Part I. Paper Assignment

1. Determine DFS coefficients of the following periodic sequences:

(a) $\tilde{x}[n] = 2\cos(\pi n/4)$

$$e^{j\frac{2\pi}{N}k_0 n} \leftrightarrow N\delta[k-k_0], e^{j\frac{2\pi}{N}k_0 n} \leftrightarrow N\delta[k-(N-k_0)]$$

$$N = 8, k_0 = 1, \tilde{X}[k] = 8(\delta[k] + \delta[k-7])$$

(b) $\tilde{x}[n] = 3\sin(0.25\pi n) + 4\cos(0.75\pi n)$

$$\cos\left(\frac{2\pi}{N}k_0 n\right) \leftrightarrow \frac{N}{2}(\delta[k-k_0] + \delta[k-(N-k_0)])$$

$$\sin\left(\frac{2\pi}{N}k_0 n\right) \leftrightarrow \frac{N}{2j}(\delta[k-k_0] - \delta[k-(N-k_0)])$$

$$\tilde{X}[k] = \frac{12}{j}(\delta[k-1] - \delta[k-7]) + 16(\delta[k-3] + \delta[k-5])$$

2. Let x[n] be an N-point sequence with an N-point DFT X[k].

(a) If N is even and if $x[n] = -x[\langle n + N/2\rangle_N]$ for all n, then show that X[k] = 0 for even k.

"N is even" means that "n + N/2" is integer.

That is, x[0] = -x[N/2], x[1] = -x[1+N/2], …, x[N/2 -1] = -x[N-1], which also means that $\sum_{n=0}^{N-1} x[n] = 0$;.

$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$, $-j\frac{2\pi}{N}kn = -j\frac{2\pi}{N}k\left(n+\frac{N}{2}\right)$ if k is even, and

$x[n] = -x[\langle n + N/2\rangle_N]$. Therefore, X[k] = 0 for even k.

(b) Show that if N=4m where m is an integer and if $x[n] = -x[\langle n + N/4\rangle_N]$ for all n, then X[k]=0 for k = 4l, $0 \le l \le \frac{N}{4} - 1$

$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$, $-j\frac{2\pi}{N}kn = -j\frac{2\pi}{N}k\left(n+\frac{N}{4}\right)$ if k is 4m, and $x[n] = -x[\langle n + N/4\rangle_N]$. Therefore, X[k] = 0 for k = 4l, $0 \le l \le \frac{N}{4} - 1$.

3. Let $x_1[n], 0 \le n \le N_1 - 1$, be an N_1-point sequence and let $x_2[n], 0 \le n \le N_2 - 1$, be an N_2-point sequence. Let $x_3[n] = x_1[n] * x_2[n]$ and let $x_4[n] = x_1[n] Ⓝ x_2[n], N \ge \max(N_1, N_2)$

(a) Show that $x_4[n] = \sum_{l=-\infty}^{\infty} x_3[n + lN]$

- $x_3[n] = \sum_{m=-\infty}^{\infty} x_1[m]\, x_2[n-m] = \sum_{m=0}^{N_1+N_2-2} x_1[m]\, x_2[n-m]$
- $x_4[n] = x_1[n] Ⓝ x_2[n] \leftrightarrow X_4[k] = X_1[k]X_2[k],\ x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, W_N^{-kn}$

$\Rightarrow x_4[n] = \frac{1}{N}\sum_{k=0}^{N-1} X_1[k]X_2[k]\,W_N^{-kn}$

$= \frac{1}{N}\sum_{k=0}^{N-1}(\sum_{a=0}^{N-1} x_1[n]W_N^{an})(\sum_{b=0}^{N-1} x_2[n]W_N^{bn})\,W_N^{-kn}$

$= \sum_{a=0}^{N-1} x_1[n]\sum_{b=0}^{N-1} x_2[n]\left(\frac{1}{N}\sum_{k=0}^{N-1} W_N^{-k(n-a-b)}\right)$

$\text{, where } \frac{1}{N}\sum_{k=0}^{N-1} W_N^{-k(n-a-b)} = \begin{cases}1, -n+a+b = rN \\ 0, \quad \text{otherwise}\end{cases}$

$\Rightarrow x_4[n] = \sum_{m=0}^{N-1} x_1[m]\,x_2[\langle n-m\rangle_N], -\infty \le n \le \infty$

$= \sum_{m=0}^{N-1} x_1[m]\sum_{r=-\infty}^{\infty} x_2[n-m+rN], N \ge \max(N_1, N_2)$

$= \sum_{r=-\infty}^{\infty}(\sum_{m=0}^{N-1} x_1[m]\,x_2[n-m+rN]), N \ge \max(N_1, N_2)$

$= \sum_{l=-\infty}^{\infty} x_3[n+lN], N \ge \max(N_1, N_2)$

(b) Let $e[n] = x_4[n] - x_3[n]$, show that

$$e[n] = \begin{cases} x_3[n+N], & \max(N_1, N_2) \le N < L \\ 0, & N \ge L \end{cases}$$

where $L = N_1 + N_2 - 1$

- $e[n] = x_4[n] - x_3[n] = \sum_{l=-\infty}^{\infty} x_3[n+lN] - x_3[n], N \ge \max(N_1, N_2)$
- For $N \ge L, x_3[n+lN] = 0$ with $(l \ne 0)$ & $(l \in \mathbb{Z})$
  $\Rightarrow e[n] = \sum_{l=-\infty}^{\infty} x_3[n+lN] - x_3[n] = x_3[n] - x_3[n] = 0$
- For $\max(N_1, N_2) \le N < L, \sum_{l=-\infty}^{\infty} x_3[n+lN] = x_3[n] + x_3[n+N]$
  $\Rightarrow e[n] = (x_3[n] + x_3[n+N]) - x_3[n] = x_3[n+N]$

(c) Verify the results in (a) and (b) for $x_1 = \{1_{n=0}, 2,3,4\}$, $x_2 = \{4_{n=0}, 3,2,1\}$, and N=5 and N=8

- $x_3[n] = \{4_{n=0}, 11,20,30,20,11,4\}$
- For N = 5
  $x_1[n] = \{1_{n=0}, 2,3,4,0\}, x_2[n] = \{4_{n=0}, 3,2,1,0\}$
  $x_{41}[n] = \sum_{m=0}^{4} x_1[m]\,x_2[\langle n-m\rangle_5] \Longrightarrow x_{41}[n] = \{15_{n=0}, 15, 20, 30, 20\}$
  $x_{42}[n] = \sum_{l=-\infty}^{\infty} x_3[n+l\times 5]$
  $\Longrightarrow x_{42}[0] = x_3[0] + x_3[5] = 15, x_{42}[1] = x_3[1] + x_3[6] = 15,$
  $x_{42}[2] = x_3[2] = 20, x_{42}[3] = x_3[3] = 30, x_{42}[4] = x_3[4] = 20$
  $\Longrightarrow x_{42}[n] = \{15_{n=0}, 15, 20, 30, 20\} = x_3[n] + x_3[n+5]$
- For N = 8
  $x_1[n] = \{1_{n=0}, 2,3,4,0,0,0,0\}, x_2[n] = \{4_{n=0}, 3,2,1,0,0,0,0\}$
  $x_{41}[n] = \sum_{m=0}^{7} x_1[m]\,x_2[\langle n-m\rangle_8]$
  $\Longrightarrow x_{41}[n] = \{4_{n=0}, 11, 20, 30, 20, 11, 4, 0\}$
  $x_{42}[n] = \sum_{l=-\infty}^{\infty} x_3[n+l\times 8]$
  $\Longrightarrow x_{42}[0] = x_3[0] = 4, x_{42}[1] = x_3[1] = 11, x_{42}[2] = x_3[2] = 20,$
  $x_{42}[3] = x_3[3] = 30, x_{42}[4] = x_3[4] = 20, x_{42}[5] = x_3[5] = 11,$

$$x_{42}[6] = x_3[6] = 4, x_{42}[7] = x_3[7] = 0$$
$$\Longrightarrow x_{42}[n] = \{4_{n=0}, 11, 20, 30, 20, 11, 4, 0\} = x_{41}[n] = x_3[n]$$

4. Let $\tilde{x}[n]$ be a periodic sequence with fundamental period N and let $\tilde{X}[k]$ be its DFS. Let $\tilde{x}_3[n]$ be periodic with period 3N consisting of three periods of $\tilde{x}[n]$ and let $\widetilde{X_3}[k]$ be its DFS. Determine $\widetilde{X_3}[k]$ in terms of $\tilde{X}[k]$.

$X[k] = \sum_{n=0}^{N-1} x[n]W_N^{kn}, W_N^{kn} = e^{j\frac{2\pi}{N}kn}$

- For $k = 3m, m \in \mathcal{Z}$
  $X_3[k] = \sum_{n=0}^{3N-1} x[n]W_{3N}^{kn} = \sum_{n=0}^{3N-1} x[n]W_{3N}^{3mn} = \sum_{n=0}^{3N-1} x[n]W_N^{mn} = \sum_{n=0}^{N-1} x[n]W_N^{mn} + \sum_{n=N}^{2N-1} x[n]W_N^{mn} + \sum_{n=2N}^{3N-1} x[n]W_N^{mn} = 3X[k]$
- For $k_1 = 3m+1, k_2 = 3m+2, m \in \mathcal{Z}$
  Euler's formula shows that the
  $$(W_{3N}^{n} + W_{3N}^{2n} + W_{3N}^{3n}) = (W_{3N}^{2n} + W_{3N}^{4n} + W_{3N}^{6n}) = 0.$$
  $\Rightarrow X_3[k_1] = \sum_{n=0}^{3N-1} x[n]W_{3N}^{(3m+1)n} = \sum_{n=0}^{3N-1} x[n]W_{3N}^{3mn}W_{3N}^{n} = 0$
  $\Rightarrow X_3[k_2] = \sum_{n=0}^{3N-1} x[n]W_{3N}^{(3m+2)n} = \sum_{n=0}^{3N-1} x[n]W_{3N}^{3mn}W_{3N}^{2n} = 0$
- $X_3[k] = \begin{cases} 3X[k], k = 3m, m \in \mathcal{Z} \\ 0, \quad \text{else} \end{cases}$

5. The first five values of the 9-point DFT of a real-valued sequence x[n] are given by
$$\{4, 2-j3, 3+j2, -4+j6, 8-j7\}$$
Without computing IDFT and then DFT but using DFT properties only, determine the DFT of each of the following sequences:

(a) $x_1[n] = x[\langle n+2\rangle_9]$
  $x[\langle n-m\rangle_N] \leftrightarrow W_N^{km}X[k] \Rightarrow X_1[k] = W_9^{-2k}X[k]$

(b) $x_2[n] = 2x[\langle 2-n\rangle_9]$
  $x[\langle n-m\rangle_N] \leftrightarrow W_N^{km}X[k], x[\langle -n\rangle_N] \leftrightarrow X[\langle -k\rangle_N]$
  $\Rightarrow X_2[k] = 2W_9^{2k}X[\langle -k\rangle_9]$

(c) $x_3[n] = x[n] ⑨ x[\langle -n\rangle_9]$
  $x[n] ⑨ h[\langle -n\rangle_N] \leftrightarrow X[k]H[\langle -k\rangle_N] \Rightarrow X_3[k] = X[k]X[\langle -k\rangle_9]$

(d) $x_4[n] = x^2[n]$
  $v[n]x[n] \leftrightarrow \frac{1}{N}V[k] Ⓝ X[k] \Rightarrow X_4[k] = \frac{1}{9}X[k] ⑨ X[k]$

(e) $x_5[n] = x[n]e^{-j4\pi n/9}$
  $x[n]e^{j\frac{2\pi}{N}k_0 n} \leftrightarrow X[k-k_0] \Rightarrow X_5[k] = X[\langle k+2\rangle_9]$