National Tsing Hua University
Department of Electrical Engineering
EE3660 Intro. to Digital Signal Processing, Spring 2020

Homework Assignment #3: Chap. 5-6

Solution

I Paper Assignment (74%)

1. (6%) Determine the system function, magnitude response, and phase response of the following systems and use the pole-zero pattern to explain the shape of their magnitude response:

(a) $y[n] = \frac{1}{4}(x[n] + x[n-1]) - \frac{1}{4}(x[n-2] + x[n-3])$

(b) $y[n] = x[n] - x[n-4] + 0.6561y[n-4]$

(a)

$$\text{System function:} \quad H(z) = \frac{Y(z)}{X(z)} = \frac{1}{4}(1 + z^{-1} - z^{-2} - z^{-3})$$

Frequency response:

$$H(e^{j\omega}) = \frac{1}{4}[(1+\cos\omega-\cos 2\omega-\cos 3\omega)+j(-\sin\omega+\sin 2\omega+\sin 3\omega)]$$

Magnitude response:

$$|H(e^{j\omega})| = \frac{1}{4}\sqrt{(1+\cos\omega-\cos 2\omega-\cos 3\omega)^2+(-\sin\omega+\sin 2\omega+\sin 3\omega)^2}$$

Phase response:

$$\angle H(e^{j\omega}) = tan^{-1}(\frac{-\sin\omega+\sin 2\omega+\sin 3\omega}{1+\cos\omega-\cos 2\omega-\cos 3\omega})$$

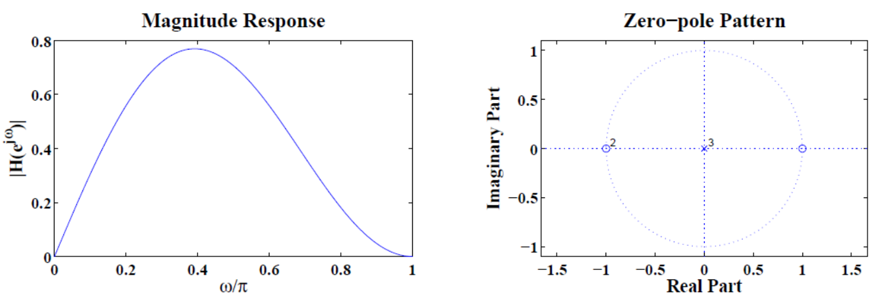


FIGURE 5.20: Magnitude response and pole-zero plot of $y[n] = \frac{1}{4}(x[n] + x[n-1]) - \frac{1}{4}(x[n-2] + x[n-3])$.

(b)

System function:

$$H(z)=\frac{Y(z)}{X(z)}=\frac{1-z^{-4}}{1-0.6561z^{-4}}=\frac{(1+z^{-2})(1+z^{-1})(1-z^{-1})}{(1+0.81z^{-2})(1+0.9z^{-1})(1-0.9z^{-1})}=\frac{(z+i)(z-i)(z+1)(z-1)}{(z+0.9i)(z-0.9i)(z+0.9)(z-0.9)}$$

Magnitude response:

$$\left|H(e^{j\omega})\right|=\frac{\sqrt{(1-\cos 4\omega)^2+(\sin 4\omega)^2}}{\sqrt{(1-0.6561\cos 4\omega)^2+(0.6561\sin 4\omega)^2}}$$

Phase response:

$$\angle H(e^{j\omega})=tan^{-1}\left(\frac{\sin 4\omega}{1-\cos 4\omega}\right)-tan^{-1}\left(\frac{0.6561\sin 4\omega}{1-0.6561\cos 4\omega}\right)$$

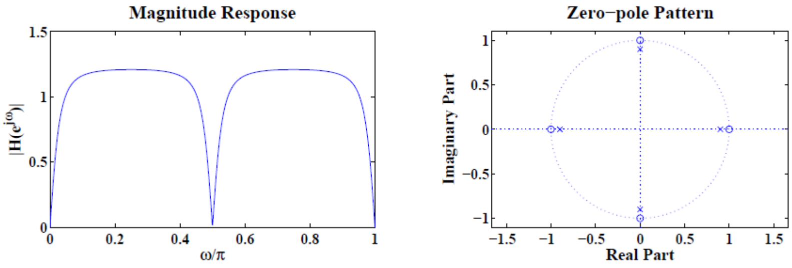


FIGURE 5.137: (a) Magnitude response (b) Pole-zero pattern.

2. (12%) Consider a periodic signal

$$x[n]=\sin(0.1\pi n)+\frac{1}{3}\sin(0.3\pi n)+\frac{1}{5}\sin(0.5\pi n)$$

For each of the following systems, determine if the system imparts (i) no distortion, (ii) magnitude distortion, and/or (iii) phase (or delay) distortion.

(a) $h[n]=\{1_{n=0},-2,3,-4,0,4,-3,2,-1\}$

(b) $y[n]=10x[n-10]$

No distortion means: constant gain, linear phase

(a) Magnitude distortion, Phase distortion

$$H(e^{j\omega})=\sum_{n=-\infty}^{\infty}h[n]e^{-j\omega n}$$
$$=(1-2\cos\omega+3\cos 2\omega-4\cos 3\omega+4\cos 5\omega-3\cos 6\omega+2\cos 7\omega-\cos 8\omega)$$
$$+j(2\sin\omega-3\sin 2\omega+4\sin 3\omega-4\sin 5\omega+3\sin 6\omega-2\sin 7\omega+\sin 8\omega)$$

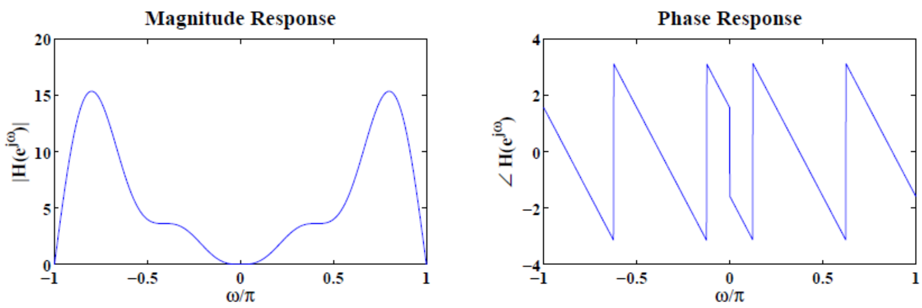


FIGURE 5.67: Magnitude and phase responses of the system.

(b) No distortion

$$\mathrm{H}(e^{j\omega}) = 10e^{-10j\omega} \quad |\mathrm{H}(e^{j\omega})| = 10,\ \angle\mathrm{H}(e^{j\omega}) = -10\omega$$

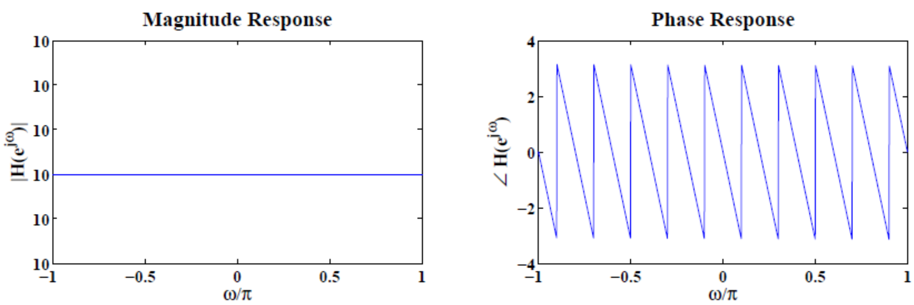


FIGURE 5.69: Magnitude and phase responses of the system.

3. (12%) An economical way to compensate for the droop distortion in S/H DAC is to use an appropriate digital compensation filter prior to DAC.

(a) Determine the frequency response of such an ideal digital filter $H_r(e^{j\omega})$ that will perform an equivalent filtering given by following $H_r(j\Omega)$

$$H_r(j\Omega) = \begin{cases} \frac{\Omega T/2}{\sin(\Omega T/2)} e^{j\Omega T/2}, & |\Omega| < \pi/T \\ 0. & \text{otherwise} \end{cases}$$

(b) One low-order FIR filter suggested in Jackson (1996) is

$$H_{FIR}(z) = -\frac{1}{16} + \frac{9}{8}z^{-1} - \frac{1}{16}z^{-2}$$

Compare the magnitude response of $H_{FIR}(e^{j\omega})$ with that of $H_r(e^{j\omega})$ above.

(c) Another low-order IIR filter suggested in Jackson (1996) is

$$H_{IIR}(z) = \frac{9}{8 + z^{-1}}$$

Compare the magnitude response of $H_{IIR}(e^{j\omega})$ with that of $H_r(e^{j\omega})$ above.

(a) Solution:

$$g_{SH}(t) = \begin{cases} 1, & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases} \xrightarrow{\text{CTFT}} G_{SH}(j\Omega) = \frac{2\sin(\Omega T/2)}{\Omega} e^{-j\Omega T/2}$$

$$H_r(j\Omega) = \begin{cases} \frac{\Omega T/2}{\sin(\Omega T/2)} \cdot e^{j\Omega T/2}, & |\Omega| < \pi/T \\ 0, & \text{otherwise} \end{cases}$$

The frequency response is:

$$H_r(e^{j\omega}) = \begin{cases} \frac{\omega/2}{\sin(\omega/2)} e^{j\omega/2} & ,|\omega| < \pi \\ 0 & ,otherwise \end{cases}$$

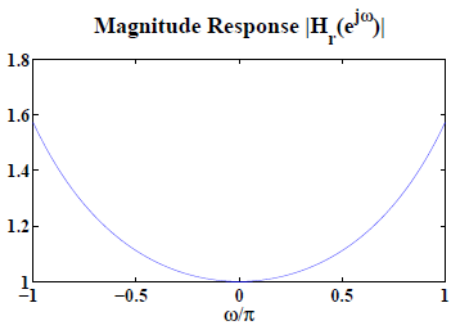


FIGURE 6.21: Magnitude response of ideal digital filter $H_r(e^{j\omega})$.

(b) Solution: The magnitude response of $H_{FIR}(e^{j\omega})$ is:

$$|H_{FIR}(e^{j\omega})| = \sqrt{(-\frac{1}{16} + \frac{9}{8}\cos\omega - \frac{1}{16}\cos 2\omega)^2 + (-\frac{9}{8}\sin\omega + \frac{1}{16}\sin 2\omega)^2}$$
$$= \sqrt{\frac{1}{16^2} + \frac{9^2}{8^2} + \frac{1}{16^2} - 4 \times \frac{9}{8} \times 116\cos\omega + \frac{2}{16^2}\cos 2\omega}$$

(c) Solution: The magnitude response of $H_{IIR}(e^{j\omega})$ is:

$$|H_{IIR}(e^{j\omega})| = \frac{9}{\sqrt{(8+\cos\omega)^2 + \sin^2\omega}} = \frac{9}{\sqrt{1 + 8^2 + 2\cos\omega}}$$

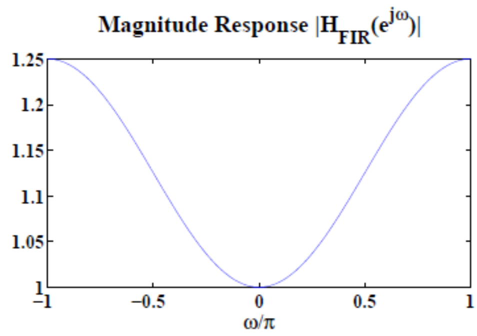


FIGURE 6.22: Magnitude response of low-order FIR filter $H_{\text{FIR}}(e^{j\omega})$.

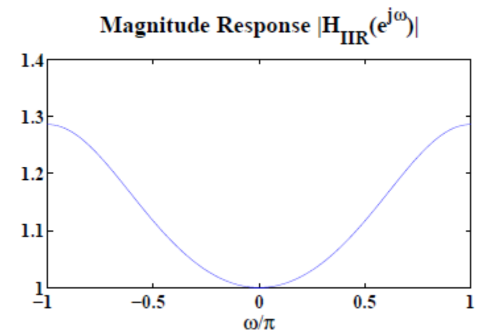


FIGURE 6.23: Magnitude response of low-order IIR filter $H_{\text{IIR}}(e^{j\omega})$.

4. (12%) Consider the following continuous-time system

$$\mathrm{H(s)} = \frac{s^4 - 6s^3 + 10s^2 + 2s - 15}{s^5 + 15s^4 + 100s^3 + 370s^2 + 744s + 720}$$

(a) Show that the system H(s) is a nonminimum phase system.

(b) Decompose H(s) into the product of minimum phase component $H_{min}(s)$ and an all pass component $H_{ap}(s)$.

(c) Briefly plot the magnitude and phase responses of H(s) and $H_{min}(s)$ in one figure and explain your plots.

(d) Briefly plot the magnitude and phase responses of $H_{ap}(s)$.

(a) Proof:

$$H(s) = \frac{(s-3)(s-2-\mathrm{j})(s-2+\mathrm{j})(s+1)}{(s+5)(s+3-3\mathrm{j})(s+3+3\mathrm{j})(s+2-2\mathrm{j})(s+2+2\mathrm{j})}$$

Hence, there are three zero on the right-hand plane which proved that the system $H(s)$ is NOT minimum phase system.

(b) Solution:

$$H(s) = H_{\min}(s) \cdot H_{\text{ap}}(s)$$

$$H_{\min}(s) = \frac{(s+3)(s+2-\mathrm{j})(s+2+\mathrm{j})(s+1)}{(s+5)(s+3-3\mathrm{j})(s+3+3\mathrm{j})(s+2-2\mathrm{j})(s+2+2\mathrm{j})}$$

$$H_{\text{ap}}(s) = \frac{(s-3)(s-2-\mathrm{j})(s-2+\mathrm{j})}{(s+3)(s+2-\mathrm{j})(s+2+\mathrm{j})}$$

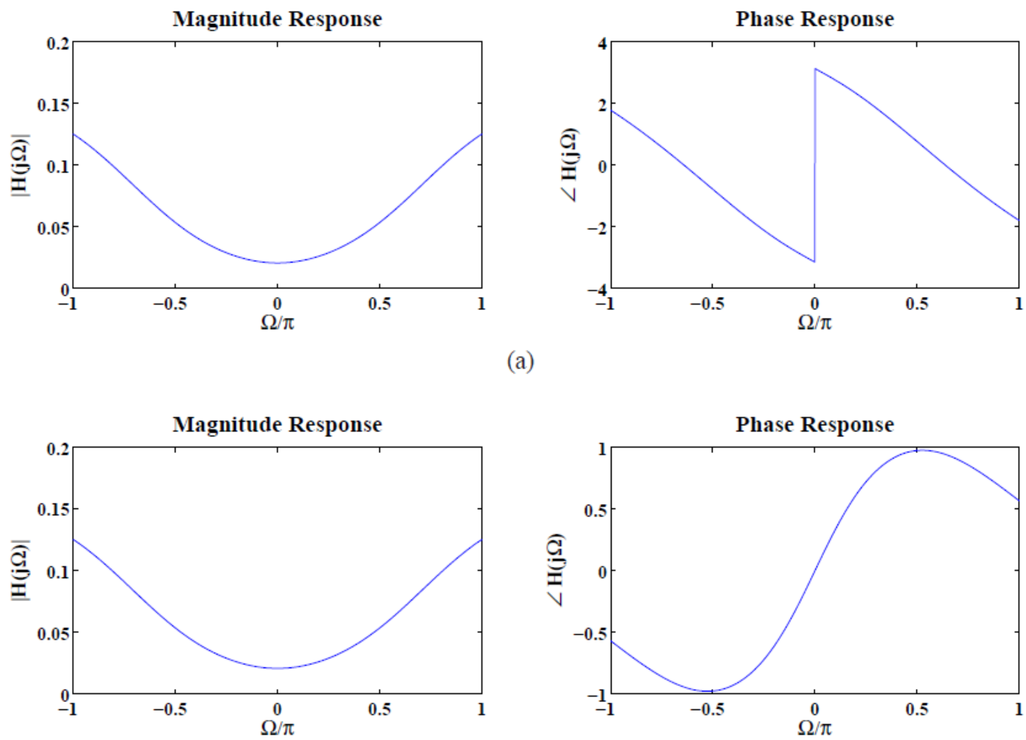


FIGURE 5.47: Magnitude and phase responses of (a) $H(s)$ and (b) $H_{\min}(s)$.

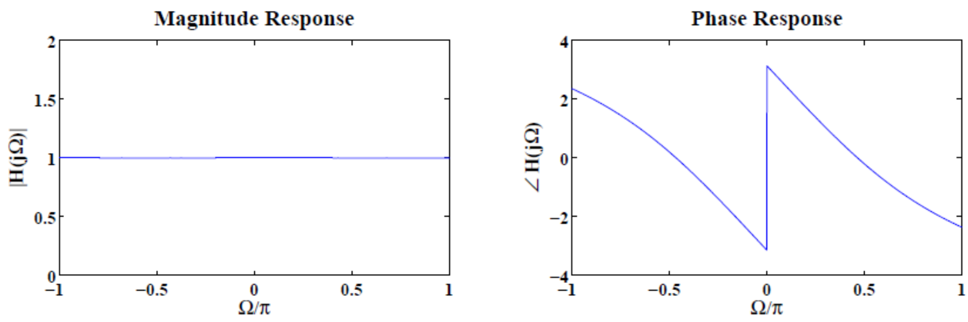


FIGURE 5.48: Magnitude and phase responses of $H_{\mathrm{ap}}(s)$.

5. (12%) We want to design a second-order IIR filter using pole-zero placement that satisfies the following requirements: (1) the magnitude response is 0 at $\omega_1 = 0$ and $\omega_3 = \pi$ (2) The maximum magnitude is 1 at $\omega_{2,4} = \pm\frac{\pi}{4}$ and (3) the magnitude response is approximately $\frac{1}{\sqrt{2}}$ at frequencies $\omega_{2,4} \pm 0.05$

(a) Determine locations of two poles and two zeros of the required filter and then compute its system function H(z).

(b) Briefly graph the magnitude response of the filter.

(c) Briefly graph phase and group-delay responses.

Determine the response with the definition of Mag/Phase response:

$$|H(\mathrm{e}^{\mathrm{j}\omega})| = |b_0| \prod_{k=1}^{M} \left|1 - z_k \mathrm{e}^{-\mathrm{j}\omega}\right| \bigg/ \prod_{k=1}^{N} \left|1 - p_k \mathrm{e}^{-\mathrm{j}\omega}\right|,$$

$$\angle H(\mathrm{e}^{\mathrm{j}\omega}) = \angle b_0 + \sum_{k=1}^{M} \angle(1 - z_k \mathrm{e}^{-\mathrm{j}\omega}) - \sum_{k=1}^{N} \angle(1 - p_k \mathrm{e}^{-\mathrm{j}\omega}),$$

Or from the Pole/Zero Map

(a) Solution:

$$\text{zeros:} \quad z_1 = e^{j0} = 1, \quad z_2 = e^{j\pi} = -1$$

$$\text{poles:} \quad p_1 = re^{j\frac{\pi}{4}}, \quad p_2 = re^{-j\frac{\pi}{4}}, \quad r \in (0,1)$$

The system function is:

$$H(z) = b_0 \frac{(1-z^{-1})(1+z^{-1})}{(1-re^{j\frac{\pi}{4}}z^{-1})(1-re^{-j\frac{\pi}{4}}z^{-1})}$$

The frequency response is:

$$H(e^{j\omega}) = b_0 \frac{(1-e^{-j\omega})(1+e^{-j\omega})}{(1-re^{j\frac{\pi}{4}}e^{-j\omega})(1-re^{-j\frac{\pi}{4}}e^{-j\omega})}$$

Constrain $|H(e^{j\omega})|_{\max} = 1$, we have

at w = +-pi/4, |H| have maximum value

$$|b_0| = \frac{(1-r)\sqrt{1+r^2}}{\sqrt{2}}$$

Choose $r = 0.95$.

(b) See plot below.

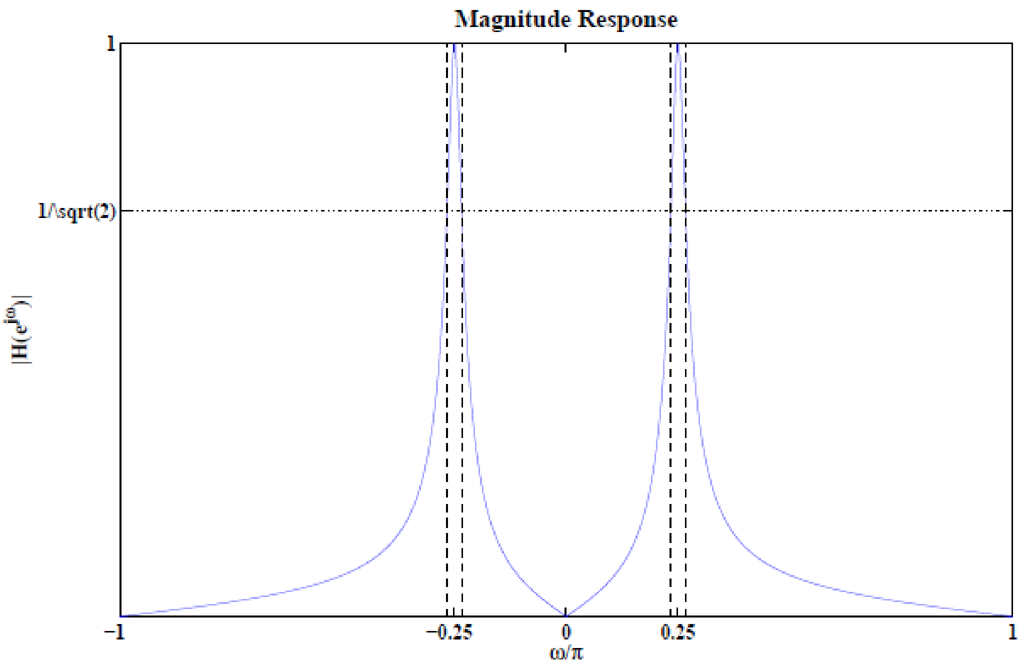


FIGURE 5.96: Magnitude response of the filter.

(c)

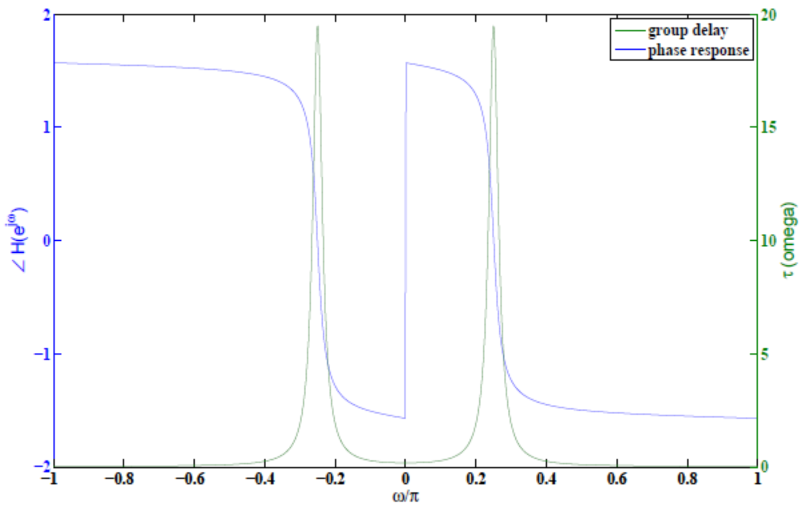


FIGURE 5.97: Phase and group-delay responses of the filter.

6. (10%) The following signals $x_c(t)$ is sampled periodically to obtained the discrete-time signal x$[n]$. For each of the given sampling rates in F_s Hz or in T period, (i) determine the spectrum $X(e^{i\omega})$ of x$[n]$; (ii) plot its magnitude and phase as a function of ω in $\frac{rad}{sam}$ and as a function of F in Hz; and (iii) explain whether $x_c(t)$ can be recovered from x$[n]$.

(a) $x_c(t) = 5e^{i40t} + 3e^{-i70t}$, with sampling period T = 0.01, 0.04, 0.1

(b) $x_c(t) = 3 + 2\sin(16\pi t) + 10\cos(24\pi t)$, with sampling rate F_s = 30, 20, 15 Hz.

(a)

$$X(j\Omega) = 10\pi\delta(\Omega - 40) + 6\pi\delta(\Omega + 70)$$

The spectra of the sampled sequence $x[n]$ is:

$$X(e^{j\omega})\big|_{\omega=\Omega T} = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_r[j(\Omega - k\Omega_s)]$$

The continuous signal $x_c(t)$ can be recovered if the sampling interval is (a) $T = 0.01$, (b) $T = 0.04$, and can NOT be recovered if the sampling interval is (c) $T = 0.1$.

T=0.01, $\Omega_s = 2\pi F_s = 200\pi$, $\omega_s = \Omega_s/100$

$$X(e^{j\omega}) = 100 \sum_{k=-\infty}^{\infty} 10\pi\delta(\Omega - 40 - 200\pi k) + 6\pi\delta(\Omega + 70 - 200\pi k)$$

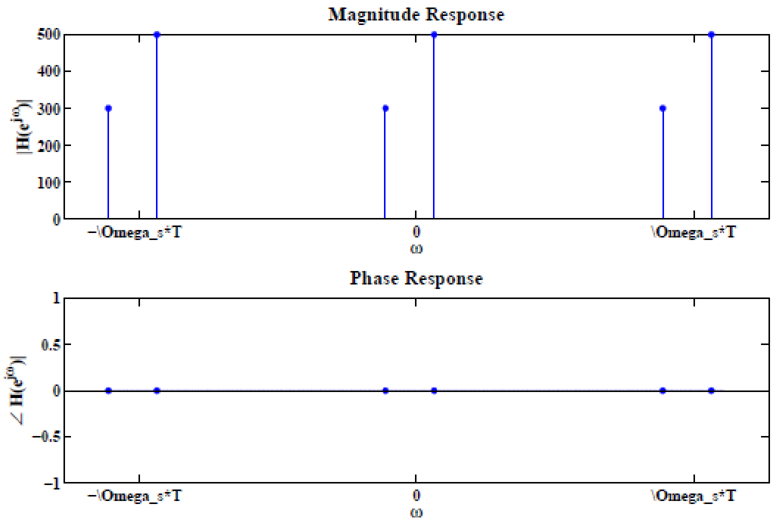


FIGURE 6.30: Magnitude and phase responses of $X(e^{j\omega})$ as a function of ω in $\frac{\text{rad}}{\text{sam}}$ when the sampling interval is $T = 0.01$.

T=0.04, $\Omega_s = 2\pi F_s = 50\pi$, $\omega_s = \Omega_s/25$

$$X(e^{j\omega}) = 25 \sum_{k=-\infty}^{\infty} 10\pi\delta(\Omega - 40 - 50\pi k) + 6\pi\delta(\Omega + 70 - 50\pi k)$$

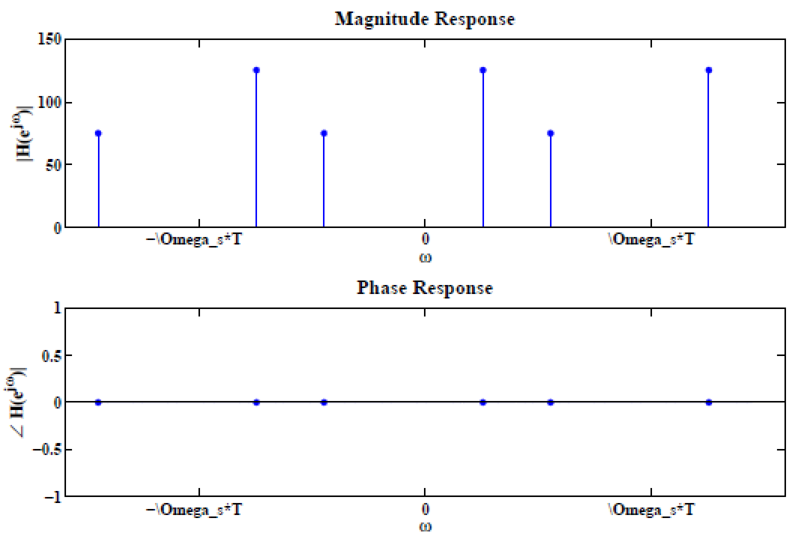


FIGURE 6.31: Magnitude and phase responses of $X(e^{j\omega})$ as a function of ω in $\frac{\text{rad}}{\text{sam}}$ when the sampling interval is $T = 0.04$.

T=0.1, $\Omega_s = 2\pi F_s = 20\pi, \quad \omega_s = \Omega_s/10$

$$X(e^{j\omega}) = 10 \sum_{k=-\infty}^{\infty} 10\pi\delta(\Omega - 40 - 20\pi k) + 6\pi\delta(\Omega + 70 - 20\pi k)$$

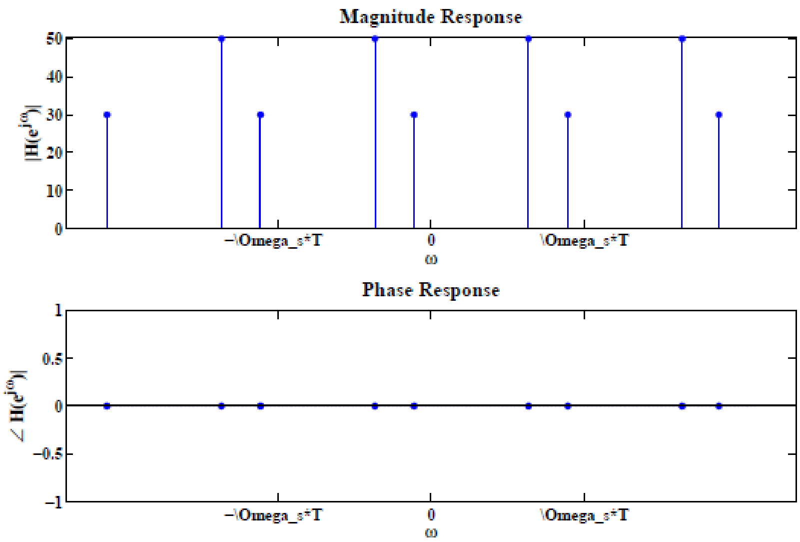


FIGURE 6.32: Magnitude and phase responses of $X(e^{j\omega})$ as a function of ω in $\frac{\text{rad}}{\text{sam}}$ when the sampling interval is $T = 0.1$.

(b)

$$Xc(j\Omega) = 6\pi + \frac{1}{j}2\pi\delta(\Omega - 16\pi) - \frac{1}{j}2\pi\delta(\Omega + 16\pi) + 10\pi\delta(\Omega - 24\pi) + 10\pi\delta(\Omega + 24\pi)$$

The spectra of the sampled sequence $x[n]$ is:

$$X(e^{j\omega})\big|_{\omega=2\pi F/F_s} = F_s \sum_{k=-\infty}^{\infty} X_c[j2\pi(F - kF_s)]$$

The signal $x_c(t)$ can be recovered from $x[n]$ if the sampling rate is (a) $F_s =$ 30 Hz, and can NOT be recovered if the sampling rate is (b) $F_s = 20$ Hz, (c) $F_s = 15$ Hz.

Fs=30KHz

$$X(e^{j\omega}) = 30 \sum_{k=-\infty}^{\infty} Xc(j\,\Omega - j60\pi k)$$

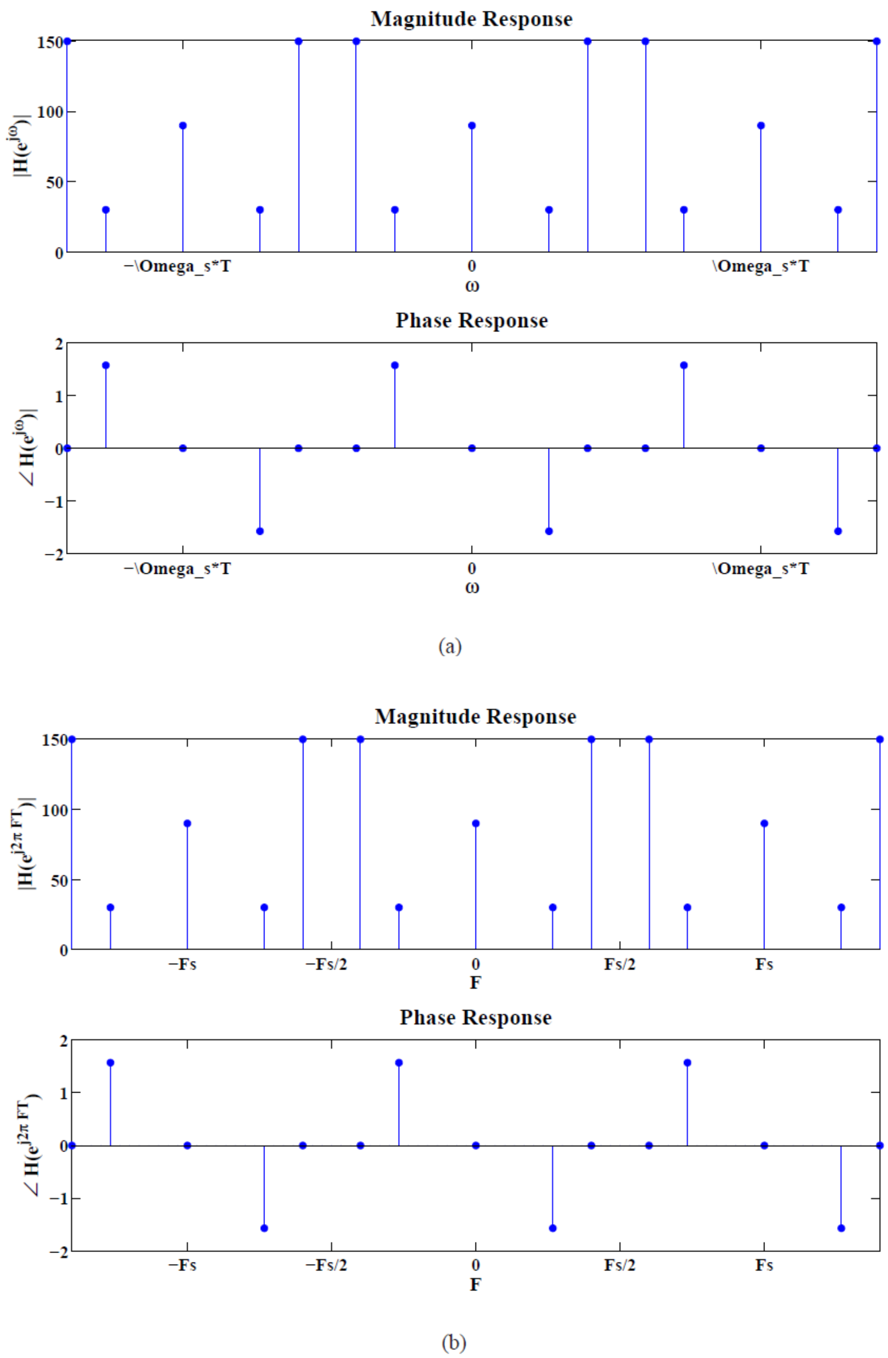
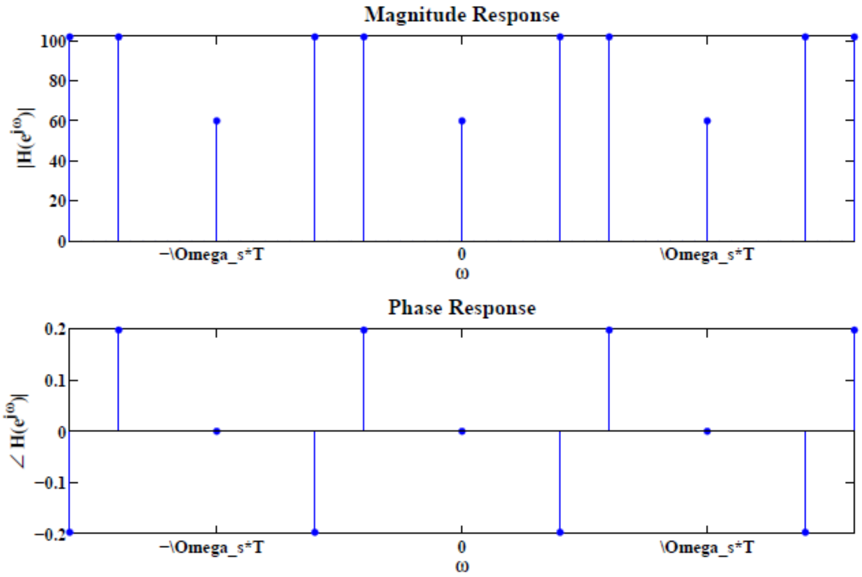


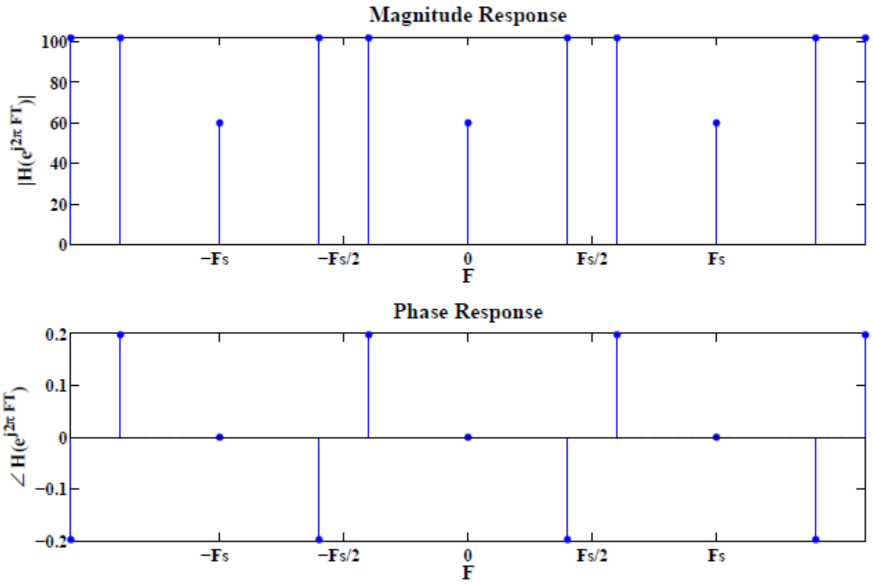
FIGURE 6.48: Spectra of $X(e^{j\omega})$ as a function of (a) ω in $\frac{\text{rad}}{\text{sam}}$ and (b) F in Hz when the sample rate is $F_s = 30$ KHz.

Fs=20KHz

$$X(e^{j\omega}) = 20 \sum_{k=-\infty}^{\infty} Xc(j\,\Omega - j40\pi k)$$



(a)

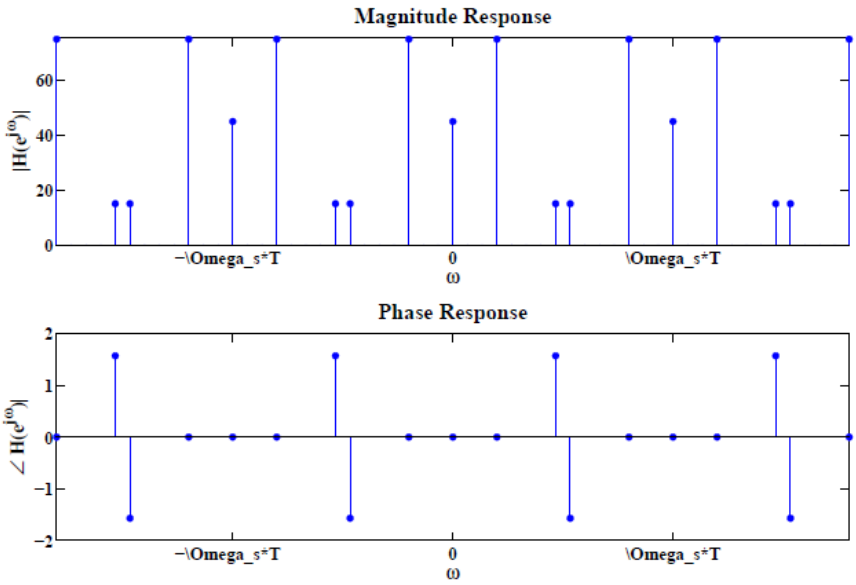


(b)

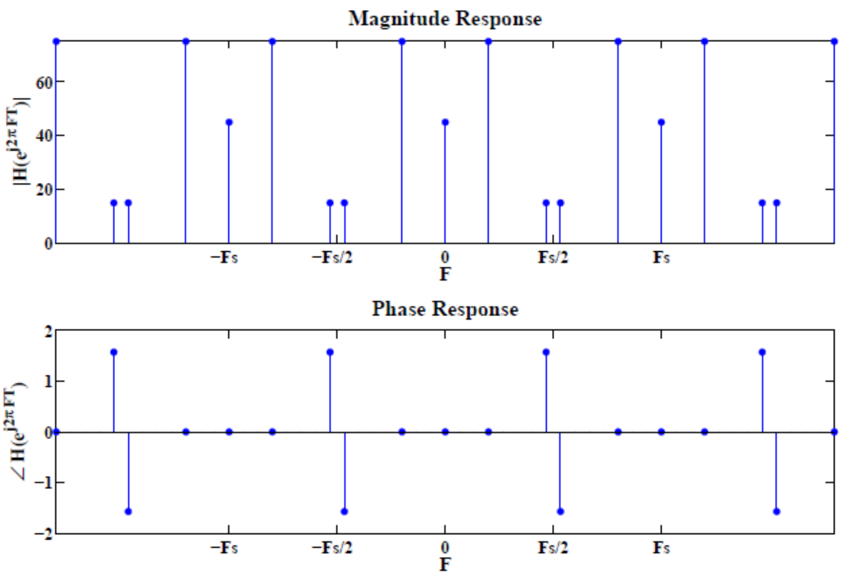
FIGURE 6.49: Spectra of $X(e^{j\omega})$ as a function of (a) ω in $\frac{\text{rad}}{\text{sam}}$ and (b) F in Hz when the sample rate is $F_s = 20$ KHz.

Fs=15KHz

$$X(e^{j\omega}) = 15 \sum_{k=-\infty}^{\infty} Xc(j\Omega - j30\pi k)$$



(a)



(b)

FIGURE 6.50: Spectra of $X(e^{j\omega})$ as a function of (a) ω in $\frac{\text{rad}}{\text{sam}}$ and (b) F in Hz when the sample rate is $F_s = 15$ KHz.

7. (8%) An 8-bit ADC has an input analog range of ±5 volts. The analog input signal is

$$x_c(t) = 2\cos(200\pi t) + 3\sin(500\pi t)$$

The converter supplies data to a computer at a rate of 2048 bits/s. The computer, without processing, supplies these data to an ideal DAC to form the reconstructed signal $y_c(t)$. Determine:

(a) the quantizer resolution (or step),

(b) the SQNR in dB,

(c) the folding frequency and the Nyquist rate,

(a) Solution:
The quantizer resolution is:

$$\frac{10\text{v}}{2^8} = 0.0390625\text{v}$$

(b) Solution:

$$\text{SQNR} = 10\log_{10}\text{SQNR} = 6.02B+1.76 = 6.02\times 8+1.76 = 49.92\text{dB}$$

(c) Solution:
The sampling rate is:

$$F_s = \frac{2^{11}}{2^3} = 2^8 \text{sam/sec}$$

The folding frequency is $F_s/2 = 2^7$.
The Nyquist rate is 500.