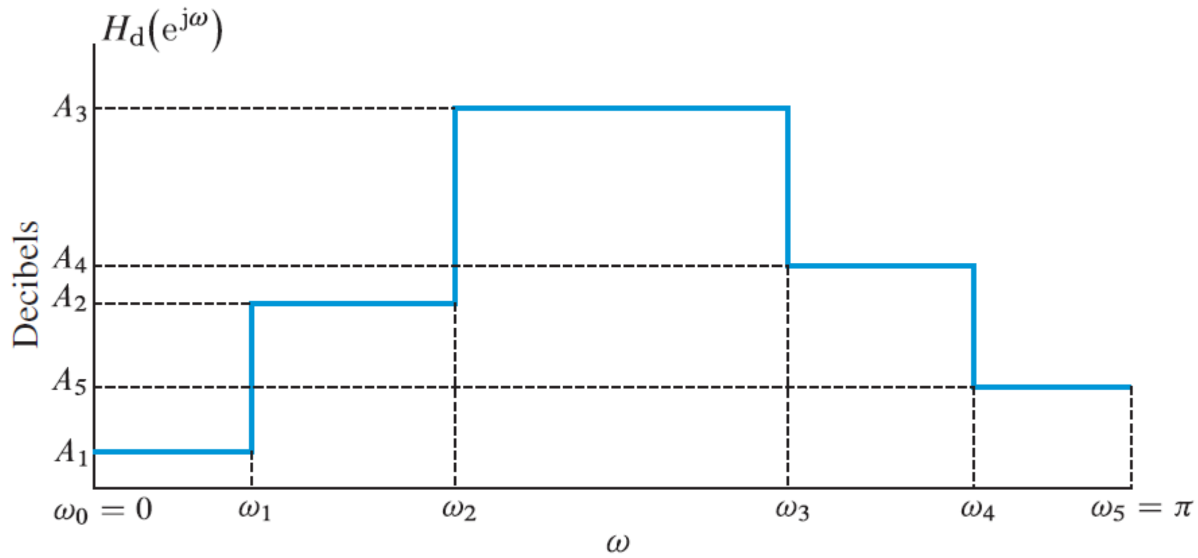
$M = 59$

Type-I

$L = 61 \quad M = 60$



Multi-band filter design



LP-band Partitions

$$h_{mb}[n] = \sum_{k=1}^{K} (A_k - A_{k+1}) \frac{\sin[\omega_k (n - M/2)]}{\pi (n - M/2)}$$

Design example of a bandpass linear-phase filter

Specification (Kaiser window)

$$|H(e^{j\omega})| \le 0.01, \qquad |\omega| \le 0.2\pi$$

$$0.99 \le |H(e^{j\omega})| \le 1.01, \qquad 0.3\pi \le |\omega| \le 0.7\pi$$

$$|H(e^{j\omega})| \le 0.01. \qquad 0.78\pi \le |\omega| \le \pi$$

$$\boxed{\omega_{c_1} = 0.25\pi}$$

$$\Delta\omega_1 = 0.3\pi - 0.2\pi = 0.1\pi$$

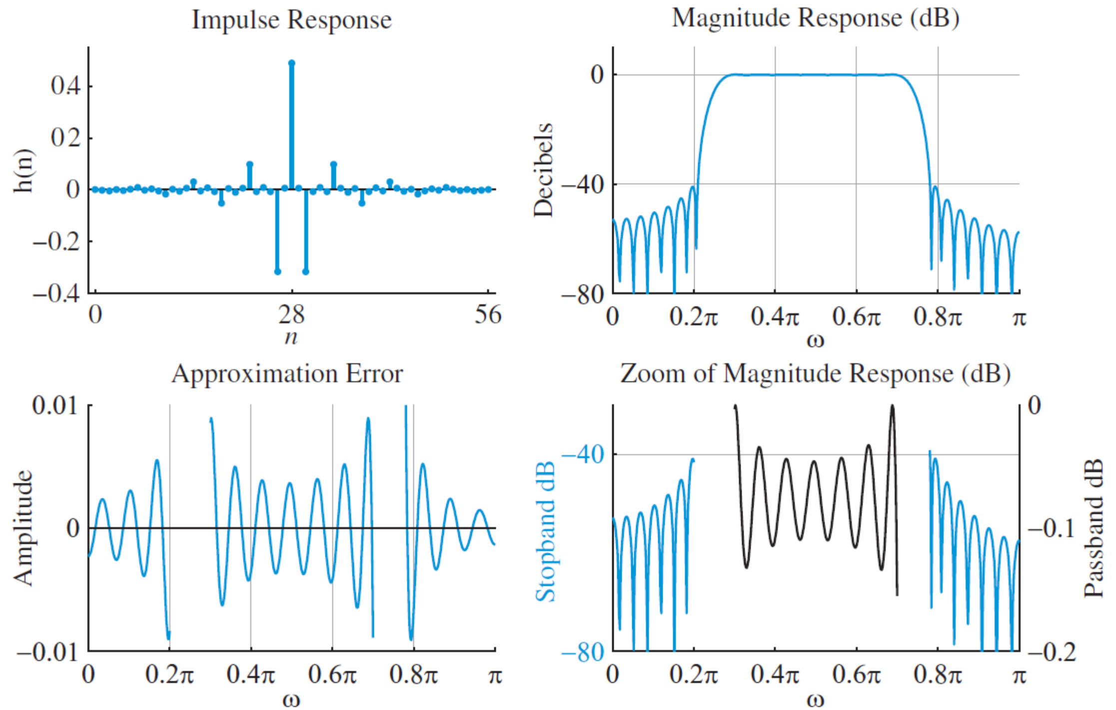
$$\boxed{\omega_{c_2} = 0.74\pi}$$

$$\Delta\omega_2 = 0.78\pi - 0.7\pi = 0.08\pi$$

$\delta = 0.01$ or $A \approx 40$ dB → $\boxed{\beta = 3.3953}$ → Type-I $\boxed{M = 56}$ ← $\Delta\omega = 0.08\pi$

$$h_{bp}[n] = \frac{\sin[\omega_{c_2}(n - M/2)]}{\pi(n - M/2)} - \frac{\sin[\omega_{c_1}(n - M/2)]}{\pi(n - M/2)}$$

(ideal)



Basic approach for FIR filter design using frequency sampling

Discrete sampling of DTFT

$$H_d[k] \triangleq H_d\left(e^{j2\pi k/L}\right). \quad k = 0, 1, \ldots, L-1$$

Inverse DFT (time-domain aliasing)

$$\tilde{h}[n] \triangleq \frac{1}{L}\sum_{k=0}^{L-1} H_d[k] W_N^{-kn} = \sum_{m=-\infty}^{\infty} h_d[n - mL]$$

(L could be large to reduce aliasing)

Windowing

$$h[n] = \tilde{h}[n]w[n]$$

$$H\left(e^{j\omega}\right) = \frac{1}{L}\sum_{k=0}^{L-1} H_d[k] W\left(e^{j(\omega - 2\pi k/L)}\right)$$

(windowing as frequency-domain interpolation)

Linear-phase FIR filter design

$$H_d[k] = A_d[k]e^{j\Psi_d[k]},$$

Amplitude Sampling

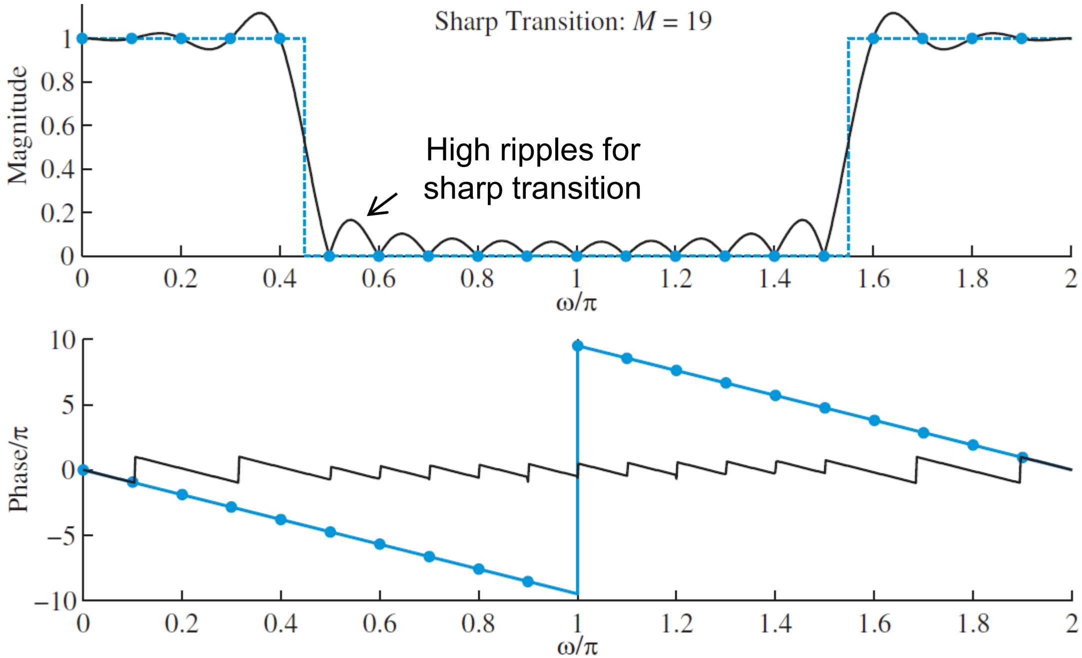
$$A_d[k] = \begin{cases} A_d(e^{j0}), & k = 0 \\ A_d(e^{j2\pi k/L}), & k = 1, 2, \ldots, L \end{cases}$$

Linear-phase enforcement (Type-I/II)

$$\Psi_d[k] = \begin{cases} -\frac{L-1}{2}\frac{2\pi}{L}k, & k = 0, 1, \ldots, Q \\ \frac{L-1}{2}\frac{2\pi}{L}(L-k), & k = Q+1, \ldots, L-1 \end{cases}$$

Phase shift = $-\frac{M}{2}\omega$

$$Q = \lfloor (L-1)/2 \rfloor$$



Rectangular windowing

Method 1: Smooth transition band approach

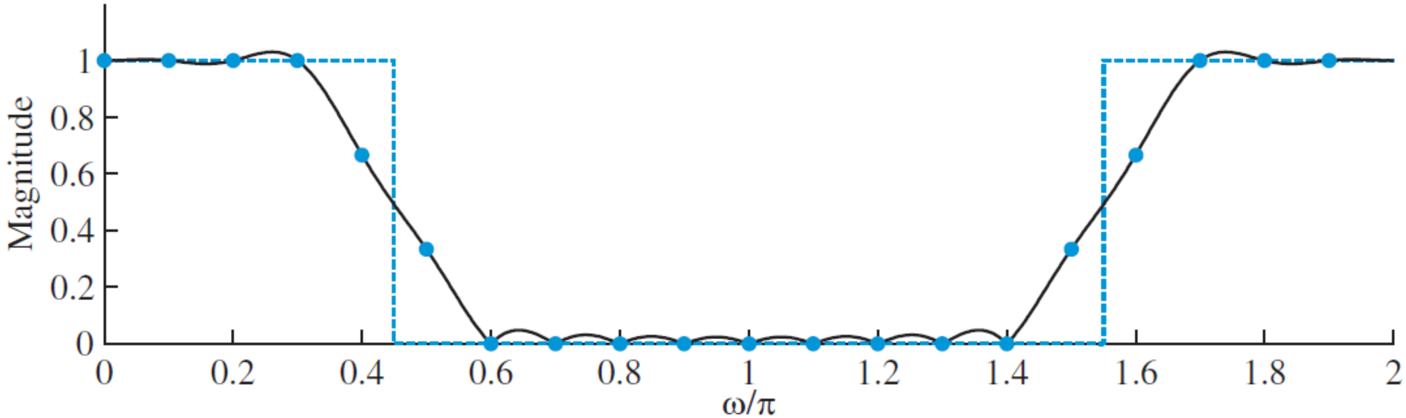
Linear roll-off

$$A_d(e^{j\omega}) = (\omega_s - \omega)/(\omega_s - \omega_p), \quad \omega_p < \omega < \omega_s$$

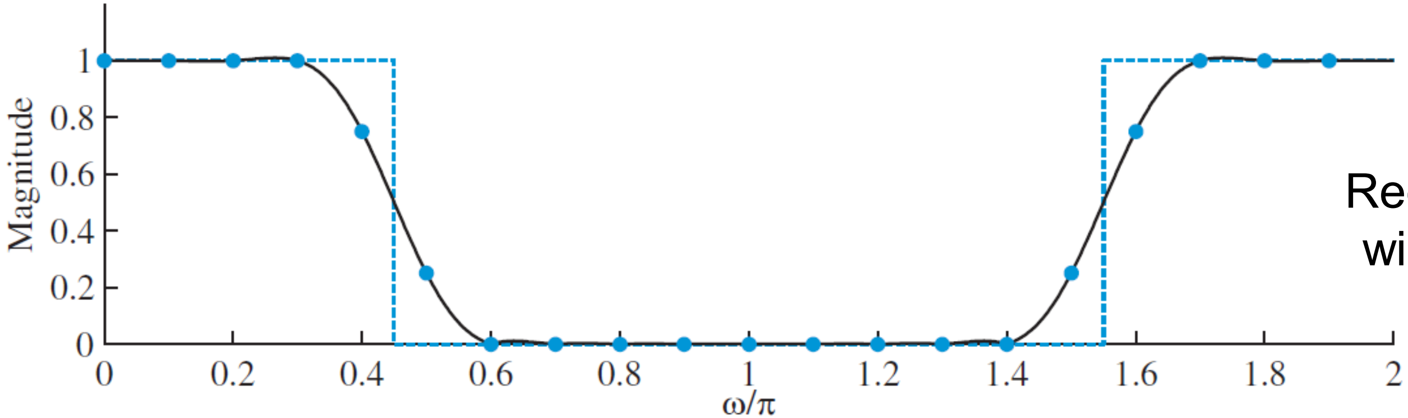
Raised-cosine roll-off (preferred)

$$A_d(e^{j\omega}) = 0.5 - 0.5\cos[\pi(\omega_s - \omega)/(\omega_s - \omega_p)], \quad \omega_p < \omega < \omega_s$$

(a) Linear Transition: $M = 19$



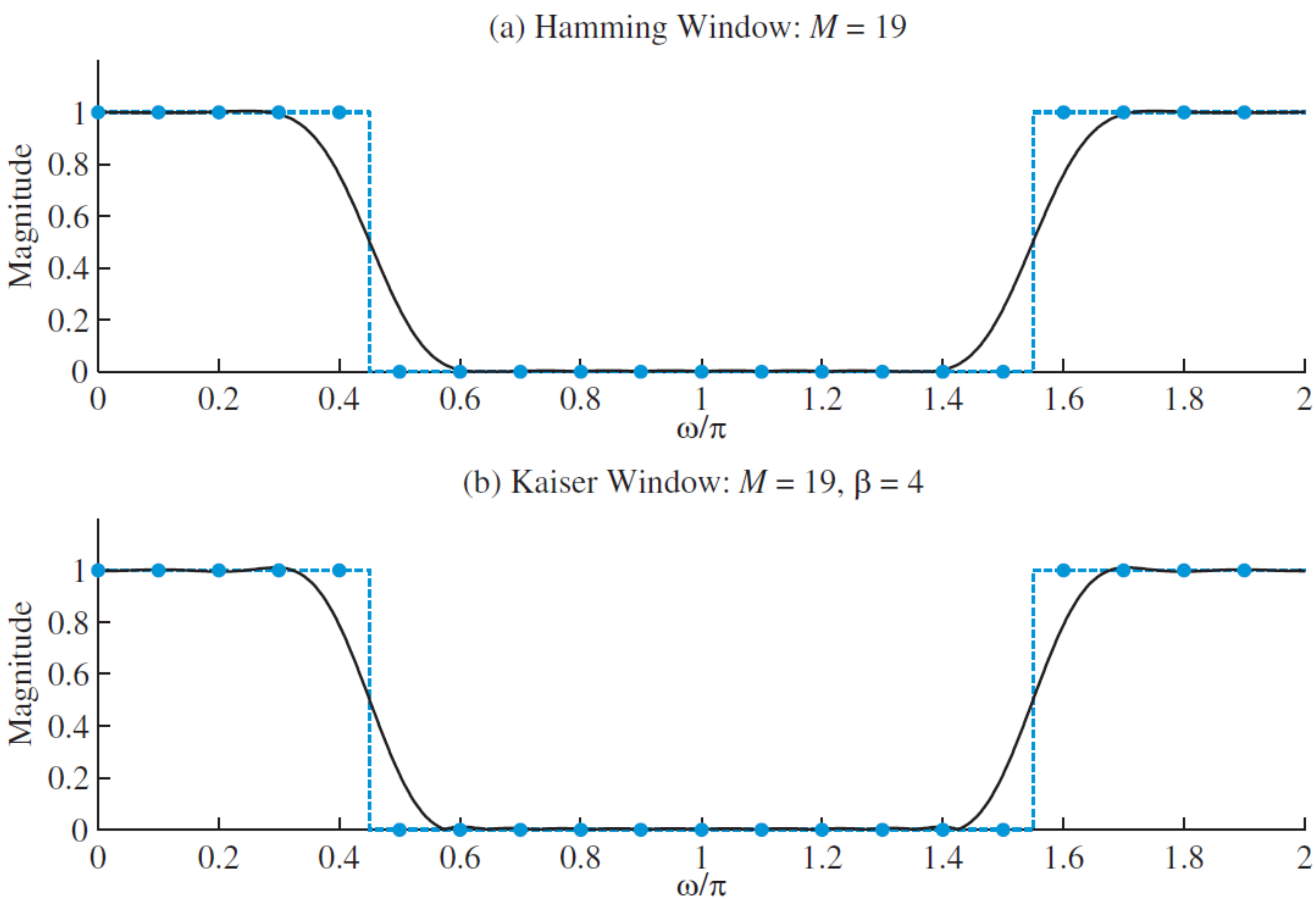
(b) Raised-Cosine Transition: $M = 19$



Rectangular windowing

Method 2: Nonrectangular windowing approach

Apply Hamming or Kaiser window after direct sampling



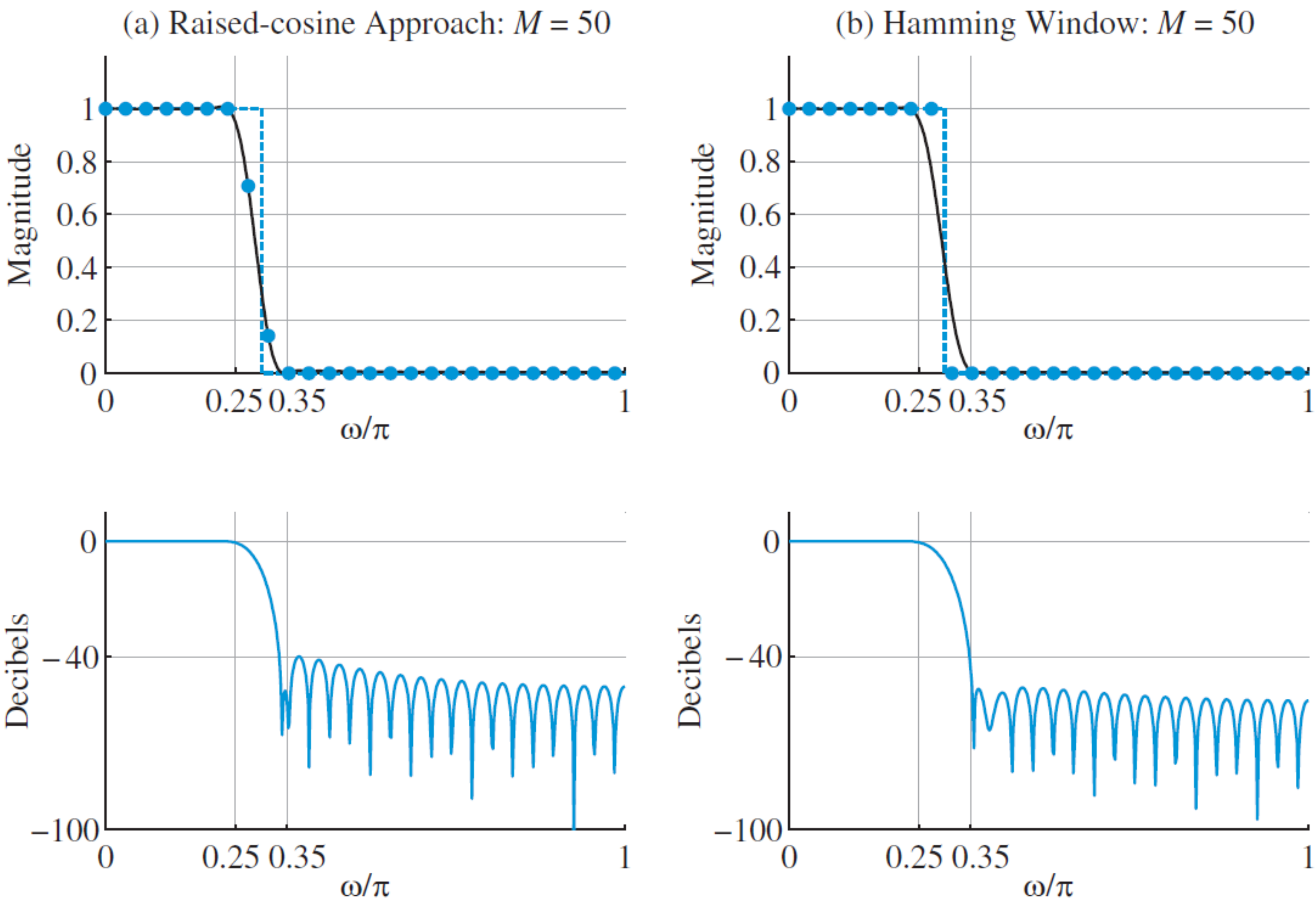
FIR filter design using frequency sampling

1. Choose the order of the filter M by placing at least two samples in the transition band.
2. For a window design approach[2] obtain samples of the desired frequency response $H_d[k]$ using (10.94). For a smooth transition band approach[1] use (10.95) or (10.96) for transition band samples in addition to (10.94) for remaining samples.
3. Compute the $(M+1)$-point IDFT of $H_d[k]$ to obtain $h[n]$. For a window design approach multiply $h[n]$ by the appropriate window function.
4. Compute log-magnitude response $H_d(e^{j\omega})$ and verify the design over passband and stopband.
5. If the specifications are not met, increase M and go back to step 1.

Iterated until spec is met

Design example of a lowpass linear-phase filter

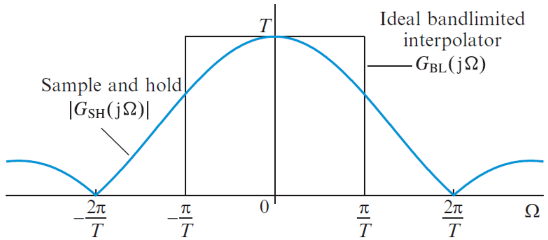
Specification $\omega_p = 0.25\pi, \quad \omega_s = 0.35\pi, \quad A_s = 40\,\text{dB}$



Note: Frequency sampling is not suited for standard LP/BP/HP filters.

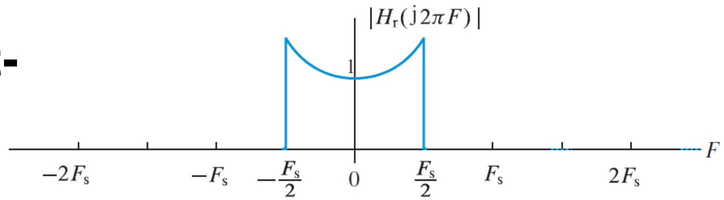
Design example of DAC equalization

S/H DAC



$$g_{SH}(t) = \begin{cases} 1, & 0 \le t \le T \\ 0. & \text{otherwise} \end{cases} \xleftrightarrow{\text{CTFT}} G_{SH}(j\Omega) = \frac{2\sin(\Omega T/2)}{\Omega} e^{-j\Omega T/2}$$

Analog post-filter



$$H_r(j\Omega) = \begin{cases} \frac{\Omega T/2}{\sin(\Omega T/2)} e^{j\Omega T/2}, & |\Omega| < \pi/T \\ 0. & \text{otherwise} \end{cases}$$

Digital full-band post-filter

$$H_d(e^{j\omega}) = \frac{\omega/2}{\sin(\omega/2)}. \quad -\pi \le \omega \le \pi$$