National Tsing Hua University
Department of Electrical Engineering
EE3660 Intro. to Digital Signal Processing, Spring 2020

Homework Assignment #4: Chap. 7

Answer

I Paper Assignment (50%)

1. (10%) Determine DFS coefficients of the following periodic sequences:

(a) $\tilde{x}[n] = 2\cos(\pi n/4)$

(b) $\tilde{x}[n] = 3\sin(0.25\pi n) + 4\cos(0.75\pi n)$

(a) Solution:

The DFT of $\tilde{x}[n] = 2\cos(\pi n/4)$ is :

$$\mathrm{X}[k] = \sum_{n=0}^{7}\left(e^{\frac{j\pi n}{4}} + e^{-\frac{j\pi n}{4}}\right)e^{-\frac{j2\pi n}{8}k}$$
$$= \sum_{n=0}^{7}\left(e^{\frac{j\pi n}{4}(1-k)} + e^{-\frac{j\pi n}{4}(1+k)}\right)$$
$$= 8\delta[k-1] + 8\delta[k-7]$$

The DFS of $\tilde{x}[n] = 2\cos(\pi n/4)$ is :

$$\tilde{X}[k] = 8\delta[\langle k\rangle_8 - 1] + 8\delta[\langle k\rangle_8 - 7]$$

(b) Solution:

The DFT of $\tilde{x}[n] = 3\sin(0.25\pi n) + 4\cos(0.75\pi n)$ is :

$$\mathrm{X}[k] = \sum_{n=0}^{7}\left[\frac{3}{2j}\left(e^{\frac{j\pi n}{4}} - e^{-\frac{j\pi n}{4}}\right) + 2\left(e^{\frac{j3\pi n}{4}} + e^{-\frac{j3\pi n}{4}}\right)\right]e^{-\frac{j2\pi n}{8}k}$$
$$= \sum_{n=0}^{7}\left[\frac{3}{2j}\left(e^{\frac{j\pi n}{4}(1-k)} - e^{-\frac{j\pi n}{4}(1+k)}\right) + 2\left(e^{\frac{j\pi n}{4}(3-k)} + e^{-\frac{j\pi n}{4}(3+k)}\right)\right]$$
$$= -12j\delta[k-1] + 12j\delta[k-7] + 16\delta[k-3] + 16\delta[k-5]$$

The DFS of $\tilde{x}[n] = 3\sin(0.25\pi n) + 4\cos(0.75\pi n)$ is :

$$\tilde{X}[k] = -12j\delta[\langle k\rangle_8 - 1] + 12j\delta[\langle k\rangle_8 - 7] + 16\delta[\langle k\rangle_8 - 3] + 16\delta[\langle k\rangle_8 - 5]$$

2. (10%) Let x[n] be an N-point sequence with an N-point DFT X[k].

(a) If N is even and if $x[n] = -x[\langle n + N/2 \rangle_N]$ for all n, then show that X[k] = 0 for even k.

(b) Show that if N=4m where m is an integer and if $x[n] = -x[\langle n + N/4 \rangle_N]$ for all n, then X[k]=0 for $k = 4\ell,\ 0 \le \ell \le \frac{N}{4} - 1$

(a) Proof:

If k is even and N is even, the correspondent DFT is:

$$\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk} = \sum_{n=0}^{N/2-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk} + \sum_{n=N/2}^{N-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk} \\
&= \sum_{n=0}^{N/2-1} \left(x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk} + x[n+\frac{N}{2}]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}(n+\frac{N}{2})k} \right) \\
&= \sum_{n=0}^{N/2-1} \left(x[n] - x[n] \right) \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk} = 0
\end{aligned}$$

(b) Proof:

If $N = 4m$, $k = 4\ell$, the correspondent DFT is:

$$\begin{aligned}
X[4\ell] &= \sum_{n=0}^{N-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(4\ell)} = \sum_{n=0}^{\frac{N}{4}-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(4\ell)} \\
&+ \sum_{n=\frac{N}{4}}^{\frac{2N}{4}-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(4\ell)} + \sum_{n=\frac{2N}{4}}^{\frac{3N}{4}-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(4\ell)} + \sum_{n=\frac{3N}{4}}^{N-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(4\ell)} \\
&= \left(\sum_{n=0}^{\frac{N}{4}-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(4\ell)} + \sum_{n=0}^{\frac{N}{4}-1} x[n+\frac{N}{4}]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}(n+\frac{N}{4})(4\ell)} \right) \\
&+ \left(\sum_{n=\frac{2N}{4}}^{\frac{3N}{4}-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(4\ell)} + \sum_{n=\frac{2N}{4}}^{\frac{3N}{4}-1} x[n+\frac{N}{4}]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}(n+\frac{N}{4})(4\ell)} \right) \\
&= \sum_{n=0}^{\frac{N}{4}-1} \left(x[n] + x[n+\frac{N}{4}] \right) \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(4\ell)} + \sum_{n=\frac{2N}{4}}^{\frac{3N}{4}-1} \left(x[n] + x[n+\frac{N}{4}] \right) \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(4\ell)} \\
&= 0
\end{aligned}$$

3. (12%) Let $x_1[\mathrm{n}], 0 \leq \mathrm{n} \leq N_1 - 1$, be an N_1-point sequence and let $x_2[\mathrm{n}], 0 \leq \mathrm{n} \leq N_2 - 1$, be an N_2-point sequence. Let $x_3[\mathrm{n}] = x_1[\mathrm{n}] * x_2[\mathrm{n}]$ and let $x_4[\mathrm{n}] = x_1[\mathrm{n}] \circledN x_2[\mathrm{n}]$, $\mathrm{N} \geq \max(N_1, N_2)$

(a) Show that

$$x_4[\mathrm{n}] = \sum_{l=-\infty}^{\infty} x_3[\mathrm{n} + l\mathrm{N}] \tag{7.209}$$

(b) Let $\mathrm{e}[\mathrm{n}] = x_4[\mathrm{n}] - x_3[\mathrm{n}]$, show that

$$\mathrm{e}[\mathrm{n}] = \begin{cases} x_3[\mathrm{n}+\mathrm{N}], & \max(N_1, N_2) \leq N < L \\ 0, & N \geq L \end{cases}$$

where $\mathrm{L} = N_1 + N_2 - 1$

(c) Verify the results in (a) and (b) for $x_1 = \{1_{n=0}, 2,3,4\}$, $x_2 = \{4_{n=0}, 3,2,1\}$, and N=5 and N=8

(a) Proof:
$X_4[K]$ can be obtained by frequency sampling of $X_3[k]$, hence in the time domain, according the aliasing equation, we have

$$x_4[n] = \sum_{\ell=-\infty}^{\infty} x_3[n + \ell N]$$

(b) Proof:
When $N \geq L$, there is no time aliasing, we conclude

$$x_4[n] = x_3[n], \quad \text{for} \quad 0 \leq n \leq L$$

When $\max(N_1, N_2) \leq N < L$, since $L = N_1 + N_2 - 1 \leq 2N - 1$, we conclude that

$$x_4[n] = x_3[n] + x_3[n+N], \quad \text{for} \quad 0 \leq n \leq N-1$$

Hence, we proved the equation

(c)

$$x_3[n] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \\ 0 & 4 & 3 & 2 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \\ 20 \\ 30 \\ 20 \\ 11 \\ 4 \end{bmatrix}$$

N=5:

$$x_4[n] = \begin{bmatrix} 1 & 0 & 4 & 3 & 2 \\ 2 & 1 & 0 & 4 & 3 \\ 3 & 2 & 1 & 0 & 4 \\ 4 & 3 & 2 & 1 & 0 \\ 0 & 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 15 \\ 15 \\ 20 \\ 30 \\ 20 \end{bmatrix}$$

$$\sum_{l=-\infty}^{\infty} x_3[\mathrm{n} + 5l] = \{15,15,20,30,20\} = x_4[\mathrm{n}]$$

$$\mathrm{e}[n] = x_4[n] - x_3[n] = \begin{bmatrix} 11 \\ 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} = x_3[\mathrm{n} + 5]$$

N=8:

$$x_4[n] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 4 & 3 & 2 \\ 2 & 1 & 0 & 0 & 0 & 0 & 4 & 3 \\ 3 & 2 & 1 & 0 & 0 & 0 & 0 & 4 \\ 4 & 3 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 3 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \\ 20 \\ 30 \\ 20 \\ 11 \\ 4 \\ 0 \end{bmatrix}$$

$$\sum_{l=-\infty}^{\infty} x_3[\mathrm{n} + 8l] = \{4,11,20,30,20,11,4,0\} = x_4[\mathrm{n}]$$

$$\mathrm{e}[n] = x_4[n] - x_3[n] = 0$$

4. (8%) Let $\tilde{x}[n]$ be a periodic sequence with fundamental period N and let $\tilde{X}[k]$ be its DFS. Let $\widetilde{x_3}[n]$ be periodic with period 3N consisting of three periods of $\tilde{x}[n]$ and let $\widetilde{X}_3[k]$ be its DFS. Determine $\widetilde{X}_3[k]$ in terms of $\tilde{X}[k]$.

(a) Solution: The DFS of $\tilde{x}[n]$ and $\tilde{x}_3[n]$ can be written as:

$$\tilde{X}[k] = X[\langle k \rangle_N] = \sum_{n=0}^{N-1} \tilde{x}[n] \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N} n \langle k \rangle_N}$$

$$\tilde{X}_3[k] = X_3[\langle k \rangle_{3N}] = \sum_{n=0}^{3N-1} \tilde{x}[n] \mathrm{e}^{-\mathrm{j}\frac{2\pi}{3N} n \langle k \rangle_{3N}}$$

We have

$$
\begin{aligned}
\tilde{X}_3[k] &= \sum_{n=0}^{3N-1} \tilde{x}[n] e^{-j\frac{2\pi}{3N} n \langle k \rangle_{3N}} \\
&= \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{3N} n \langle k \rangle_{3N}} + \sum_{n=N}^{2N-1} \tilde{x}[n] e^{-j\frac{2\pi}{3N} n \langle k \rangle_{3N}} + \sum_{n=2N}^{3N-1} \tilde{x}[n] e^{-j\frac{2\pi}{3N} n \langle k \rangle_{3N}} \\
&= \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{3N} n \langle k \rangle_{3N}} + \sum_{n=0}^{N-1} \tilde{x}[n+N] e^{-j\frac{2\pi}{3N} (n+N) \langle k \rangle_{3N}} \\
&\quad + \sum_{n=0}^{N-1} \tilde{x}[n+2N] e^{-j\frac{2\pi}{3N} (n+2N) \langle k \rangle_{3N}} \\
&= \left(\sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N} n \langle k \rangle_{N}} \right) \cdot \left(1 + e^{-j\frac{2\pi}{3} \langle k \rangle_{3N}} + e^{-j\frac{4\pi}{3} \langle k \rangle_{3N}} \right) \\
&= 3\tilde{X}[k/3]
\end{aligned}
$$

5. (10%) The first five values of the 9-point DFT of a real-valued sequence x[n] are given by

$$\{4, 2 - j3, 3 + j2, -4 + j6, 8 - j7\}$$

Without computing IDFT and then DFT but using DFT properties only, determine the DFT of each of the following sequences:

(a) $x_1[n] = x[\langle n+2 \rangle_9]$

(b) $x_2[n] = 2x[\langle 2-n \rangle_9]$

(c) $x_3[n] = x[n] ⑨ x[\langle -n \rangle_9]$

(d) $x_4[n] = x^2[n]$

(e) $x_5[n] = x[n] e^{-j4\pi n/9}$

Solution:
From the symmetry property of DFT of real-valued sequence, we can conclude the 9-point DFT as

$$\{4, 2 - j3, 3 + j2, -4 + j6, 8 - j7, 8 + j7, -4 - j6, 3 - j2, 2 + j3\}$$

(a) By applying the time-shifting property, the DFT of $x_1[n]$ is:

$$X_1[k] = W_9^{-2k} X[k]$$

(b) By applying the folding and time-shifting properties, the DFT of $x_2[n]$ is:

$$X_2[k] = 2W_9^{-2k} X^*[k]$$

(c) By applying the correlation property, the DFT of $x_3[n]$ is:

$$X_3[k] = X[k]X^*[k] = |X[k]|^2$$

(d) By applying the windowing property, the DFT of $x_4[n]$ is:

$$X_4[k] = \frac{1}{9}X[k] \textcircled{9} X[k]$$

(e) By applying the frequency-shifting property, the DFT of $x_5[n]$ is:

$$X_5[k] = X[\langle k+2 \rangle_9]$$