National Tsing Hua University
Department of Electrical Engineering
EE3660 Intro. to Digital Signal Processing, Spring 2020

Homework Assignment #1: Chap. 2-4

Solution

I Paper Assignment (100%)

1. (12%) Determine whether the following systems are linear, time-invariant, causal, and stable.

(a) $y[n] = x[-n]$

(b) $y[n] = \cos(\pi n)x[n]$

(c) $y[n] = \sum_{k=n-1}^{k=\infty} x[k]$

Solution

(a) $y[n] = x[-n]$

(i) Linear：

$y[n] = L(x[n]) = x[-n],$

$Let\, y3[n] = L(a^* x1[n] + b^* x2[n])$

$= a^* x1[-n] + b^* x2[-n]$

$= a^* L(x1[n]) + b^* L(x2[n])$

$= a^* y1[n] + b^* y2[n]$

It’s a linear system.

(ii) Time-invariant：

This is a “flip” function.

$L(x[n-n0]) = x[-n-n0]$

$\neq y[n-n0] = x[-(n-n0)] = x[-n+n0]$

It’s not a time-invariant system.

(iii) Causal：

When n < 0, the values of y[n] depend on the future values of x[n]

It's not a causal system.

(iv) Stable：

$| y[n] | \leq | x[-n] | \leq M$,

It's a stable system.

(b) $y[n] = \cos(\pi n)x[n]$

(i) Linear：

$y[n] = L(x[n]) = \cos(\pi n)x[n]$,

$Let\ y3[n] = L(a^*x1[n] + b^*x2[n])$

$= \cos(\pi n)(a^*x1[n] + b^*x2[n])$

$= a^*\cos(\pi n)x1[n] + b^*\cos(\pi n)x2[n]$

$= a^*L(x1[n]) + b^*L(x2[n])$

$= a^*y1[n] + b^*y2[n]$

It's a linear system.

(ii) Time-invariant：

$y[n] = L(x[n]) = (-1)^n x[n]$,

$L(x[n-1]) = (-1)^n x[n-1]$

$\neq y[n-1] = (-1)^{n-1}x[n-1]$

It's not a time-invariant system.

(iii) Causal：

It's each output only depends on the current value of x[n].

It's a causal system.

(iv) Stable：

$\cos(\pi n) = (-1)^n,\ n \in Z$

This system only changes the sign of x[n], and doesn't change the magnitude of it.

It's a stable system.

(c) $\mathrm{y}[n] = \sum_{k=\mathrm{n}-1}^{k=\infty} \mathrm{x}[k]$

(i) Linear：

$\mathrm{y}[n] = L(x[n]) = \sum_{k=\mathrm{n}-1}^{k=\infty} \mathrm{x}[k]$

$Let\, y3[n] = L(a^* x1[n] + b^* x2[n])$

$= \sum_{k=\mathrm{n}-1}^{k=\infty} (a^* x1[k] + b^* x2[k])$

$= a^* \sum_{k=\mathrm{n}-1}^{k=\infty} \mathrm{x1}[k] + b^* \sum_{k=\mathrm{n}-1}^{k=\infty} \mathrm{x2}[k]$

$= a^* L(x1[n]) + b^* L(x2[n])$

$= a^* y1[n] + b^* y2[n]$

It's a linear system.

(ii) Time-invariant：

$\mathrm{y}[n] = L(x[n]) = \sum_{k=\mathrm{n}-1}^{k=\infty} \mathrm{x}[k],$

$L(x[n - n0]) = \sum_{k=\mathrm{n}-\mathrm{n}0-1}^{k=\infty} \mathrm{x}[k],$

$= \mathrm{y}[\mathrm{n} - \mathrm{n}0]$

It's a time-invariant system.

(iii) Causal：

It's not causal, since it sums forward in time.

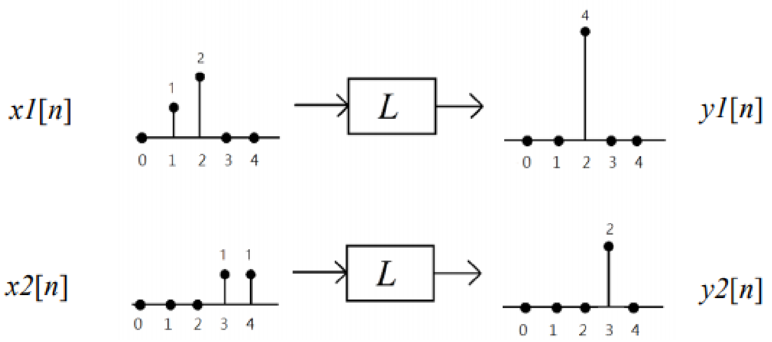
It's not a causal system.

(iv) Stable：

Let x[n] = δ[n], $\sum_{k=\mathrm{n}-1}^{k=\infty} \delta[k] = \infty$

It's not a stable system.

2. (10%) A linear system L generates output signals: $y1[n]$, $y2[n]$ in response to the input signals $xl[n]$, $x2[n]$ respectively. Determine whether the system L is time invariant.



Solution

Suppose system L is time invariant,

$$\delta[n] = x1[n+2] - x2[n+4] = \{\underset{\uparrow}{1}, 0, 0, 0, 0\}$$

Due to L is linear,

$$h[n] = y1[n+2] - y2[n+4] = \{-2, \underset{\uparrow}{4}, 0, 0, 0\}$$

$$x1[n] = \delta[n-1] + 2*\delta[n-2]$$

$$\hat{y}1[n] = h[n-1] + 2*\ h[n-2] = \{0, \underset{\uparrow}{-2}, 4, 0, 0\} + \{0, \underset{\uparrow}{0}, -4, 8, 0\}$$

$$= \{0, \underset{\uparrow}{-2}, 0, 8, 0\} \neq y1[n] = \{\underset{\uparrow}{0}, 0, 4, 0, 0\}$$

L is not time invariant system.

3. (12%) Given the z-transform pair x[n] ↔ X(z) = $\frac{1}{(1-2z^{-1})}$ with ROC: |z| < 2, use the z-transform properties to determine the z-transform of the following sequences:

(a) $y[n] = (\frac{1}{3})^n\, x[n]$

(b) $y[\mathrm{n}] = x[n]*x[-n]$ *(* denotes convolution)*

(c) $y[n] = nx[n]$

Solution

(a) $y[n] = (\frac{1}{3})^n\, x[n]$

Scaling：

$Y(z) = X(3z) = \frac{1}{1-\frac{2}{3}z^{-1}}$, ROC：| z | < $\frac{2}{3}$

(a) $y[\mathrm{n}] = x[n]*x[-n]$

Folding and convolution：

$Y(z) = X(z)X\left(\frac{1}{z}\right) = \frac{1}{1-2z^{-1}}\frac{1}{1-2z} = \frac{-1/2}{1-\frac{5}{2}z^{-1}+z^{-2}}$, ROC：0.5 < | z | < 2

(b) $y[n] = nx[n]$

Differentiation：

$Y(z) = -z\frac{dX(z)}{dz} = \frac{2z^{-1}}{1-4z^{-1}+4z^{-2}}$, ROC：| z | < 2

4. (12%) A causal LTI system has impulse response h[n], for which the z-transform is

$$H(z) = \frac{1 + z^{-1}}{(1 - 0.5z^{-1})(1 + 0.25z^{-1})}$$

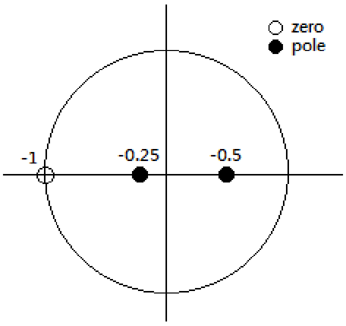
(a) Draw the pole-zero plot of H(z) and specify its ROC.

(b) Explain whether the system is stable?

(c) Find the impulse response h[n] of the system.

Solution

(a) Draw the pole-zero plot of H(z) and specify its ROC.



Due to H(z) is a causal system, ROC should include | z | = ∞

ROC：| z | > 0.5

(b) Explain whether the system is stable?

Due to the ROC includes the unit circle, this system is stable.

(c) Find the impulse response h[n] of the system.

$$H(z) = \frac{1+z^{-1}}{(1-0.5z^{-1})(1+0.25z^{-1})} = \frac{2}{(1-0.5z^{-1})} - \frac{1}{(1+0.25z^{-1})}$$

$$h[n] = 2\left(\frac{1}{2}\right)^n u[n] - \left(-\frac{1}{4}\right)^n u[n]$$

5. (15%) Use the method of partial fraction expansion to determine the sequences corresponding to the following z-transforms:

(a) $X(z) = \frac{z}{z^3 + 2z^2 + \frac{5}{4}z + \frac{1}{4}}$, $|z| > 1$.

(b) $X(z) = \frac{z}{(z^2 - \frac{1}{3})^2}$, $|z| < 0.5$.

Solution

(a) $X(z) = \frac{z}{z^3 + 2z^2 + \frac{5}{4}z + \frac{1}{4}}$, $|z| > 1$.

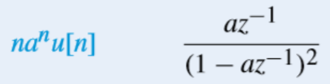
$$\mathrm{X(z)} = \frac{z^{-2}}{(1+z^{-1})(1+0.5z^{-1})^2} \ ,$$

$$\mathrm{X(z)} = \frac{A}{1+z^{-1}} + \frac{B}{1+0.5z^{-1}} + \frac{C}{(1+0.5z^{-1})^2} \ ,$$

$$\mathrm{A} = (1+z^{-1})\mathrm{X(z)}|_{z^{-1}=-1} = 4,$$
$$\mathrm{C} = (1+0.5z^{-1})^2\mathrm{X(z)}|_{z^{-1}=-0.5} = -4,$$
$$z^{-2} = (1+0.5z^{-1})^2\mathrm{A} + (1+z^{-1})(1+0.5z^{-1})\mathrm{B} + (1+z^{-1})\mathrm{C}$$
$$z^{-2} = (1+0.5z^{-1})^2 4 + (1+z^{-1})(1+0.5z^{-1})\mathrm{B}\text{-}(1+z^{-1})4$$
$$z^{-2} = z^{-2} + 0.5z^{-2}\mathrm{B},\ \ \mathrm{B} = 0 \ \text{, (focus on } z^{-2} \text{ coefficients)}$$

$$\mathrm{X(z)} = \frac{4}{1+z^{-1}} + \frac{-4}{(1+0.5z^{-1})^2}$$

$$\mathrm{X(z)} = \frac{4}{1+z^{-1}} + \frac{8*(\text{-}0.5)z^{-1}}{(1+0.5z^{-1})^2}z$$



$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$

$$\mathrm{x[n]} \ = 4(-1)^n u[n] + 8(n+1)(-0.5)^{n+1}u[n+1]$$
$$\mathrm{x[n]} \ = 4(-1)^n u[n] - 4(n+1)(-0.5)^n u[n]$$
$$\mathrm{x[n]} \ = 4(-1)^n u[n] - 4(-0.5)^n u[n] - 4n(-0.5)^n u[n]$$

(b) $X(z) = \frac{z}{(z^2 - \frac{1}{3})^2}$, $|z| < 0.5$.

$$\mathrm{X(z)} = \frac{z}{(z-\frac{1}{\sqrt{3}})^2(z+\frac{1}{\sqrt{3}})^2} = \frac{A}{z-\frac{1}{\sqrt{3}}} + \frac{B}{(z-\frac{1}{\sqrt{3}})^2} + \frac{C}{z+\frac{1}{\sqrt{3}}} + \frac{D}{(z+\frac{1}{\sqrt{3}})^2} \ ,$$

$$\mathrm{B} = (z-\tfrac{1}{\sqrt{3}})^2\mathrm{X(z)}|_{z=\frac{1}{\sqrt{3}}} = \frac{\sqrt{3}}{4},$$

$$\mathrm{D} = (z+\frac{1}{\sqrt{3}})^2\mathrm{X(z)}|_{z=-\frac{1}{\sqrt{3}}} = \frac{-\sqrt{3}}{4},$$

$$\mathrm{z} = (z-\frac{1}{\sqrt{3}})(z+\frac{1}{\sqrt{3}})^2\mathrm{A} + (z+\frac{1}{\sqrt{3}})^2\mathrm{B} + (z-\frac{1}{\sqrt{3}})^2(z+\frac{1}{\sqrt{3}})\mathrm{C} + (z-\frac{1}{\sqrt{3}})^2\mathrm{D}$$

$0 = \frac{-1}{3}\frac{1}{\sqrt{3}}\mathrm{A} + \frac{1}{3}\frac{\sqrt{3}}{4} + \frac{1}{3}\frac{1}{\sqrt{3}}\mathrm{C} + \frac{1}{3}\frac{-\sqrt{3}}{4}$, $\mathrm{A} = \mathrm{C}$, (focus on constant coefficients)

$0 = \frac{1}{\sqrt{3}}\mathrm{z}^2\mathrm{A} + \mathrm{z}^2\frac{\sqrt{3}}{4} + \frac{1}{\sqrt{3}}\mathrm{z}^2\mathrm{C} + \mathrm{z}^2\frac{-\sqrt{3}}{4}$, $\mathrm{A} = \text{-}\mathrm{C}$, (focus on z^2 coefficients)

$\mathrm{A} = \mathrm{C} = 0,$

$$\mathrm{X(z)} = \frac{\frac{\sqrt{3}}{4}}{(z-\frac{1}{\sqrt{3}})^2} + \frac{\frac{-\sqrt{3}}{4}}{(z+\frac{1}{\sqrt{3}})^2},$$

$$\mathrm{X(z)} = \frac{\frac{3}{4}\frac{1}{\sqrt{3}}z^{-1}}{(1-\frac{1}{\sqrt{3}}z^{-1})^2}z^{-1} + \frac{\frac{3}{4}\frac{-1}{\sqrt{3}}z^{-1}}{(1+\frac{1}{\sqrt{3}}z^{-1})^2}z^{-1} \quad ,$$

$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$

$$\mathrm{x[n]} = \frac{-3}{4}(n-1)(\frac{1}{\sqrt{3}})^{n-1}u[-(n-1)-1] + \frac{-3}{4}(n-1)(\frac{-1}{\sqrt{3}})^{n-1}u[-(n-1)-1]$$

$$\mathrm{x[n]} = \frac{-3}{4}(n-1)(\frac{1}{\sqrt{3}})^{n-1}u[-n] - \frac{3}{4}(n-1)(\frac{-1}{\sqrt{3}})^{n-1}u[-n]$$

6. (12%) A function called autocorrelation for a real-valued, absolutely summable sequence x[n], is defined as

$$r_{xx}[\ell] \triangleq \sum_n x[n]x[n-\ell].$$

Let X(z) be the z-transform of x[n] with ROC $\alpha < |z| < \beta$.

(a) Show that the z-transform of $r_{xx}[\ell]$ is given by $R_{xx}(z) = X(z)X(z^{-1})$.

(b) Let $x[n] = a^n u[n]$, $|a| < 1$. Determine $R_{xx}(z)$ and sketch its pole-zero plot and the ROC.

Solution

(a) Show that the z-transform of $r_{xx}[\ell]$ is given by $R_{xx}(z) = X(z)X(z^{-1})$.

$$r_{xx}[\ell] \triangleq \sum_n x[n]x[n-\ell] = \sum_m x[m+l]x[m] = x[l] * x[-l]$$

By applying the folding property, the z-transform of sequence $x[-l]$ is $X(z^{-1})$ with ROC：$\beta -1 < | z | < \alpha-1$. Hence, we prove $R_{xx}(z) = X(z)X(z^{-1})$

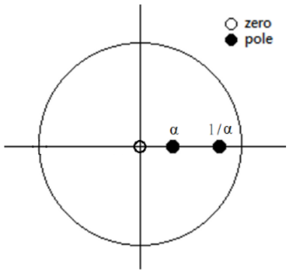
ROC：$\max\{\alpha, \beta-1\} < | z | < \min\{\beta, \alpha-1\}$

(b) Let $x[n] = a^n u[n]$, $|a| < 1$. Determine $R_{xx}(z)$ and sketch its pole-zero plot and the ROC.

$X(z) = \frac{1}{1-\alpha z^{-1}}$, ROC：$| z | > | \alpha |$

$X(z^{-1}) = \frac{-\alpha z^{-1}}{1-\alpha^{-1}z^{-1}}$, ROC：$| z | < | \alpha^{-1} |$

$R_{zz}(z) = X(z)X(z^{-1}) = \frac{-\alpha z^{-1}}{1-(\alpha+\alpha^{-1})z^{-1}+z^{-2}}$, ROC：$| \alpha | < | z | < | \alpha^{-1} |$



7. (12%) Determine the DTFT of following signals:

(a) $x1[n] = (\frac{1}{4})^n cos(\frac{\pi n}{4}) u[n-2]$

(b) $x2[n] = sin(0.1\pi n)(u[n] - u[n-10])$

Solution

DTFT：$X(e^{jw}) = \sum_{n=-\infty}^{n=\infty} x[n] e^{-jwn}$

(a) $x1[n] = (\frac{1}{4})^n cos(\frac{\pi n}{4}) u[n-2]$

$$X1(e^{jw}) = \sum_{n=2}^{n=\infty} (\frac{1}{4})^n cos(\frac{\pi n}{4}) e^{-jwn}$$

$$= \sum_{n=2}^{n=\infty} (\frac{1}{4})^n (\frac{1}{2}) (e^{j\frac{\pi n}{4}} + e^{j\frac{-\pi n}{4}}) e^{-jwn}$$

$$= \sum_{n=2}^{n=\infty} (\frac{1}{4})^n (\frac{1}{2}) (e^{-j\left(w-\frac{\pi}{4}\right)n} + e^{-j\left(w+\frac{\pi}{4}\right)n})$$

$$= (\frac{1}{4})^2 (\frac{1}{2}) (e^{-j\left(w-\frac{\pi}{4}\right)2} + e^{-j\left(w+\frac{\pi}{4}\right)2}) + (\frac{1}{4})^3 (\frac{1}{2}) (e^{-j\left(w-\frac{\pi}{4}\right)3} + e^{-j\left(w+\frac{\pi}{4}\right)3}) + \ldots$$

$$= (\frac{1}{2})^5 (\frac{e^{-j\left(w-\frac{\pi}{4}\right)2}}{1-\frac{1}{4}e^{-j\left(w-\frac{\pi}{4}\right)}} + \frac{e^{-j\left(w+\frac{\pi}{4}\right)2}}{1-\frac{1}{4}e^{-j\left(w+\frac{\pi}{4}\right)}})$$

(b) $x2[n] = sin(0.1\pi n)(u[n] - u[n-10])$

$$X2(e^{jw}) = \sum_{n=0}^{n=9} sin(0.1\pi n)\, e^{-jwn}$$

$$= \sum_{n=0}^{n=9} (\frac{1}{2j}) (e^{j0.1\pi n} - e^{-j0.1\pi n}) e^{-jwn}$$

$$= \sum_{n=0}^{n=9} (\frac{1}{2j}) (e^{j(0.1\pi - w)n} - e^{-j(0.1\pi + w)n})$$

$$= (\frac{1}{2j}) (\frac{1-e^{j10(0.1\pi - w)}}{1-e^{j(0.1\pi - w)}} - \frac{1-e^{j10(-0.1\pi - w)}}{1-e^{j(-0.1\pi - w)}})$$

8. (15%) Let $x[n]$ and $y[n]$ denote complex sequences and $X(e^{j\omega})$ and $Y(e^{j\omega})$ their respective Fourier transforms

(a) Determine, in terms of x[n] and y[n], the sequence whose Fourier transform is $X(e^{j\omega})Y^*(e^{j\omega})$. *(See lecture slide ch4 p21. You can use ∗ as convolution operator)*

(b) Using the result in part (a), show that

$$\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega. \qquad \text{(eq.7b)}$$

(*eq.7b is a more general form of Parseval's theorem, as mentioned in lecture slide ch4 p17.*)

(c) Using (eq.7b), determine the value of the sum

$$\sum_{-\infty}^{\infty} \frac{sin(\pi n/5)}{2\pi n}\frac{sin(\pi n/3)}{7\pi n}$$

(*Hint: Check what is the expression of an inverse Fourier transform of a rectangular pulse mentioned in lecture slide ch4 p23.*)

Solution

(a) Determine, in terms of x[n] and y[n], the sequence whose Fourier transform is $X(e^{j\omega})Y^*(e^{j\omega})$.

The Fourier transform of $y^*[-n]$ is $Y^*(e^{j\omega})$,

$$G(e^{j\omega}) = X(e^{j\omega})Y^*(e^{j\omega})$$

$$g[n] = x[n] * y^*[-n]$$

(b) Using the result in part (a), show that

$$\sum_{n=-\infty}^{\infty} x[n]y * [n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y * (e^{j\omega})d\omega. \qquad \text{(eq.7b)}$$

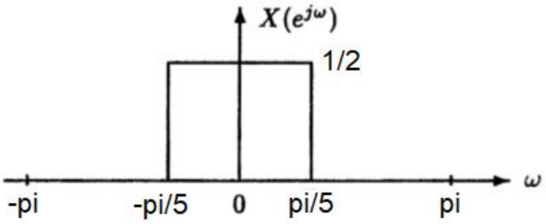
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})e^{j\omega n}d\omega = \sum_{n=-\infty}^{\infty} (x[n] * y^*[-n])\, e^{-j\omega n}$$

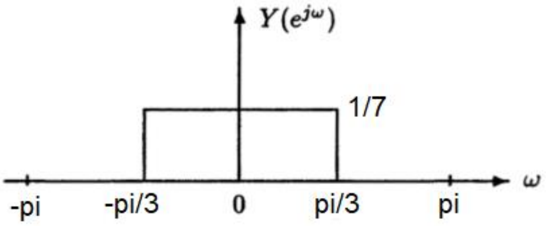
$$= \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} x[k]y^*[k-n]\, e^{-j\omega n}$$

$$\text{for } n = 0, \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega = \sum_{k=-\infty}^{\infty} x[k]y^*[k]$$

(c) Using (eq.7b), determine the value of the sum

$$\sum_{-\infty}^{\infty} \frac{sin(\pi n/5)}{2\pi n}\frac{sin(\pi n/3)}{7\pi n}$$





$\mathrm{x[n]} = \frac{sin(\pi n/5)}{2\pi n}$, $\mathrm{y^*[n]} = \frac{sin(\pi n/3)}{7\pi n}$

$\sum_{-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})dw$

$= \frac{1}{2\pi}[(\frac{1}{2})(\frac{1}{7})(\frac{2\pi}{5})] = \frac{1}{70}$