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1. linear, time-invariant, causal, and stable

(a) $y[n] = x[-n]$: linear, not time-invariant, not causal, stable #

(i) linear

Let $x_1[n] \to y_1[n] = x_1[-n]$ $\quad a x_1[n] + b x_2[n] \Rightarrow a x_1[-n] + b x_2[-n] = a y_1[n] + b y_2[n]$

$x_2[n] \to y_2[n] = x_2[-n]$

(ii) not time-invariant

Let $y_1[n] = x_1[-n]$ $\quad y_1[n-n_0] = x_1[-n+n_0]$

$x_2[n] = x_2[n-n_0]$, $y_2[n] = x_2[-n] = x_2[-n-n_0] \neq y_1[n-n_0]$

(iii) not causal

Output depends on input at future when $n<0$

(iv) Stable

If $|x[n]| < B \quad \forall n$, then $|y[n]| = |x[-n]| < B$

(b) $y[n] = \cos(\pi n) x[n]$: linear, time-invariant, causal, stable #

(i) linear

Let $x_1[n] \to y_1[n] = \cos(\pi n) x_1[n]$ $\quad a x_1[n] + b x_2[n] \to \cos(\pi n) a x_1[n] + \cos(\pi n) b x_2[n]$

$x_2[n] \to y_2[n] = \cos(\pi n) x_2[n]$ $\quad = a y_1[n] + b y_2[n]$

(ii) not time-invariant

Let $y_1[n] = \cos(\pi n) x_1[n]$ $\quad y_1[n-n_0] = \cos(\pi(n-n_0)) x_1[n-n_0]$

$x_2[n] = x_1[n-n_0]$, $y_2[n] = \cos(\pi n) x_2[n] = \cos(\pi n) x_2[n-n_0] \neq y_1[n-n_0]$

(iii) Causal

(iv) stable : If $|x[n]| < B$, then $|y[n]| = |\cos(\pi n) x[n]| < |x[n]| < B$

(c) $y[n] = \sum_{k=n-1}^{\infty} x[k]$: linear, time-invariant, not causal, unstable #

(i) linear

Let $x_1[n] \to y_1[n] = \sum_{k=n-1}^{\infty} x_1[k]$ $\quad a x_1[n] + b x_2[n] \to a y_1[n] + b y_2[n]$

$x_2[n] \to y_2[n] = \sum_{k=n-1}^{\infty} x_2[k]$

(ii) time-invariant

Let $y_1[n] = \sum_{k=n-1}^{\infty} x_1[k]$ $\quad y_1[n-n_0] = \sum_{k=n-1-n_0}^{\infty} x_1[k]$

$x_2[n] = x_1[n-n_0]$, $y_2[n] = \sum_{k=n-1}^{\infty} x_2[k] = \sum_{k=n-1}^{\infty} x_1[k-n_0] = \sum_{k=n-n_0-1}^{\infty} x_1[k] = y_1[n-n_0]$

(iii) not causal

(iv) unstable : If $|x[n]| < B$, then $|y[n]| = \left|\sum_{k=n-1}^{\infty} x[k]\right| = \infty$

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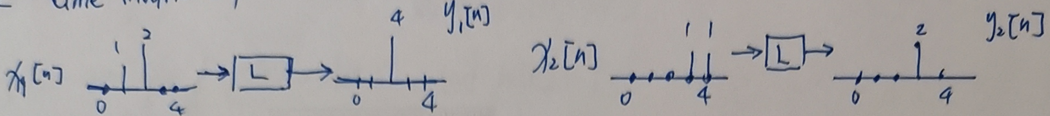
(linear)

2. L time-invariant ?



- $x_1[n]=a_1+2a_2$ $\quad y_1[n]=4a_2$, $x_2[n]=a_3+a_4$ $\quad y_2[n]=2a_3$
- If L is time-invariant, $x_1[n-2]=a_3+2a_4 \to y_1[n-2]=4a_4$

  Due to L is a linear system, $\underbrace{x_1[n-2]-x_2[n]}_{x_3[n]}=a_4 \to \underbrace{y_1[n-2]-y_2[n]}_{y_3[n]}=4a_4-2a_3$

  Then $x_3[n]+x_3[n+1]=a_3+a_4=x_2[n] \to y_3[n]+y_3[n+1]=4a_4-2a_3+4a_3-2a_3 \neq y_2[n]$
- Therefore, L isn't a time-invariant system #

3. $x[n] \leftrightarrow X(z)=\frac{1}{1-2z^{-1}}$ w/ ROC: $|z|<2$

(a) $y[n]=\left(\frac{1}{3}\right)^n x[n]$ $\qquad x[n]=-2^n u[-n-1]$

$\Rightarrow Y(z)=\frac{1}{1-\frac{2}{3}z^{-1}}$ with ROC: $|z|<\frac{2}{3}$ #

(b) $y[n]=x[n]*x[-n]$

$\Rightarrow Y(z)=X_1(z)X_2(z)$, ROC $\supset (R_1\cap R_2)$, $X_1(z)=\frac{1}{1-2z^{-1}}$ w/ ROC: $|z|<2$

$x[-n] \to X_1(z^{-1})$ w/ ROC: $|z|>\frac{1}{2}$, $X_2(z)=\frac{1}{1-2z}$

$\Rightarrow Y(z)=\frac{1}{1-2z^{-1}-2z+4}=\frac{z}{-2z^2+5z-2}$ with ROC: $\frac{1}{2}<|z|<2$ #

(c) $y[n]=nx[n]$

$\Rightarrow Y(z)=-z\frac{d(1-2z^{-1})^{-1}}{dz}=-z\times-(1-2z^{-1})^{-2}\times 2z^{-2}=\frac{2z}{(z-2)^2}$ with ROC: $|z|<2$ #

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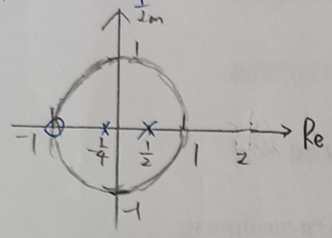
4. Causal impulse response $h[n]$, $H(z) = \dfrac{1+z^{-1}}{(1-0.5z^{-1})(1+0.25z^{-1})}$

(a) pole-zero plot (b) stable? (c) $h[n] = ?$

(a)



zero : -1

pole : $\frac{1}{2}, -\frac{1}{4}$

Causal → $h[n] = a^n u[n]$

$\Rightarrow (|z| > \frac{1}{2}) \cap (|z| > \frac{1}{4})$

$\Rightarrow$ ROC : $|z| > \frac{1}{2}$ #

(b) This is a stable system because R.O.C. include $|z| = 1$ #

(c) $H(z) = \dfrac{2}{1-\frac{1}{2}z^{-1}} + \dfrac{-1}{1+\frac{1}{4}z^{-1}} \Rightarrow h[n] = 2\left(\frac{1}{2}\right)^n u[n] - \left(-\frac{1}{4}\right)^n u[n]$ #

5. (a) $X(z) = \dfrac{z}{z^3+2z^2+\frac{5}{4}z+\frac{1}{4}}$, $|z|>1$

$z^3+2z^2+\frac{5}{4}z+\frac{1}{4} = z(z+1)^2 + \frac{1}{4}(z+1) = (z+1)\left(\frac{1}{4}+z^2+z\right) = (z+1)\left(z+\frac{1}{2}\right)^2$

$\hookrightarrow X(z) = \dfrac{4z}{z+1} + \dfrac{-4z}{z+\frac{1}{2}} + \dfrac{-4\times\left(-\frac{1}{2}\right)z}{\left(z+\frac{1}{2}\right)^2} \rightarrow x[n] = 4(-1)^n u[n] + (-4)\left(-\frac{1}{2}\right)^n u[n] + (-4)\left(-\frac{1}{2}\right)^n n\,u[n]$ #

(b) $X(z) = \dfrac{z}{\left(z^2-\frac{1}{3}\right)^2}$, $|z| < 0.5$ $\sqrt{\frac{1}{3}} = 0.577$ $\left(z^2-\frac{1}{3}\right)^2 = \left(z-\frac{1}{\sqrt{3}}\right)^2\left(z+\frac{1}{\sqrt{3}}\right)^2$

$\hookrightarrow X(z) = \dfrac{-\frac{1}{4}z}{z-\frac{1}{\sqrt{3}}} + \dfrac{\frac{1}{4}\times\frac{1}{\sqrt{3}}z}{\left(z-\frac{1}{\sqrt{3}}\right)^2} + \dfrac{\frac{1}{4}z}{z+\frac{1}{\sqrt{3}}} + \dfrac{-\frac{1}{4}\times\frac{-1}{\sqrt{3}}z}{\left(z+\frac{1}{\sqrt{3}}\right)^2} \rightarrow x[n] = \left(-\frac{1}{4}\right)\left(\frac{1}{\sqrt{3}}\right)^n u[n] + \left(\frac{1}{4}\right)\left(\frac{1}{\sqrt{3}}\right)^n n\,u[n] + \left(\frac{1}{4}\right)\left(\frac{-1}{\sqrt{3}}\right)^n u[n] + \left(-\frac{1}{4}\right)\left(\frac{-1}{\sqrt{3}}\right)^n n\,u[n]$ #

6. Autocorrelation for real-value, absolutely summable sequence $x[n]$

- $\gamma_{xx}[l] \triangleq \sum_n x[n]x[n-l]$, Let $X(z)$ w/ ROC : $\alpha < |z| < \beta$

(a) Show $R_{xx}(z) = X(z)X(z^{-1})$ (b) Let $x[n] = a^n u[n]$, $|a|<1$, Determine $R_{xx}(z)$

(a) $x_1[n] * x_2[n] \overset{z}{\longleftrightarrow} X_1(z)X_2(z)$ $\gamma_{xx}[l] \triangleq \sum_n x[n]x[n-l]$; $X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$, $X(z^{-1}) = \sum_n x[n]z^{n}$

$x[-n] \overset{z}{\longleftrightarrow} X(z^{-1})$

- Assume $x_1[n] = x[-n]$ → $\gamma_{xx}[l] \triangleq \sum_n x[n]x[n-l] = \sum_n x[n]x_1[l-n] = x[n] * x_1[n]$

$x_1[l-n] = x[n-l] \Rightarrow R_{xx}(z) = X(z)X_1(z) = X(z)X(z^{-1})$ #

- $X(z^{-1})$ w/ ROC : $\frac{1}{\alpha} > |z| > \frac{1}{\beta}$

- ROC of $R_{xx}(z)$: $(\alpha<|z|<\beta) \cap \left(\frac{1}{\beta}<|z|<\frac{1}{\alpha}\right) \rightarrow \max\left(\alpha,\frac{1}{\beta}\right) < |z| < \min\left(\beta,\frac{1}{\alpha}\right)$ #

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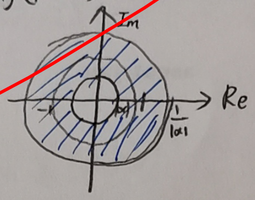
6、(b) $x[n]=\alpha^n u[n],\ |\alpha|<1 \rightarrow X(z)=\frac{z}{z-\alpha}$ with ROC : $|z|>|\alpha|$

$R_{xx}(z)=X(z)X(z^{-1})=\frac{z}{(z-\alpha)(1-\alpha z)}$ w/ zero : 0 , pole : $\alpha, \frac{1}{\alpha}$ #



7, Determine DTFT

(a) $x_1[n]=(\frac{1}{4})^n\cos(\frac{\pi n}{4})u[n-2]$ Modulation : $x[n]\cos(w_c n)\leftrightarrow \frac{1}{2}X(e^{j(w+w_c)})+\frac{1}{2}X(e^{j(w-w_c)})$

$a^n u[n]\leftrightarrow \frac{1}{1-ae^{-jw}}$, $x[n-n_0]\leftrightarrow X(e^{jw})e^{-jwn_0}$, $\cos(w_0 n)\leftrightarrow \pi\sum_{k=-\infty}^{+\infty}\{\delta(w-w_0-2\pi k)+\delta(w+w_0-2\pi k)\}$

$x[n]=(\frac{1}{4})^n u[n-2] \leftrightarrow X(e^{jw})=\frac{e^{-j2w}}{1-\frac{1}{4}e^{-jw}}$ $x_1[n]=x[n]\cos(\frac{\pi}{4}n)$

$\Rightarrow X_1(e^{jw})=\frac{1}{2}X(e^{j(w+\frac{\pi}{4})})+\frac{1}{2}X(e^{j(w-\frac{\pi}{4})})=(\cancel{\frac{1}{2}}e^{-j(2w+\frac{\pi}{2})})/(1-\frac{1}{4}e^{-j(w+\frac{\pi}{4})})+(\cancel{\frac{1}{2}}e^{-j(2w-\frac{\pi}{2})})/(1-\frac{1}{4}e^{-j(w-\frac{\pi}{4})})$ #

$3\frac{1}{2}$

(b) $x_3[n]=\sin(0.1\pi n)(u[n]-u[n-10])$ $\sin(w_c n)=\frac{1}{2j}(e^{jw_c n}-e^{-jw_c n})$

• $x[n-n_0]\leftrightarrow X(e^{jw})e^{-jwn_0}$. If $y[n]=\frac{1}{j}e^{jw_c n}x[n]\leftrightarrow Y(e^{jw})=\frac{1}{j}\sum_n e^{jw_c n}x[n]e^{-jwn}=\frac{1}{j}X(e^{j(w-w_c)})$

$\Rightarrow \sin(w_c n)x[n]\leftrightarrow \frac{1}{2j}(X(e^{j(w-w_c)})-X(e^{j(w+w_c)}))$

• $x[n]=u[n]-u[n-10] \leftrightarrow X(e^{jw})=(1-e^{-jw})^{-1}(1-e^{-j10w})$

$\Rightarrow X_3(e^{jw})=\frac{1}{2j}[(1-e^{-j(10w-\pi)})/(1-e^{-j(w-0.1\pi)})-(1-e^{-j(10w+\pi)})/(1-e^{-j(w+0.1\pi)})]$ #

8. $x[n]\Leftrightarrow X(e^{jw})$, $y[n]\Leftrightarrow Y(e^{jw})$ (a) $X(e^{jw})Y^*(e^{jw})$ (b) show $\sum_n x[n]y^*[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{jw})Y^*(e^{jw})dw$

(c) determine $\sum_n \frac{\sin(\pi n/5)}{2\pi n}\frac{\sin(\pi n/3)}{7\pi n}$

$X(e^{jw})=\sum_n x[n]e^{-jwn}$

$x[n]=\frac{1}{2\pi}\int_{2\pi}X(e^{jw})e^{jwn}dw$

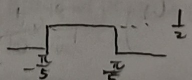
$X^*(e^{jw})\leftrightarrow x^*[-n]$

(a) $X(e^{jw})Y^*(e^{jw})\leftrightarrow x[n]*y^*[-n]=\sum_{k=-\infty}^{+\infty}x[k]y^*[k-n]$ #

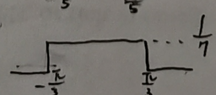
(b) $\sum_n x[n]y^*[n]=\sum_n x[n][\frac{1}{2\pi}\int_{2\pi}Y^*(e^{jw})e^{jwn}dw]=\frac{1}{2\pi}\int_{2\pi}[\sum_n x[n]e^{-jwn}]Y^*(e^{jw})dw$

$=\frac{1}{2\pi}\int_{2\pi}X(e^{jw})Y^*(e^{jw})dw$ #

(c) $x[n]=\frac{\sin(\pi n/5)}{2\pi n}=\frac{1}{2}\frac{\sin(\pi n/5)}{\pi n}\longleftrightarrow$



$y[n]=\frac{\sin(\pi n/3)}{7\pi n}=\frac{1}{7}\frac{\sin(\pi n/3)}{\pi n}\longleftrightarrow$



$\Rightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{jw})Y^*(e^{jw})=\frac{1}{14}\times(\frac{\pi}{5}-\frac{-\pi}{5})\times\frac{1}{2\pi}=\frac{1}{70}$ #