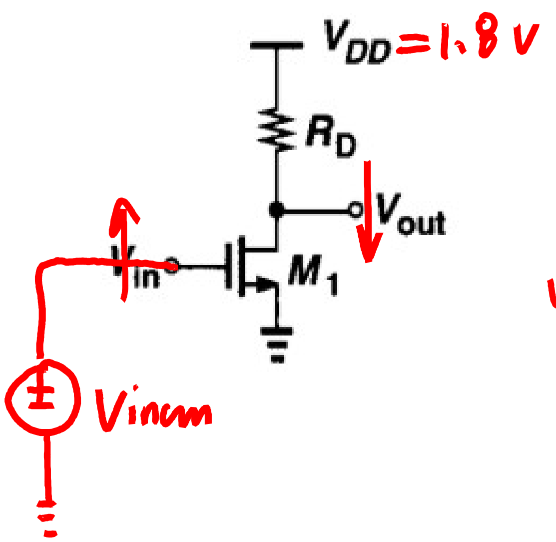
EE4280 Lecture 1: Nonlinearity

Ping-Hsuan Hsieh (謝秉璇)
Delta Building R908
EXT 42590
phsieh@ee.nthu.edu.tw

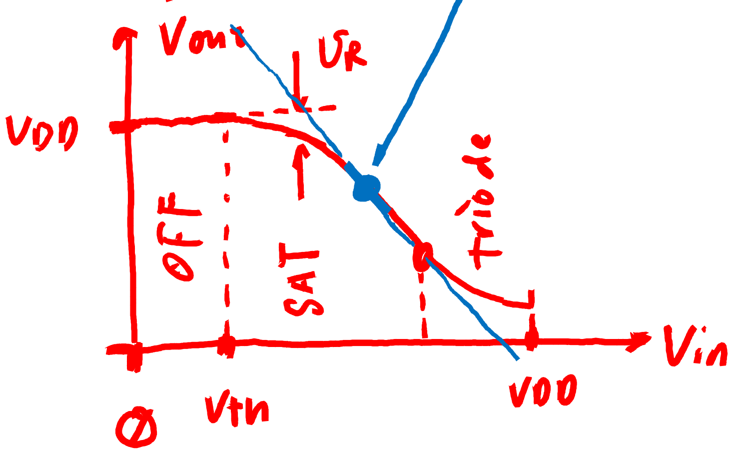
Nonlinearity

Nonlinear characteristic deviates from a straight line as the input swing increases



(Vin, Vout) = (0.9, 1.1) one bias point

in general



$A_v = \frac{\partial V_{out}}{\partial V_{in}}$

$= g_m R$

$\parallel$

$f(V_{in})$

Gain being a function of Vin (bias point)

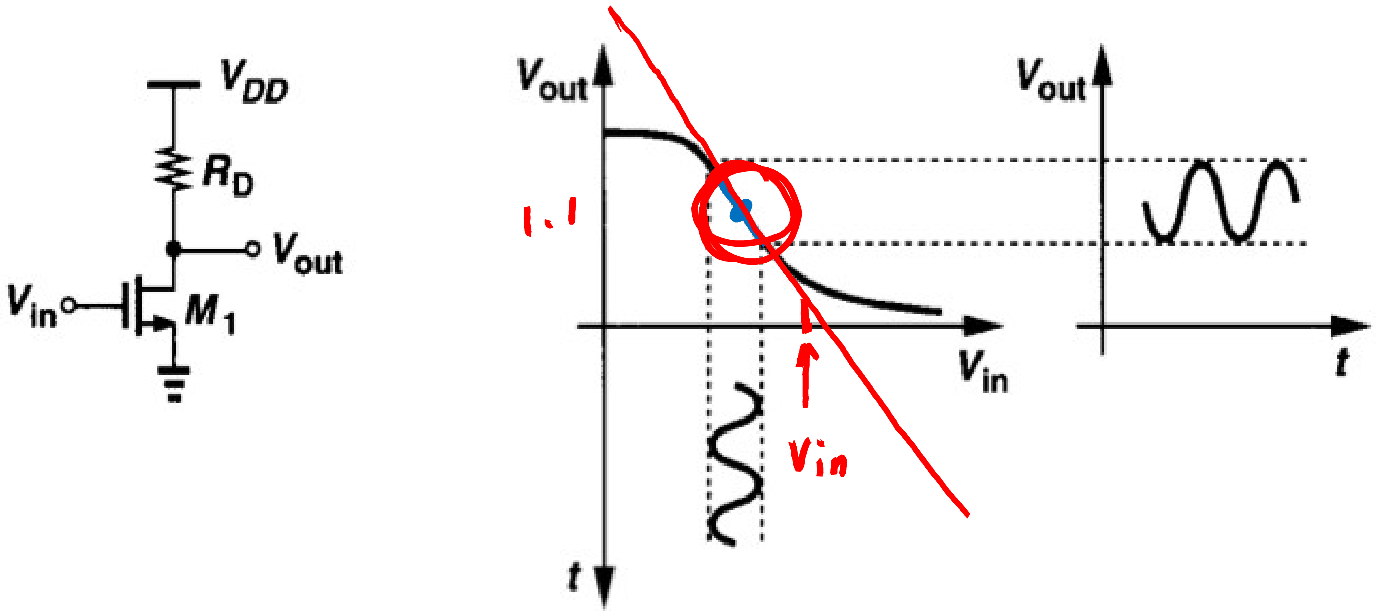
shows that the original curve is nonlinear

Nonlinearity

$y = f(x)$ expand @ $x = 0.9$

Nonlinear characteristic deviates from a straight line as the input swing increases

$\Rightarrow y = 1.1 + (A_v)\cdot(x - 0.9)$

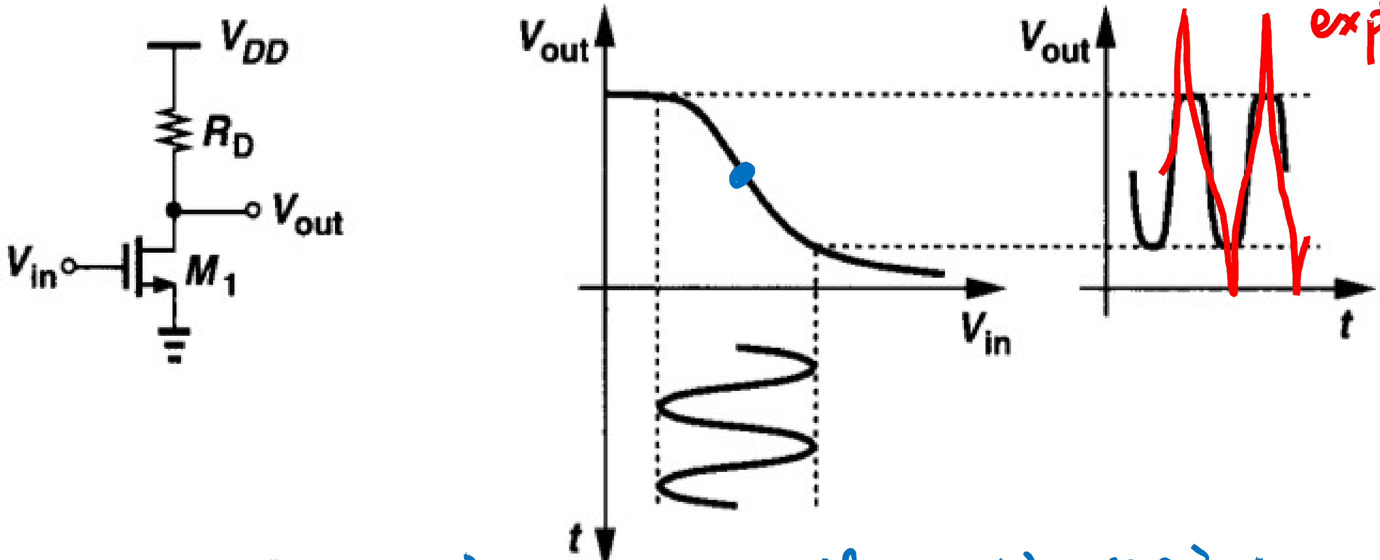


- **For small input swing, the output is a reasonable replica of the input**
  - Small-signal gain is related to the slope at a given bias point

Nonlinearity

Nonlinear characteristic deviates from a straight line as the input swing increases

waveform distortion
→ harmonic generation
expansion

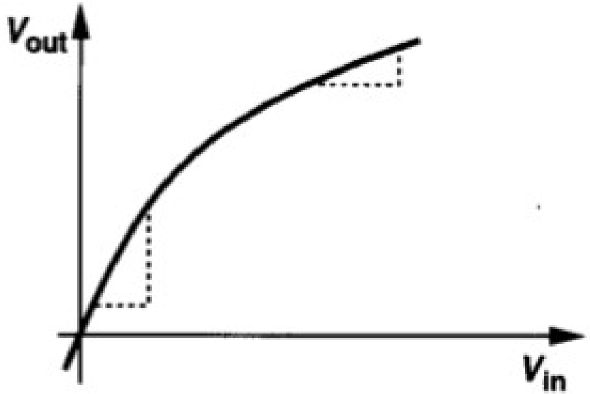


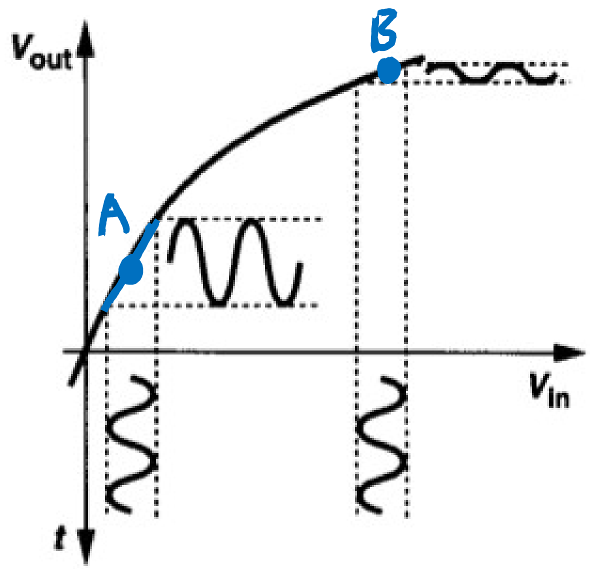
large-signal operation → change of operating regions

- **For small input swing, the output is a reasonable replica of the input**
- **For large input swings, most amplifiers experience gain compression (instead of expansion)**
  - The output exhibits "saturated" levels due to supply voltage or bias current

Nonlinearity

Can be viewed as variation of the slope (small-signal gain) with the input level (common-mode)





To Quantify Nonlinearity (I)

- **Taylor Expansion**

$$y = f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots$$

@ $x = x_0$

$$y - f(x_0) = \alpha_1 \cdot (x-x_0) + \alpha_2 (x-x_0)^2 + \alpha_3 (x-x_0)^3 + \cdots$$

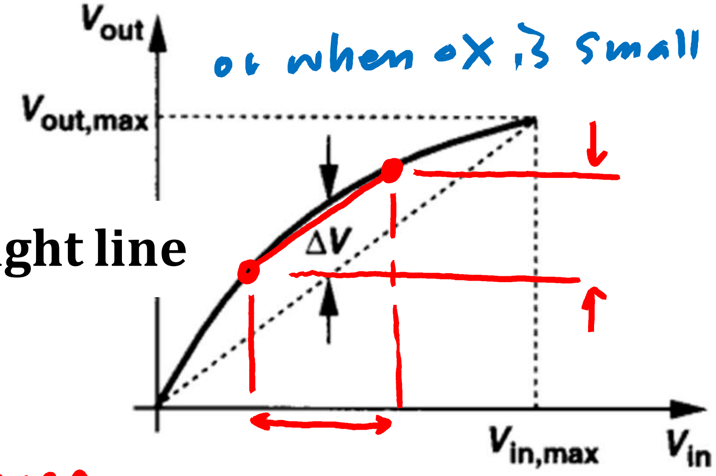
$$\Delta y = \underbrace{\alpha_1}_{(-g_m R)} \Delta x + \alpha_2 \cdot \Delta x^2 + \alpha_3 \cdot \Delta x^3 + \cdots$$

linear approximation OK when $\alpha_2, \alpha_3, \ldots$ are small or when Δx is small

$\frac{\Delta V}{V_{inrange}}$ or $\frac{\Delta V}{V_{outrange}}$

#1

- **Maximum deviation from an ideal straight line** $\Delta V / V_{out,\max}$ **for certain** $V_{in,\max}$



#2 gain variation as well

$A_{v,max}$, $A_{v,min}$, ΔV_{range}

To Quantify Nonlinearity (II)

- **A single tone test**

$$\Delta x = A\cos\omega_1 t \qquad x(t) = x_0 + A\cdot\cos\omega_1 t$$

$$\Delta y = \alpha_1 A\cos\omega_1 t + \underline{\alpha_2 A^2\cos^2\omega_1 t} + \underline{\alpha_3 A^3\cos^3\omega_1 t} + \ldots$$

$$\frac{\alpha_2\cdot A^2}{2}(1+\cos 2\omega_1 t) \qquad \alpha_3 A^3\left(\frac{3}{4}\cos\omega_1 t + \frac{1}{4}\cos 3\omega_1 t\right)$$

$$= \left(\alpha_1 A + \frac{3}{4}\alpha_3\cdot A^3\right)\cos\omega_1 t$$

$$+ \left(\frac{\alpha_2\cdot A^2}{2}\right)\cos 2\cdot\omega_1 t$$

$$+ \frac{1}{4}\alpha_3\cdot A^3\cos 3\omega_1 t + \ldots \qquad \propto \frac{\alpha_2}{\alpha_1}A^2$$

- **Total harmonic distortion**

$$THD = \frac{\left(\frac{\alpha_2\cdot A^2}{2}\right)^2 + \left(\frac{1}{4}\alpha_3 A^3\right)^2}{\left(\alpha_1 A + \frac{3}{4}\alpha_3\cdot A^3\right)^2}$$

To Quantify Nonlinearity (III)

- **1-dB compression point**

$$20\log\left(\frac{\cancel{\frac{3}{4}\alpha_3 A_{1dB}^3}}{\cancel{\alpha_1 A_{1dB}}}\right) = 1\,dB$$

$\alpha_1 \cdot A_{1dB}$

$\alpha_1 \cdot A_{1dB} + \frac{3}{4}\alpha_3 \cdot A_{1dB}^3$

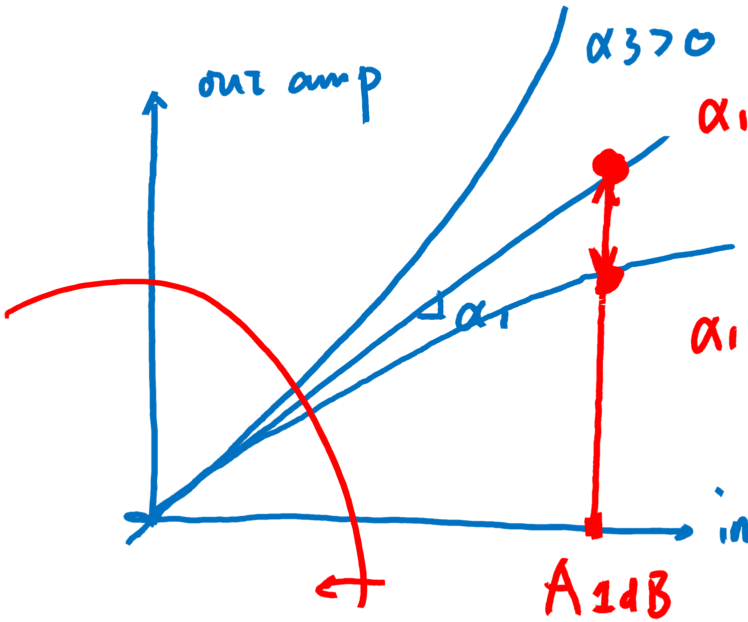
The signal at ω_1

input amplitude of A

$$\Delta y = \underbrace{\left(\alpha_1 A + \frac{3}{4}\alpha_3 A^3\right)}_{\text{output amplitude}} \cos\omega_1 t$$

$$\Rightarrow A_{1dB} = \sqrt{\left(-0.11 \cdot \frac{4}{3} \frac{\alpha_1}{\alpha_3}\right)}$$

The input level where the gain has dropped by 1dB



$\alpha_3 > 0$ expanding behavior

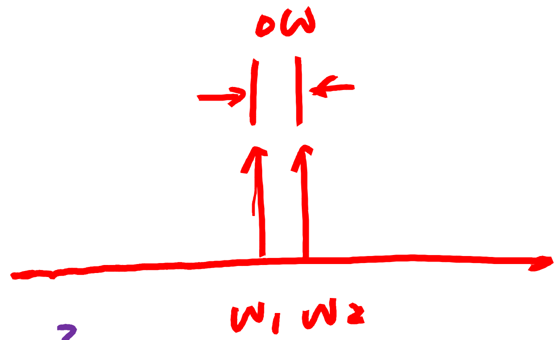
$\alpha_1 \cdot A_{1dB}$

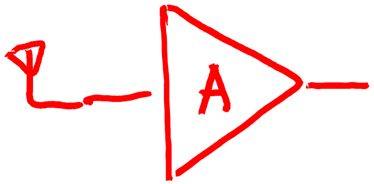
$\alpha_3 < 0$ compressing behavior

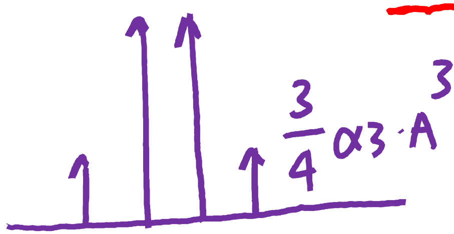
$$\underbrace{\alpha_1 A_{1dB} + \frac{3}{4}\alpha_3 \cdot A_{1dB}^3}_{\text{error}} = 0.89125\,\alpha_1 \cdot A_{1dB}$$

two-tone test (RF application)

$x(t) = A_1 \cdot \cos \omega_1 t + A_2 \cdot \cos \omega_2 t$

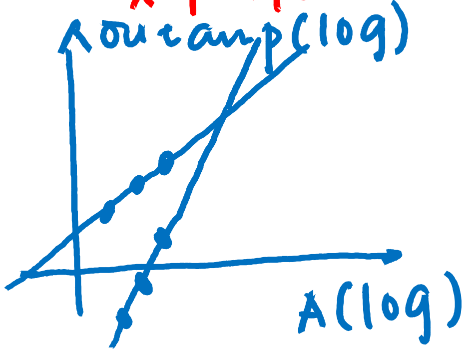




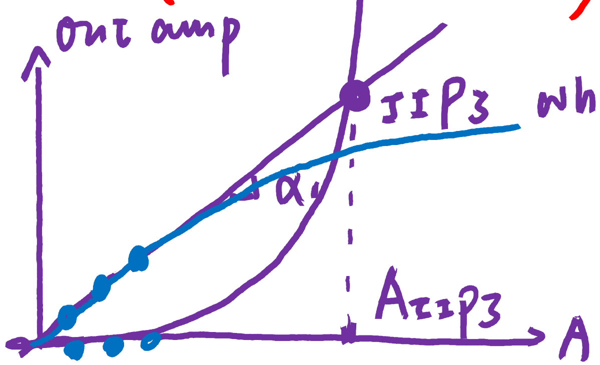


$y(t) = \alpha_1 \cdot x(t) + \alpha_2 \cdot x^2(t) + \alpha_3 \cdot x^3(t) + \cdots$

其中会產生 $\frac{3 \cdot \alpha_3 A_1^2 A_2}{4} \left(\cos(2\omega_1 - \omega_2)t \right) + \frac{3}{4}\alpha_3 A_1 A_2^2 \cos(2\omega_2 - \omega_1)t$



$\Leftarrow$



IIP3 when $\alpha_1 \cdot A_{IIP3} = \frac{3}{4}\alpha_3 \cdot A_{IIP3}^3$

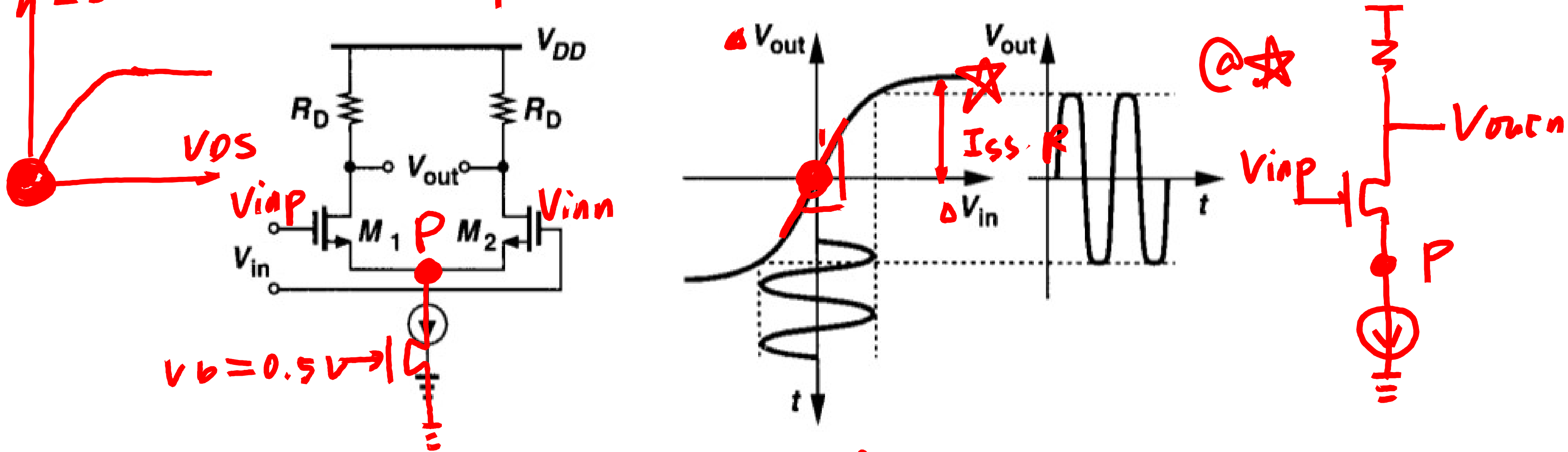
$\Rightarrow A_{IIP3} = \sqrt{\frac{4}{3}\frac{|\alpha_1|}{|\alpha_3|}}$

Nonlinearity of Differential Circuits (I)

- **Differential circuits exhibit an "odd-symmetric" input/output characteristics, i.e.,** $f(-x) = -f(x)$

$A_v = \frac{\Delta V_{out}}{\Delta V_{in}} = g_m \cdot R$

$V_{incm} = V_{inp} = V_{inn} = 1.0\,V$



$\Delta V_{out} = f(\Delta V_{in})$

- **For the Taylor expansion to be an odd function, all the even-order terms musts be zero.**

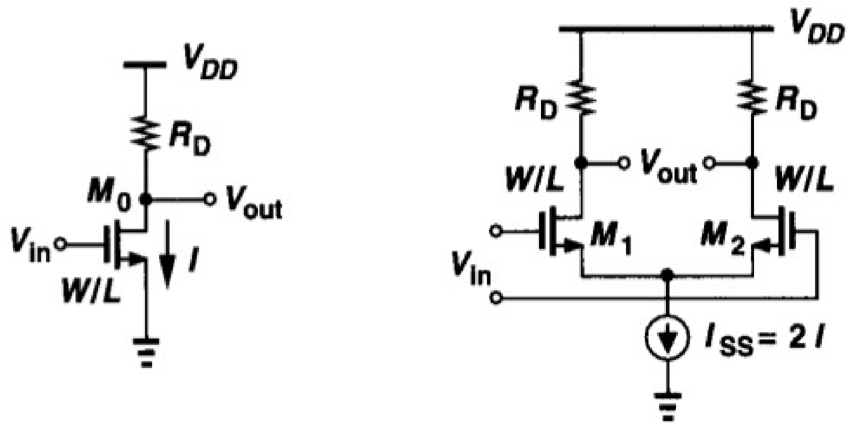
$y = f(x)$

differential operation: $f(-x) = -f(x)$

$$\Delta y = \alpha_1 \Delta x + \alpha_3 (\Delta x)^3 + \alpha_5 (\Delta x)^5 + \ldots$$

- **A differential circuit produces no even-order harmonics**

Nonlinearity in Differential Circuits (II)



- **Single-ended and differential amplifiers with the same voltage gain**

$$|A_v| \approx g_m R_D$$
$$= \mu_n C_{ox} \frac{W}{L} (V_{GS} - V_{TH}) R_D$$

$V_{GS\emptyset} + V_M \cdot \cos W_0 t$

- **For the single-ended case:**

$V_{DD} - V_{out}(t) = R_D \left(\frac{1}{2} \mu \, Cox \frac{W}{L} (V_{in}(t) - V_{th})^2 \right)$

$= f(V_{in})$

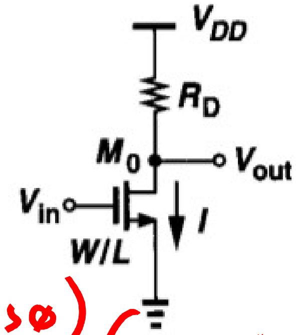
$$V_{DD} - V_{out} = I_D \cdot R_D$$

With $$V_{in} = V_{GS} + V_m \cos \omega t$$

We have $$V_{DD} - V_{out} = \frac{1}{2} \mu_n C_{ox} \frac{W}{L} \left(V_{GS} + V_m \cos \omega t - V_{th} \right)^2 R_D$$

Nonlinearity for Differential Circuits (III)

- **For the single-ended case:**



$$\cancel{V_{DD}} \overset{V_{DD}-}{-} V_{out} = \frac{1}{2}\mu_n C_{ox}\frac{W}{L}\big(\underbrace{V_{GS} + V_m\cos\omega t}_{V_{in}} - V_{th}\big)^2 R_D$$

$$V_{out} = f(V_{in})$$

$$= f(V_{GS\emptyset}) + \frac{f'(V_{GS\emptyset})}{1!}(V_{in} - V_{GS\emptyset}) + \frac{f''(V_{GS\emptyset})}{2!}(V_{in} - V_{GS\emptyset})^2$$

$$= 1.1V + f'(V_{GS\emptyset})\cdot V_m \cdot \cos\omega_0 t + \frac{1}{2} f''(V_{GS\emptyset})(V_m \cdot \cos\omega_0 t)^2$$

$$\mu C_{ox}\frac{W}{L}(V_{in} - V_{th}) R_D \qquad \frac{1}{2}\mu C_{ox}\frac{W}{L}\cdot R_D\left(V_m^2 \cos^2\omega_0 t\right)$$

$$\frac{1}{2}(1 + \cos 2\omega_0 t)$$

$@\,\omega_0: \ \mu C_{ox}\frac{W}{L}(V_{GS\emptyset} - V_{th})\cdot R_D \cdot V_m$

$@\,2\cdot\omega_0: \ \frac{1}{4}\mu C_{ox}\frac{W}{L}\cdot R_D \cdot V_m^2$

- **The second harmonic distortion:** with V_m of $20\% \cdot V_{ov}$

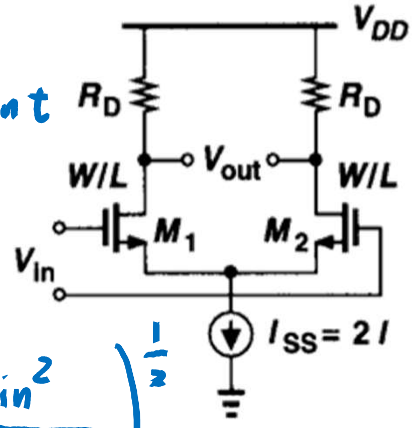
$$\frac{A_{HD2}}{A_F} = \frac{V_m}{4(V_{GS\emptyset} - V_{th})\uparrow} = 5\%$$

Nonlinearity for Differential Circuits (IV)

- **For the differential amplifier:**

$$\Delta V_{out} = (I_{D1} - I_{D2}) R_D$$

@ around equilibrium point



From Chapter 4: $$= \frac{1}{2}\mu_n C_{ox}\frac{W}{L}\Delta V_{in}\sqrt{\underline{4(V_{GS}-V_{th})^2} - \Delta V_{in}^2}$$

$$\Delta V_{out} = f(\Delta V_{in})$$

$$= R \frac{1}{2} \mu C_{ox} \frac{W}{L} \cdot \Delta V_{in} \cdot 2 \cdot (V_{GS} - V_{in}) \left(1 - \frac{\Delta V_{in}^2}{4 \cdot (V_{ov})^2}\right)^{\frac{1}{2}}$$

$$(1+\Delta)^a \cong 1 + a \cdot \Delta$$

$$\cong R \mu C_{ox} \frac{W}{L} \Delta V_{in} \cdot V_{ov} \left(1 - \frac{1}{8}\frac{\Delta V_{in}^2}{V_{ov}^2}\right)$$

$$= R \mu C_{ox} \frac{W}{L} \cdot V_{ov} \Delta V_{in} - \frac{1}{8} R \cdot \mu C_{ox} \frac{W}{L} \frac{1}{V_{ov}} \cdot \Delta V_{in}^3$$

Single-Tone test. $\Delta V_{in} = V_m \cdot \cos \omega_0 t$

@ ω_0 $\mu C_{ox} \frac{W}{L} V_{ov} \cdot R \cdot V_m$

@ $3\omega_0$: $\frac{1}{32} \mu C_{ox} \frac{W}{L} \frac{R}{V_{ov}} \cdot V_m^3$

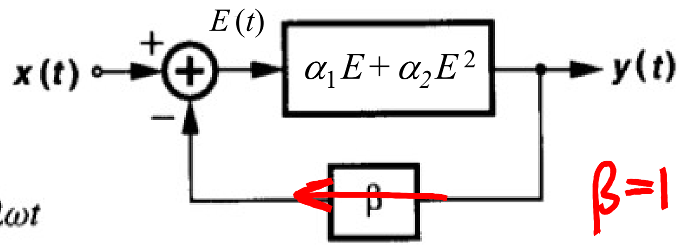
- **The third harmonic distortion:**

$$\frac{A_{HD3}}{A_F} \approx \frac{1}{32}\frac{V_m^2}{V_{ov}^2} = 0.125\%$$

Effect of Negative Feedback on Nonlinearity (I)

- Negative feedback makes the closed-loop gain relatively independent of the op amp's open-loop gain → Gain Desensitization
- Nonlinearity can be viewed as small-signal gain variation with input level → suppressed by negative feedback as well

- Consider an open-loop gain of



$\beta = 1$

With $x(t) = V_m \cos\omega t$

and if the output can be approximated as $y \approx a\cos\omega t + b\cos 2\omega t$

求出 a and b as a function of V_m, α_1, α_2, β

If linear, $\alpha_2 = 0$

amplitude of output $= \dfrac{\alpha_1}{1+\alpha_1 \cdot \beta} \cdot V_m$

linear gain

$E(t) = x(t) - y(t) = V_m \cdot \cos\omega t - a\cos\omega t - b\cos(2\omega t)$

$y(t) =$

$= \alpha_1 \cdot E + \alpha_2 E^2$

@ω_0 $\alpha_1(V_m - a) - \alpha_2(V_m - a) \cdot b = a$

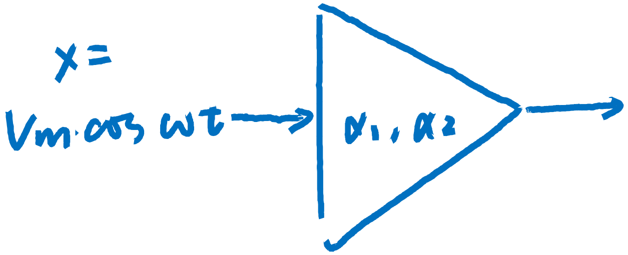
@$2\omega_0$ $\left(-\alpha_1 \cdot b + \dfrac{\alpha_2 (V_m - a)^2}{2}\right) = b$

Effect of Negative Feedback on Nonlinearity (II)

With
$$\begin{cases} a = (\alpha_1 - \cancel{\alpha_2 \beta b})(V_m - \beta a) \\ b = -\alpha_1 \beta b + \dfrac{\alpha_2 (V_m - \beta a)^2}{2} \end{cases}$$

$$a = \left(\frac{\alpha_1}{1+\alpha_1\beta}\right) \cdot V_m$$

$$b = \alpha_2 V_m^2 / 2(1+\alpha_1\beta)^3$$

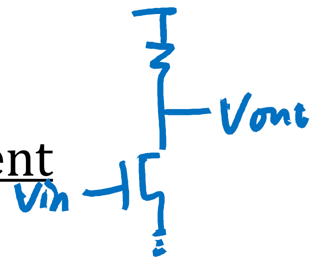


- **The second harmonic distortion:** $\dfrac{A_{HD2}}{A_F} = \dfrac{b}{a} = \dfrac{\alpha_2 V_m}{2\alpha_1 (1+\alpha_1\beta)^2}$
- **Compared to the open-loop case:** $\dfrac{b}{a} = \dfrac{\alpha_2 V_m}{2\alpha_1}$

With the same input swing: FB structure provides an improv. of $(1+\alpha_1\beta)^2$

With the same output swing: you need $(1+\alpha_1\beta)\times$ larger input for the FB structure → improv. of $(1+\alpha_1\beta)$

Linearization Technique (I)



- **To reduce the dependence of gain on input level**

  → To reduce the dependence of gain on transistor bias current

  two-step operation from $V_{in} \xrightarrow{v} I_D \rightarrow V_{out}$

  $f(g_m, R_S)$ R_{out}

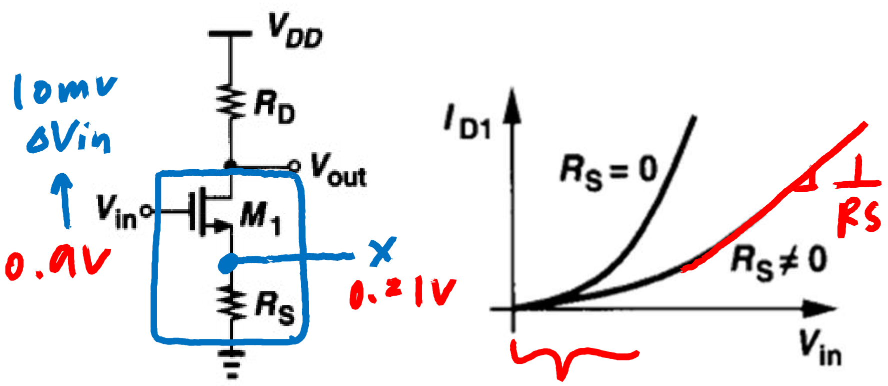
- **Source degeneration effectively reduce the signal swing on V_{GS}**

$\boxed{G_m} = \frac{g_m}{1+g_m R_S}$

① when Vin is small
or when bias condition weak
or when $1 \gg g_m R_S$ $G_m \rightarrow g_m$

② when $1 \ll g_m R_S$

$G_m \rightarrow \frac{1}{R_S}$



$$\frac{\Delta I_D}{\Delta V_{in}} = \frac{g_m}{1+g_m R_S}$$ series combination of $\boxed{\frac{1}{g_m} + R_S}$

@ a specific bias point

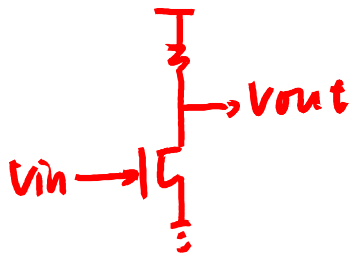
- **Trade-off between linearity, noise, power dissipation, and gain**

Linearization Technique (~~II~~) (I) Example

- **For an output swing that corresponds to** I_D variation from $0.75I_1$ to $1.25I_1$

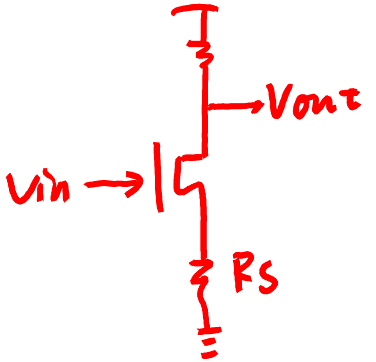
Amin Amax

  - For common-source without degeneration:



$$\frac{A_{max}}{A_{min}} = \frac{g_{m\,max}}{g_{m\,min}} = \frac{\sqrt{1.25}}{\sqrt{0.75}}$$

  - For source degeneration of $g_m R_S = 2$ at $\boxed{I_D = I_1}$



$$\frac{A_{max}}{A_{min}} = \frac{G_{m.max}}{G_{m.min}} = \frac{\dfrac{g_{m\,max}}{1+g_{m\,max}R_S}}{\dfrac{g_{m\,min}}{1+g_{m\,min}R_S}} = \frac{\dfrac{\sqrt{1.25}\,g_{m0}}{1+\sqrt{1.25}\,g_{m0}R_S}}{\dfrac{\sqrt{0.75}\,g_{m0}}{1+\sqrt{0.75}\,g_{m0}R_S}}$$

$$= \frac{\sqrt{1.25}}{\sqrt{0.75}}\ 0.84 \rightarrow 16\%\ \text{improvement}$$

Linearization Technique (III)

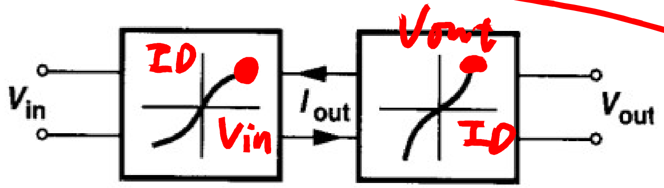
- **Post correction**
  - A common-source amplifier is in fact a voltage-to-current converter followed by a current-to-voltage converter

$$\Delta V_{in} \rightarrow \Delta I_D \rightarrow \Delta V_{out}$$

so that

$\Delta V_{out} = g(f(\Delta V_{in}))$

$= A \cdot f^{-1}(f(\Delta V_{in})) = A \cdot \Delta V_{in}$



For $\boxed{\Delta I_D = f(\Delta V_{in})}$

If we have $\Delta V_{in} = A \cdot f^{-1}(\Delta I_D)$

$\Delta V_{out} = g(\Delta I_D)$

$\Delta V_{out} = A \cdot f^{-1}(\Delta I_D)$

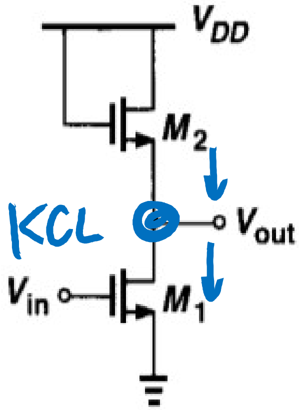
$f(\Delta V_{out}) = A \cdot \Delta I_D$

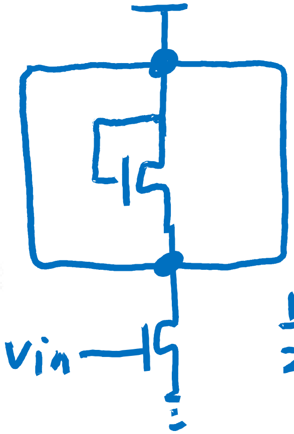
to be more specific

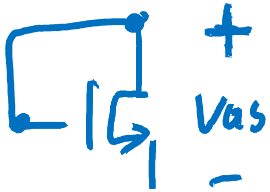
we want $g(\cdot) = A \cdot f^{-1}(\cdot)$

Linearization Technique (IV)

- **Common-source with diode-connected load**







$I_{D1} = I_{D2}$

$$\frac{1}{2}\mu C_{ox}\left(\frac{W}{L}\right)_1 (V_{in} - V_{th})^2 = \frac{1}{2}\mu C_{ox}\left(\frac{W}{L}\right)_2 (V_{DD} - V_{out} - V_{th})^2$$

$f(V_{out})$

$$\Rightarrow \frac{\partial V_{out}}{\partial V_{in}} = -\sqrt{\frac{\left(\frac{W}{L}\right)_1}{\left(\frac{W}{L}\right)_2}}$$

- **Some of the design considerations include**
  - Body-effect that degrades the linearity
  - Limited voltage headroom V_{th}