# EE4280 Lecture 1: Nonlinearity

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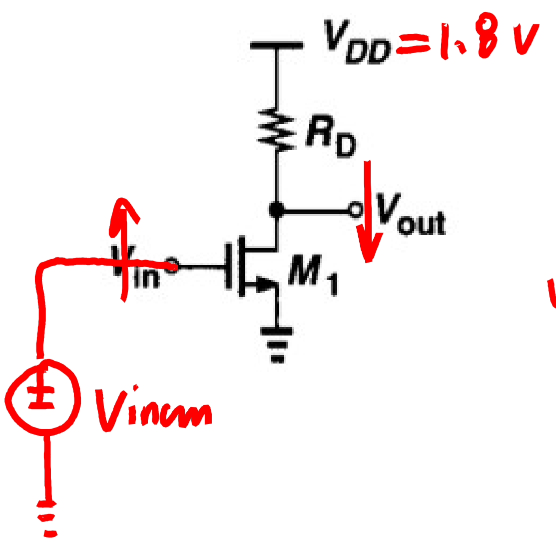
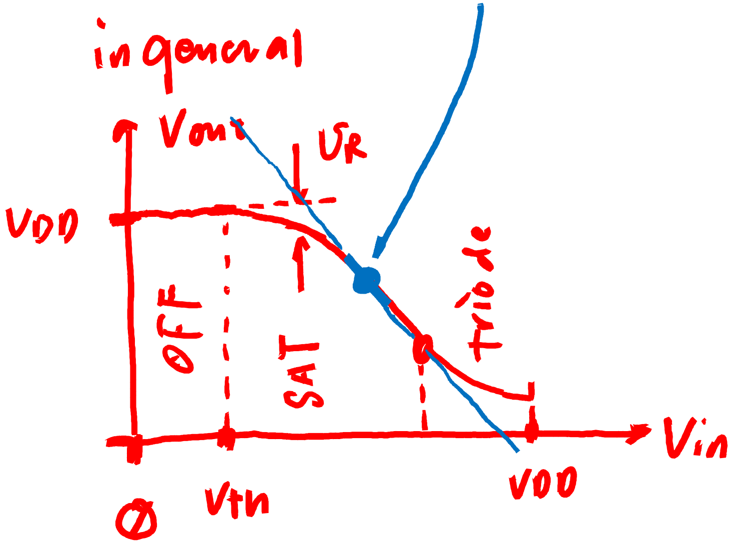
# Nonlinearity

**Nonlinear characteristic deviates from a straight line as the input swing increases**




(Vin, Vout)=(0.9, 1.1) one bias point




$$A_v = \frac{\partial V_{out}}{\partial V_{in}} = g_m R = f(V_{in})$$

Gain being a function of Vin (bias point)

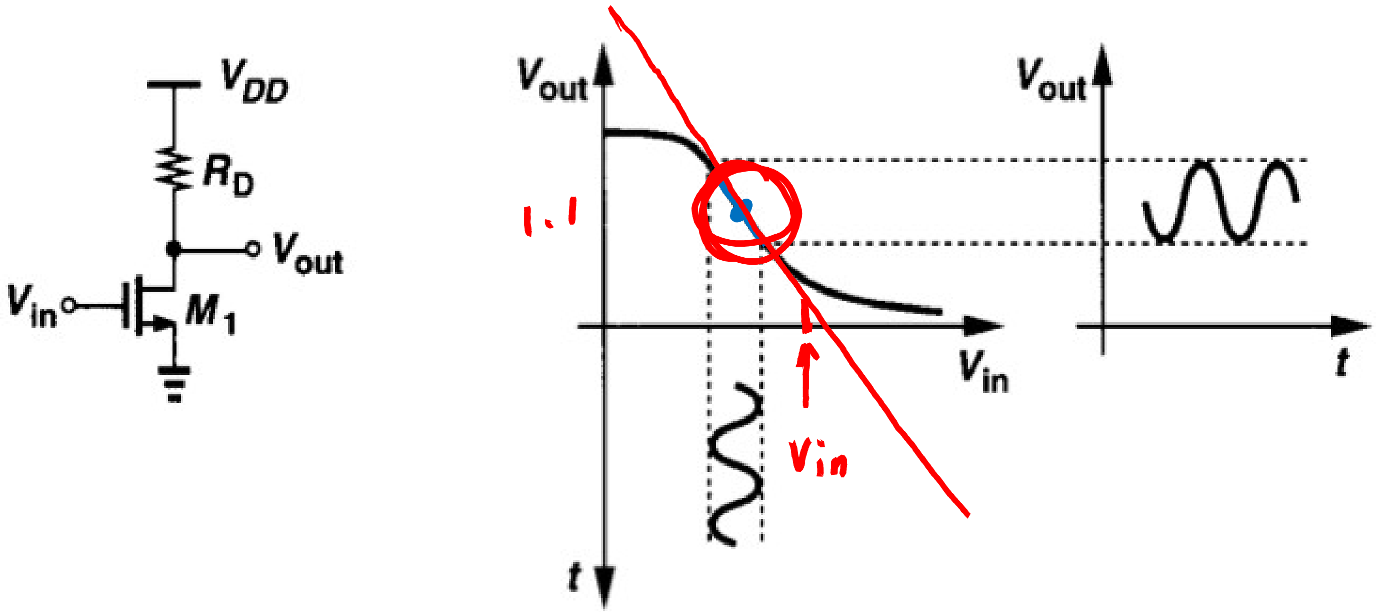
shows that the original curve is nonlinear

# Nonlinearity

$y = f(x)$ expand @ $x = 0.9$

**Nonlinear characteristic deviates from a straight line as the input swing increases**

$\rightarrow y = 1.1 + (A_v)\cdot(x - 0.9)$



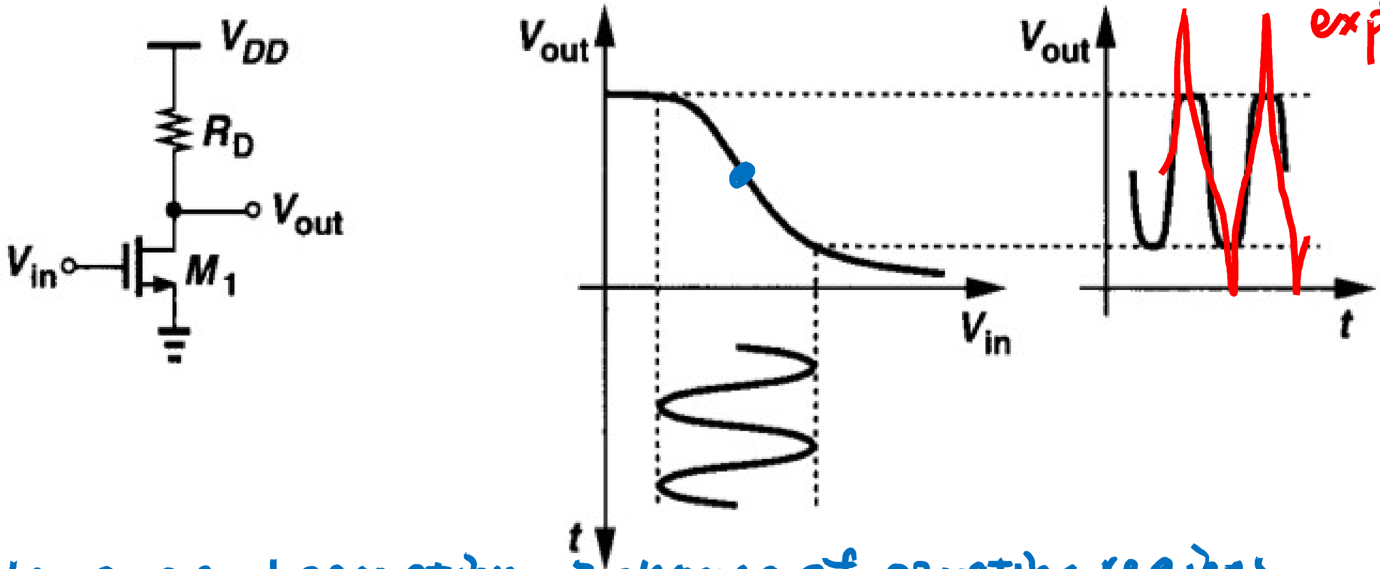

- **For small input swing, the output is a reasonable replica of the input**
  - Small-signal gain is related to the slope at a given bias point

# Nonlinearity

**Nonlinear characteristic deviates from a straight line as the input swing increases**

waveform distortion
→ harmonic generation

expansion

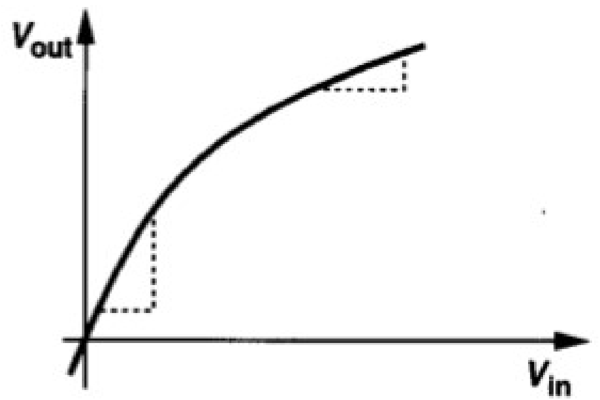


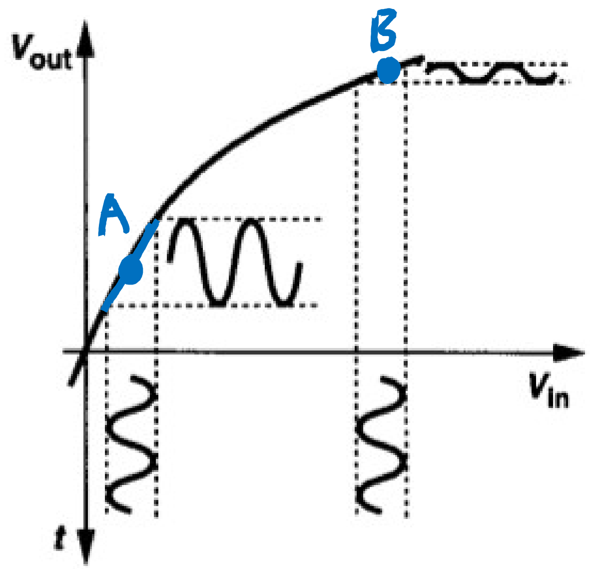

large-signal operation → change of operating regions

- **For small input swing, the output is a reasonable replica of the input**
- **For large input swings, most amplifiers experience gain compression (instead of expansion)**
  - The output exhibits "saturated" levels due to supply voltage or bias current

# Nonlinearity

**Can be viewed as variation of the slope (small-signal gain) with the input level (common-mode)**



# To Quantify Nonlinearity (I)

- **Taylor Expansion**

$$y = f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots$$

@ $x = x_0$

$$y - f(x_0) = \alpha_1 \cdot (x-x_0) + \alpha_2 (x-x_0)^2 + \alpha_3 (x-x_0)^3 + \cdots$$

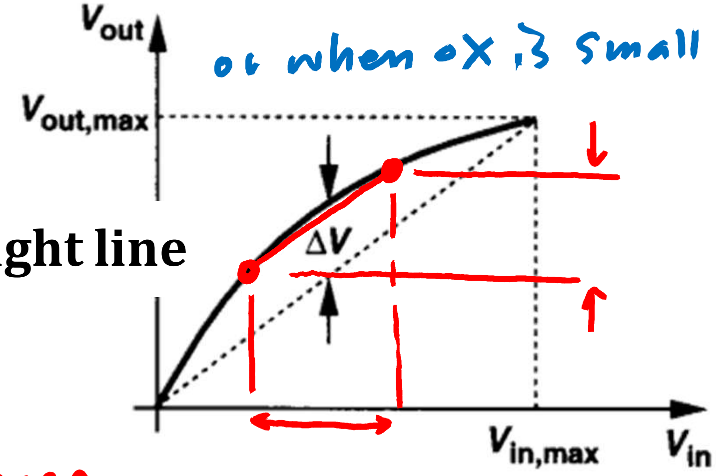
$$\Delta y = \underset{(-g_m R)}{\alpha_1} \Delta x + \alpha_2 \cdot \Delta x^2 + \alpha_3 \cdot \Delta x^3 + \cdots$$

linear approximation OK when $\alpha_2, \alpha_3, \ldots$ are small or when $\Delta x$ is small

$\frac{\Delta V}{V_{inrange}}$ or $\frac{\Delta V}{V_{outrange}}$

#1

- **Maximum deviation from an ideal straight line** $\Delta V / V_{out,\max}$ **for certain** $V_{in,\max}$




#2 gain variation as well

$A_{v,max}$, $A_{v,min}$, $\Delta V$range

# To Quantify Nonlinearity (II)

- **A single tone test**

$$\Delta x = A\cos\omega_1 t \qquad x(t) = x_0 + A\cdot\cos\omega_1 t$$

$$\Delta y = \alpha_1 A\cos\omega_1 t + \underline{\alpha_2 A^2\cos^2\omega_1 t} + \underline{\alpha_3 A^3\cos^3\omega_1 t} + \ldots$$

$$\frac{\alpha_2\cdot A^2}{2}(1+\cos 2\omega_1 t) \qquad \alpha_3 A^3\left(\frac{3}{4}\cos\omega_1 t + \frac{1}{4}\cos 3\omega_1 t\right)$$

$$= \left(\alpha_1 A + \frac{3}{4}\alpha_3\cdot A^3\right)\cos\omega_1 t$$

$$+ \left(\frac{\alpha_2\cdot A^2}{2}\right)\cos 2\cdot\omega_1 t$$

$$+ \frac{1}{4}\alpha_3\cdot A^3\cos 3\omega_1 t + \ldots$$

$$\propto \frac{\alpha_2}{\alpha_1}A^2$$

- **Total harmonic distortion**

$$THD = \frac{\left(\frac{\alpha_2\cdot A^2}{2}\right)^2 + \left(\frac{1}{4}\alpha_3 A^3\right)^2}{\left(\alpha_1 A + \frac{3}{4}\alpha_3\cdot A^3\right)^2}$$

# To Quantify Nonlinearity (III)

- **1-dB compression point**

$$20\log\left(\frac{\cancel{\frac{3}{4}\alpha_3 A_{1dB}^3}\ \alpha_1\cdot A_{1dB}}{\cancel{\alpha_1 A_{1dB}}\ \alpha_1\cdot A_{1dB}+\frac{3}{4}\alpha_3\cdot A_{1dB}^3}\right)=1\,dB$$

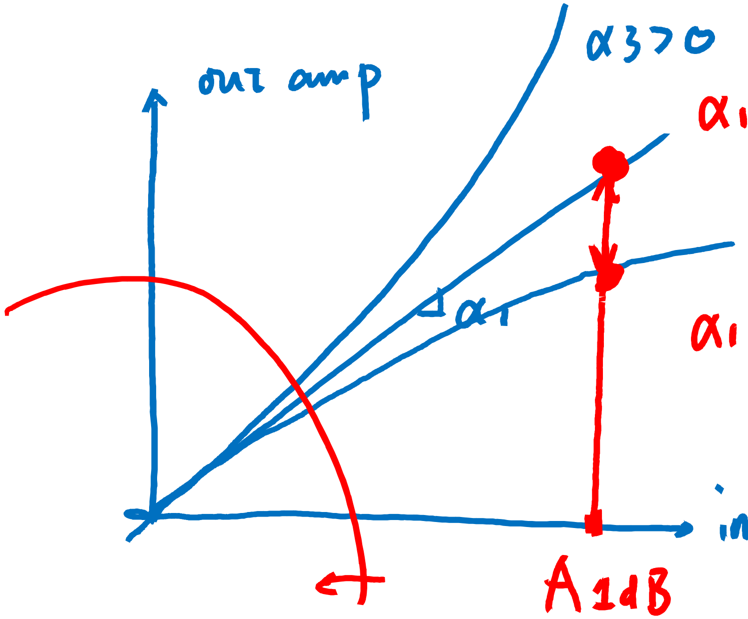
**The signal at** $\omega_1$

input amplitude of A

$$\Delta y=\underbrace{\left(\alpha_1 A+\frac{3}{4}\alpha_3 A^3\right)}_{\text{output amplitude}}\cos\omega_1 t$$

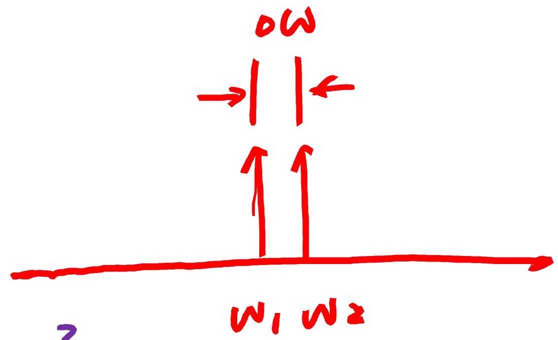
$$\Rightarrow A_{1dB}=\sqrt{\left(-0.11\cdot\cancel{\frac{3}{4}}\,\frac{4}{3}\,\frac{\alpha_1}{\alpha_3}\right)}$$

**The input level where the gain has dropped by 1dB**

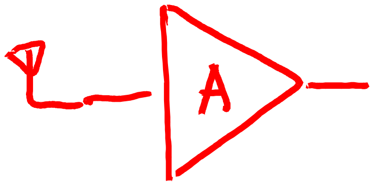


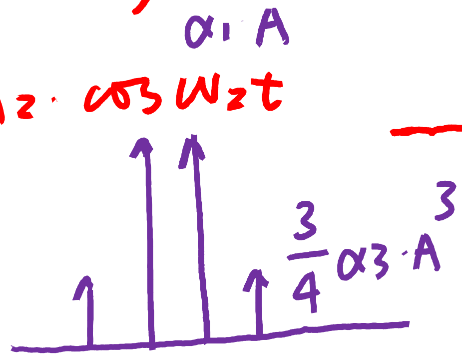

$$\underbrace{\alpha_1 A_{1dB}+\frac{3}{4}\alpha_3\cdot A_{1dB}^3}_{\text{error}}=0.89125\,\alpha_1\cdot A_{1dB}$$

# two-tone test (RF application)



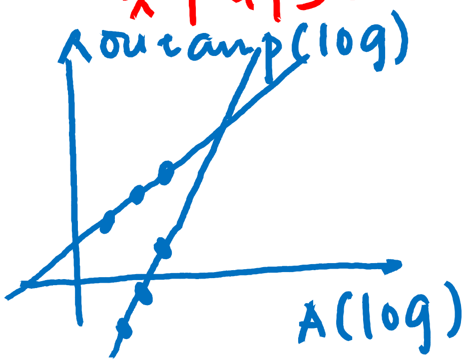

$$x(t) = A_1 \cdot \cos \omega_1 t + A_2 \cdot \cos \omega_2 t$$




$$y(t) = \alpha_1 \, x(t) + \alpha_2 \cdot x^2(t) + \alpha_3 \cdot x^3(t) + \cdots$$

其中会產生 $\frac{3 \cdot \alpha_3 A_1^2 A_2}{4} \left( \cos(2\omega_1 - \omega_2) t \right) + \frac{3}{4} \alpha_3 A_1 A_2^2 \cos(2\omega_2 - \omega_1) t$

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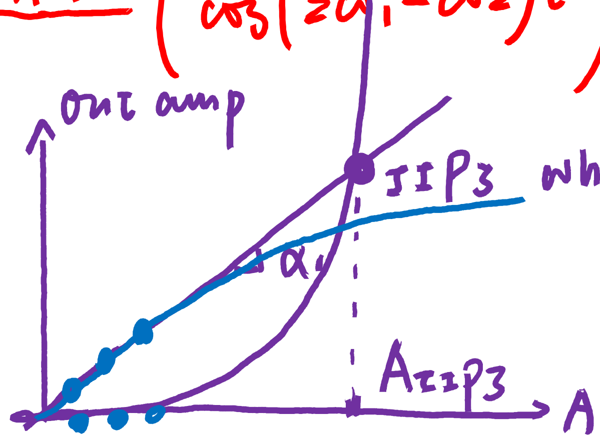



IIP3 when $\alpha_1 \cdot A_{IIP3} = \frac{3}{4} \alpha_3 \cdot A_{IIP3}^3$

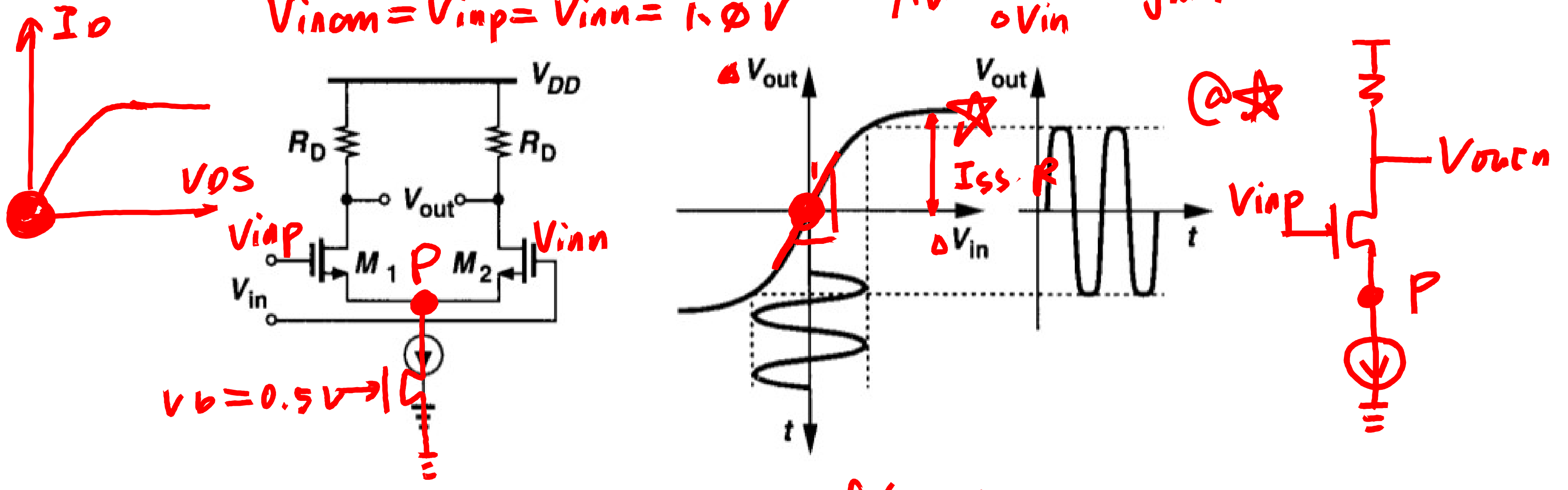
$$\Rightarrow A_{IIP3} = \sqrt{\frac{4}{3} \frac{|\alpha_1|}{|\alpha_3|}}$$

# Nonlinearity of Differential Circuits (I)

- **Differential circuits exhibit an "odd-symmetric" input/output characteristics, i.e.,** $f(-x) = -f(x)$

$A_v = \frac{\Delta V_{out}}{\Delta V_{in}} = g_m \cdot R$

$V_{incm} = V_{inp} = V_{inn} = 1.0V$




$\Delta V_{out} = f(\Delta V_{in})$

- **For the Taylor expansion to be an odd function, all the even-order terms musts be zero.**

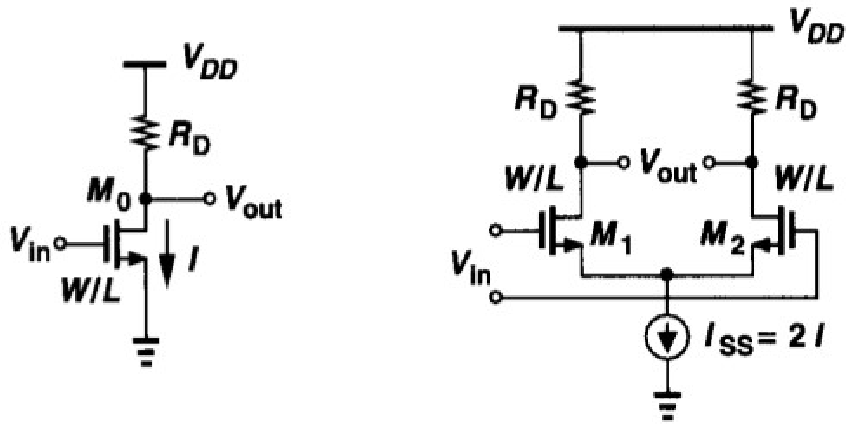
$y = f(x)$

differential operation: $f(-x) = -f(x)$

$$\Delta y = \alpha_1 \Delta x + \alpha_3 (\Delta x)^3 + \alpha_5 (\Delta x)^5 + \ldots$$

- **A differential circuit produces no even-order harmonics**

# Nonlinearity in Differential Circuits (II)




- **Single-ended and differential amplifiers with the same voltage gain**

$$|A_v| \approx g_m R_D$$
$$= \mu_n C_{ox} \frac{W}{L}(V_{GS} - V_{TH}) R_D$$

- **For the single-ended case:**

$$V_{DD} - V_{out} = I_D \cdot R_D$$

With $V_{in} = V_{GS} + V_m \cos\omega t$

We have $V_{DD} - V_{out} = \frac{1}{2}\mu_n C_{ox}\frac{W}{L}\left(V_{GS} + V_m\cos\omega t - V_{th}\right)^2 R_D$

$V_{GS0} + V_m \cdot \cos W_0 t$

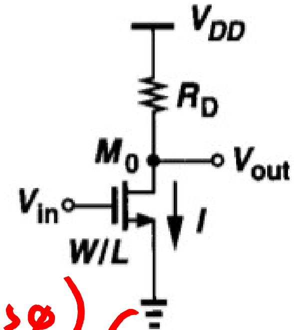
$V_{DD} - V_{out}(t) = R_D\left(\frac{1}{2}\mu\, C_{ox}\frac{W}{L}\left(V_{in}(t) - V_{th}\right)^2\right)$

$= f(V_{in})$

# Nonlinearity for Differential Circuits (III)

- **For the single-ended case:**

$$\cancel{V_{DD}} \; V_{DD} - V_{out} = \frac{1}{2}\mu_n C_{ox}\frac{W}{L}\left(\underbrace{V_{GS} + V_m \cos\omega t}_{V_{in}} - V_{th}\right)^2 R_D$$




$$V_{out} = f(V_{in})$$

$$= f(V_{GS0}) + \frac{f'(V_{GS0})}{1!}(V_{in} - V_{GS0}) + \frac{f''(V_{GS0})}{2!}(V_{in} - V_{GS0})^2$$

$$= 1.1V + \underbrace{f'(V_{GS0})}_{\mu C_{ox}\frac{W}{L}(V_{in}-V_{th})R_D} \cdot V_m \cos\omega_0 t + \underbrace{\frac{1}{2} f''(V_{GS0})}_{\frac{1}{2}\mu C_{ox}\frac{W}{L} R_D}(V_m \cos\omega_0 t)^2$$

$$\frac{1}{2}\mu C_{ox}\frac{W}{L} \cdot R_D \left(V_m^2 \underbrace{\cos^2\omega_0 t}_{\frac{1}{2}(1+\cos 2\omega_0 t)}\right)$$

@ $\omega_0$: $\mu C_{ox}\frac{W}{L}(V_{GS0} - V_{th}) \cdot R_D \cdot V_m$

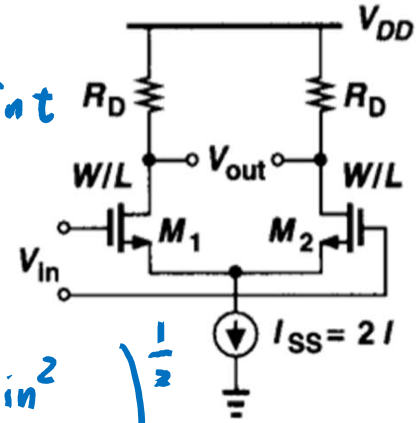
@ $2\cdot\omega_0$: $\frac{1}{4}\mu C_{ox}\frac{W}{L} \cdot R_D \cdot V_m^2$

- **The second harmonic distortion:** with $V_m$ of $20\% \cdot V_{ov}$

$$\frac{A_{HD2}}{A_F} = \frac{V_m}{4(V_{GS0} - V_{th})} = 5\%$$

# Nonlinearity for Differential Circuits (IV)

- **For the differential amplifier:**




$$\Delta V_{out} = (I_{D1} - I_{D2}) R_D$$

@ around equilibrium point

From Chapter 4: $$= \frac{1}{2}\mu_n C_{ox}\frac{W}{L}\Delta V_{in}\sqrt{4(V_{GS}-V_{th})^2 - \Delta V_{in}^2}$$

$$\Delta V_{out} = f(\Delta V_{in})$$

$$= R\,\frac{1}{2}\mu C_{ox}\frac{W}{L}\cdot\Delta V_{in}\cdot 2\cdot(V_{GS}-V_{in})\left(1-\frac{\Delta V_{in}^2}{4\cdot(V_{ov})^2}\right)^{\frac{1}{2}}$$

$$(1+\Delta)^a \cong 1 + a\Delta$$

$$\cong R\,\mu C_{ox}\frac{W}{L}\,\Delta V_{in}\cdot V_{ov}\left(1-\frac{1}{8}\frac{\Delta V_{in}^2}{V_{ov}^2}\right)$$

$$= R\,\mu C_{ox}\frac{W}{L}\cdot V_{ov}\,\Delta V_{in} - \frac{1}{8}R\cdot\mu C_{ox}\frac{W}{L}\frac{1}{V_{ov}}\cdot\Delta V_{in}^3$$

Single-Tone test: $\Delta V_{in} = V_m \cdot \cos \omega_0 t$

@ $\omega_0$: $\mu C_{ox}\frac{W}{L} V_{ov} \cdot R \cdot V_m$

@ $3\omega_0$: $\frac{1}{32}\mu C_{ox}\frac{W}{L}\frac{R}{V_{ov}}\cdot V_m^3$

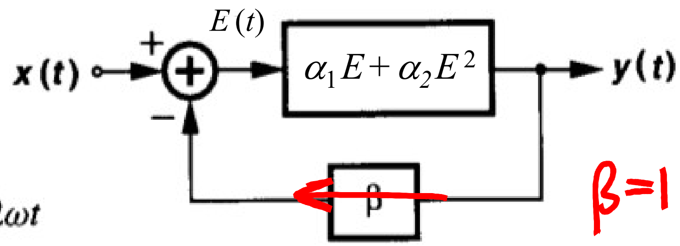
- **The third harmonic distortion:**

$$\frac{A_{HD3}}{A_F} \approx \frac{1}{32}\frac{V_m^2}{V_{ov}^2} = 0.125\%$$

# Effect of Negative Feedback on Nonlinearity (I)

- **Negative feedback makes the closed-loop gain relatively independent of the op amp's open-loop gain → Gain Desensitization**
- **Nonlinearity can be viewed as small-signal gain variation with input level → suppressed by negative feedback as well**

- **Consider an open-loop gain of**




$\beta=1$

With $x(t)=V_m\cos\omega t$

and if the output can be approximated as $y \approx a\cos\omega t + b\cos 2\omega t$

找出 a and b as a function of $V_m, \alpha_1, \alpha_2$.

$E(t)=x(t)-y(t)=V_m\cdot\cos\omega t-a\cos\omega t-b\cos(2\omega t)$

$y(t)=$ @$\omega_0$ $\alpha_1(V_m-a)-\alpha_2(V_m-a)b = a$

$=\alpha_1\cdot E+\alpha_2 E^2$ @$2\omega_0$ $\left(-\alpha_1 b+\frac{\alpha_2(V_m-a)^2}{2}\right) = b$