7

Tunneling Phenomena

7-1 (a) The reflection coefficient is the ratio of the reflected intensity to the incident wave intensity, or $R = \frac{|(1/2)(1-i)|^2}{|(1/2)(1+i)|^2}$. But $|1-i|^2 = (1-i)(1-i)^* = (1-i)(1+i) = |1+i|^2 = 2$, so that $R = 1$ in this case.

(b) To the left of the step the particle is free. The solutions to Schrödinger's equation are $e^{\pm ikx}$ with wavenumber $k = \left(\frac{2mE}{\hbar^2}\right)^{1/2}$. To the right of the step $U(x) = U$ and the equation is $\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}(U-E)\psi(x)$. With $\psi(x) = e^{-kx}$, we find $\frac{d^2\psi}{dx^2} = k^2\psi(x)$, so that $k = \left[\frac{2m(U-E)}{\hbar^2}\right]^{1/2}$. Substituting $k = \left(\frac{2mE}{\hbar^2}\right)^{1/2}$ shows that $\left[\frac{E}{(U-E)}\right]^{1/2} = 1$ or $\frac{E}{U} = \frac{1}{2}$.

(c) For 10 MeV protons, $E = 10\text{ MeV}$ and $m = \frac{938.28\text{ MeV}}{c^2}$. Using $\hbar = 197.3\text{ MeV fm}/c\left(1\text{ fm} = 10^{-15}\text{ m}\right)$, we find

$$\delta = \frac{1}{k} = \frac{\hbar}{(2mE)^{1/2}} = \frac{197.3\text{ MeV fm}/c}{\left[(2)\left(938.28\text{ MeV}/c^2\right)(10\text{ MeV})\right]^{1/2}} = 1.44\text{ fm}.$$

7-2 (a) To the left of the step the particle is free with kinetic energy E and corresponding wavenumber $k_1 = \left(\frac{2mE}{\hbar^2}\right)^{1/2}$:

$$\psi(x) = Ae^{ik_1x} + Be^{-ik_1x} \qquad x \le 0$$

To the right of the step the kinetic energy is reduced to $E-U$ and the wavenumber is now $k_2 = \left[\frac{2m(E-U)}{\hbar^2}\right]^{1/2}$

$$\psi(x) = Ce^{ik_2x} + De^{-ik_2x} \qquad x \ge 0$$

with $D=0$ for waves incident on the step from the left. At $x=0$ both ψ and $\frac{d\psi}{dx}$ must be continuous: $\psi(0)=A+B=C$

$$\left.\frac{d\psi}{dx}\right|_0 = ik_1(A-B) = ik_2C .$$

(b) Eliminating C gives $A+B=\frac{k_1}{k_2}(A-B)$ or $A\left(\frac{k_1}{k_2}-1\right)=B\left(\frac{k_1}{k_2}+1\right)$. Thus,

$$R=\left|\frac{B}{A}\right|^2=\frac{(k_1/k_2-1)^2}{(k_1/k_2+1)^2}=\frac{(k_1-k_2)^2}{(k_1+k_2)^2}$$
$$T=1-R=\frac{4k_1k_2}{(k_1+k_2)^2}$$

(c) As $E\to U$, $k_2\to 0$, and $R\to 1$, $T\to 0$ (no transmission), in agreement with the result for any energy $E<U$. For $E\to\infty$, $k_1\to k_2$ and $R\to 0$, $T\to 1$ (perfect transmission) suggesting correctly that very energetic particles do not *see* the step and so are unaffected by it.

7-3 With $E=25\text{ MeV}$ and $U=20\text{ MeV}$, the ratio of wavenumber is $\frac{k_1}{k_2}=\left(\frac{E}{E-U}\right)^{1/2}=\left(\frac{25}{25-20}\right)^{1/2}=\sqrt{5}=2.236$. Then from Problem 7-2 $R=\frac{\left(\sqrt{5}-1\right)^2}{\left(\sqrt{5}+1\right)^2}=0.146$ and $T=1-R=0.854$. Thus, 14.6% of the incoming particles would be reflected and 85.4% would be transmitted. For electrons with the same energy, the transparency and reflectivity of the step are unchanged.

7-4 The reflection coefficient for this case is given in Problem 7-2 as

$$R=\left|\frac{B}{A}\right|^2=\frac{(k_1/k_2-1)^2}{(k_1/k_2+1)^2}=\frac{(k_1-k_2)^2}{(k_1+k_2)^2} .$$

The wavenumbers are those for electrons with kinetic energies $E=54.0\text{ eV}$ and $E-U=54.0\text{ eV}+10.0\text{ eV}=64.0\text{ eV}$:

$$\frac{k_1}{k_2}=\left(\frac{E}{E-U}\right)^{1/2}=\left(\frac{54\text{ eV}}{64\text{ eV}}\right)^{1/2}=0.918\,6 .$$

Then, $R=\frac{(0.918\,6-1)^2}{(0.918\,6+1)^2}=1.80\times10^{-3}$ is the fraction of the incident beam that is reflected at the boundary.

7-5 (a) The transmission probability according to Equation 7.9 is

$$\frac{1}{T(E)}=1+\left[\frac{U^2}{4E(U-E)}\right]\sinh^2\alpha L \text{ with } \alpha=\frac{[2m(U-E)]^{1/2}}{\hbar}$$. For $E<<U$, we find

$(\alpha L)^2\approx\frac{2mUL^2}{\hbar^2}>>1$ by hypothesis. Thus, we may write $\sinh\alpha L\approx\frac{1}{2}e^{\alpha L}$. Also

$U-E\approx U$, giving $\frac{1}{T(E)}\approx 1+\left(\frac{U}{16E}\right)e^{2\alpha L}\approx\left(\frac{U}{16E}\right)e^{2\alpha L}$ and a probability for

transmission $P=T(E)=\left(\frac{16E}{U}\right)e^{-2\alpha L}$.

(b) Numerical Estimates: $\left(\hbar=1.055\times10^{-34}\text{ Js}\right)$

1) For $m=9.11\times10^{-31}$ kg, $U-E=1.60\times10^{-21}$ J, $L=10^{-10}$ m;

$\alpha=\frac{[2m(U-E)]^{1/2}}{\hbar}=5.12\times10^{8}\text{ m}^{-1}$ and $e^{-2\alpha L}=0.90$

2) For $m=9.11\times10^{-31}$ kg, $U-E=1.60\times10^{-19}$ J, $L=10^{-10}$ m; $\alpha=5.12\times10^{9}\text{ m}^{-1}$ and $e^{-2\alpha L}=0.36$

3) For $m=6.7\times10^{-27}$ kg, $U-E=1.60\times10^{-13}$ J, $L=10^{-15}$ m; $\alpha=4.4\times10^{14}\text{ m}^{-1}$ and $e^{-2\alpha L}=0.41$

4) For $m=8$ kg, $U-E=1$ J, $L=0.02$ m; $\alpha=3.8\times10^{34}\text{ m}^{-1}$ and $e^{-2\alpha L}=e^{-1.5\times10^{33}}\approx 0$

7-6 Equation 7.9 gives for the transmission probability $\frac{1}{T(E)}=1+\left[\frac{U^2}{4E(U-E)}\right]\sinh^2\alpha L$. For 0.1% transmission $T(E)=0.001$, and the resulting equation must be solved for E using $L=1$ nm and $U=5$ eV. We adopt $x=\frac{E}{U}$ as the unknown and write

$\alpha L=\frac{\left[2mL^2(U-E)\right]^{1/2}}{\hbar}=\left[\frac{\left(2mUL^2\right)^{1/2}}{\hbar}\right](1-x)^{1/2}$ and $\frac{U^2}{E(U-E)}=\frac{1}{x(1-x)}$. For this case

$\frac{\left(2mUL^2\right)^{1/2}}{\hbar}=\frac{\left[(2)\left(511\times10^3\text{ eV}/c^2\right)(5.00\text{ eV})(1\text{ nm})^2\right]^{1/2}}{197.3\text{ eV nm}/c}=11.46$ so the equation for x

becomes $1\,000=1+\frac{1}{4x(1-x)}\sinh^2\left[11.46(1-x)^{1/2}\right]$. The root is $x=0.997\,2$ implying

$E=xU=(0.997\,2)(5\text{ eV})=4.986\text{ eV}$.

7-7 The continuity requirements from Equation 7.8 are

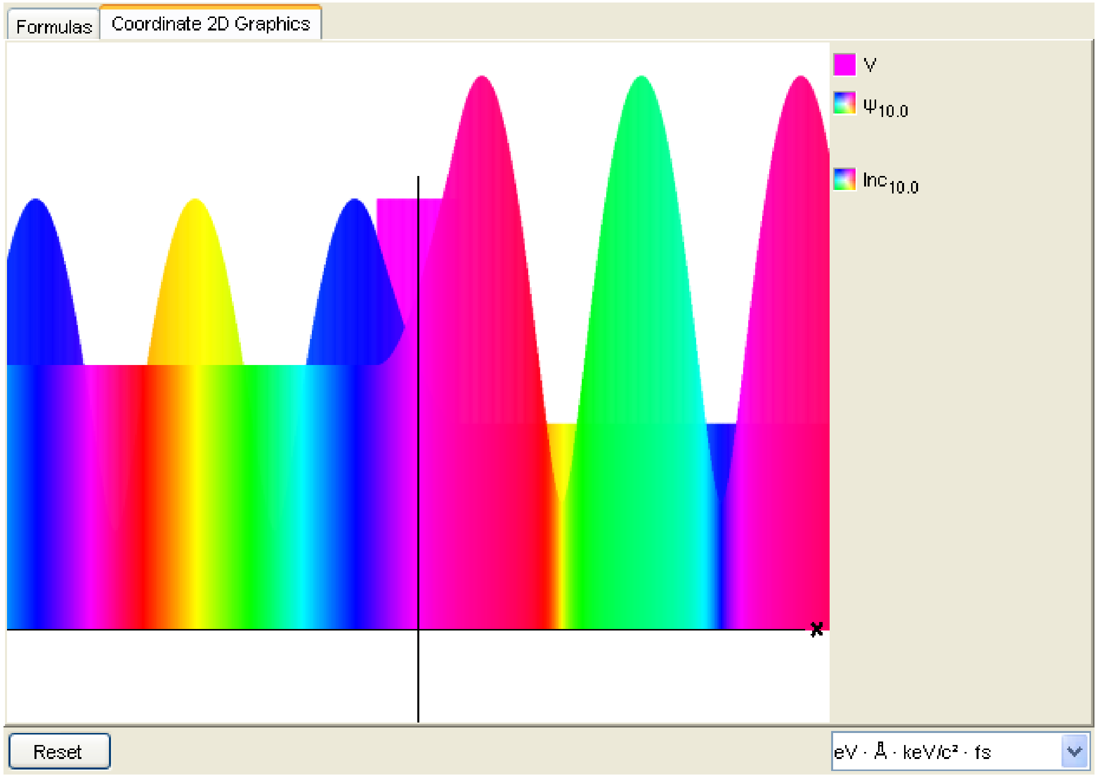
$A+B=C+D$	[continuity of Ψ at $x=0$]
$ikA-ikB=\alpha D-\alpha C$	$\left[\text{continuity of }\frac{\partial\Psi}{\partial x}\text{ at }x=0\right]$
$Ce^{-\alpha L}+De^{+\alpha L}=Fe^{ikL}$	[continuity of Ψ at $x=L$]
$\alpha De^{+\alpha L}-\alpha Ce^{-\alpha L}=ikFe^{ikL}$	$\left[\text{continuity of }\frac{\partial\Psi}{\partial x}\text{ at }x=L\right]$

To isolate the transmission amplitude $\frac{F}{A}$, we must eliminate from these relations the unwanted coefficients *B*, *C*, and *D*. Dividing the second line by *ik* and adding to the first eliminates *B*, leaving *A* in terms of *C* and *D*. In the same way, dividing the fourth line by α and adding the result to the third line gives *D* (in terms of *F*), while subtracting the result from the third line gives *C* (in terms of *F*). Combining these results finally yields *A*:
$A = \frac{1}{4}Fe^{ikL}\left\{\left[2-\left(\frac{\alpha}{ik}+\frac{ik}{\alpha}\right)\right]e^{+\alpha L}+\left[2+\left(\frac{\alpha}{ik}+\frac{ik}{\alpha}\right)\right]e^{-\alpha L}\right\}$. The transmission probability is
$T = \left|\frac{F}{A}\right|^2$. Making use of the identities $e^{\pm\alpha L} = \cosh\alpha L \pm \sinh\alpha L$ and $\cosh^2\alpha L = 1+\sinh^2\alpha L$, we obtain

$$\frac{1}{T} = \left|\frac{A}{F}\right|^2 = \frac{1}{4}\left|2\cosh\alpha L + i\left(\frac{\alpha}{k}-\frac{k}{\alpha}\right)\sinh\alpha L\right|^2 = \cosh^2\alpha L + \frac{1}{4}\left(\frac{\alpha}{k}-\frac{k}{\alpha}\right)^2 \sinh\alpha L$$

$$= 1+\frac{1}{4}\left[\frac{U-E}{E}+\frac{E}{U-E}+2\right]\sinh^2\alpha L = 1+\frac{1}{4}\left[\frac{U^2}{E(U-E)}\right]\sinh^2\alpha L$$

7-8 The Java applet for this problem is available from our companion website (http://info.brookscole.com/mp3e QMTools Simulations → Problem 7.8). The applet models an electron (mass 511 keV/c^2) in a potential *V(x)* describing a square barrier of height 10.0 eV and width 1.00 Å on the interval [−5.0 Å, 5.0 Å]. The listing to the right of the graph includes a placeholder for one stationary state of the electron with energy 10.0 eV. Follow the applet instructions to display this state, using both the (default) color-for-phase scheme for complex-valued waveforms, as well as separate plots for the real and imaginary parts. Notice that for this case the waveform (real and imaginary parts) is linear in the barrier region. Use the zoom and scroll features as necessary to enhance the linearity. [The Schrödinger equation for $E = U$ prescribes $\mathrm{d}^2\psi/\mathrm{d}x^2 = 0$, with solution $\psi(x) = Cx + D$.] The waveform to the right of the barrier is the transmitted wave at this energy (return to the color-for-phase display and note the appearance of the plot in this region); the waveform on the left is a mixture of incident and reflected waves. Follow the applet instructions to display the incident wave. At this stage the applet should resemble the screenshot below:



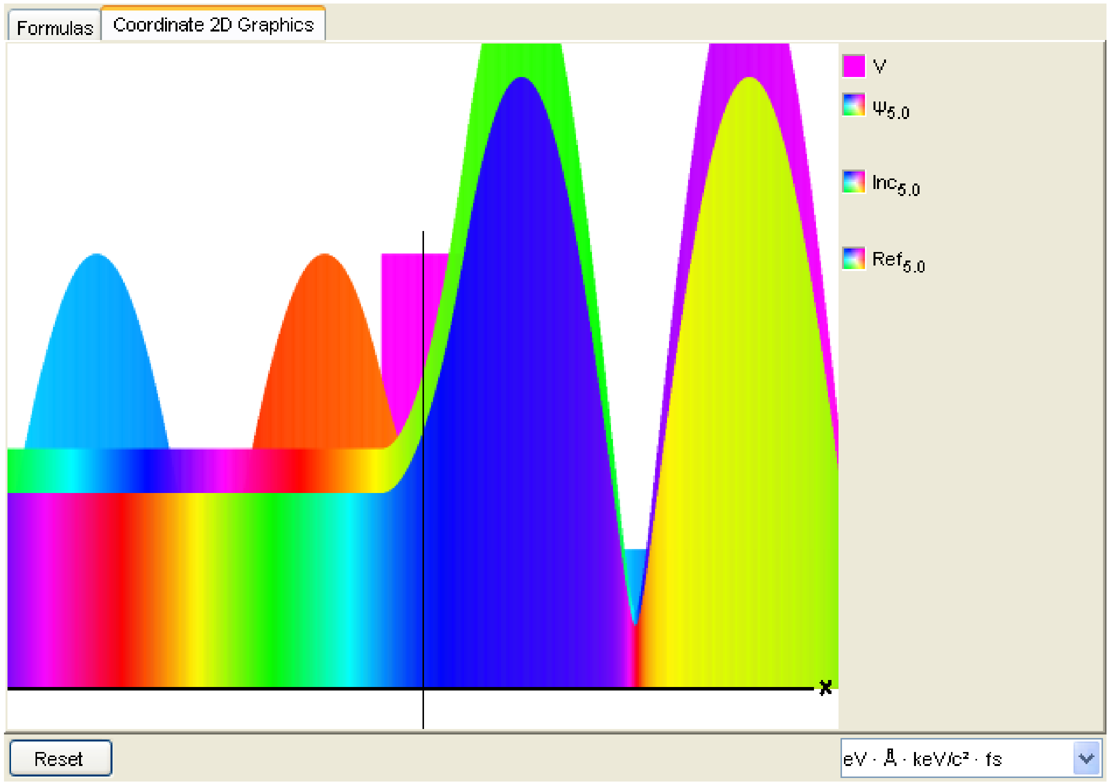
Activate the "Trace" feature to investigate the wave amplitudes at various locations. Note that the incident wave amplitude A is unity (any discrepancy is due to the limitation inherent in representing decimal numbers in machine-readable, i.e., binary form). The transmitted wave amplitude is $F = 0.777946$, giving for the transmission coefficient at this energy $|F/A|^2 = 0.60520$. This should be compared with the prediction of Equation 7.9.

For $E \to U$, $\alpha \to 0$, $\sinh \alpha L \to \alpha L$, and $T \to \{1+[(mUL^2)/2\hbar^2]\}^{-1}$, so the exact numerical result for this case is

$$T = \{1+(511\times 10^3 \text{ eV}/c^2)(10 \text{ eV})(1\text{Å})^2 /[2(1.973\times 10^3 \text{ eV}\cdot\text{Å}/c^2\}^{-1} = 0.60374$$

Classically, a particle incident with $E = U$ would come to rest at the leading edge of the barrier and remain there indefinitely. In this unique case, the particle is neither reflected nor transmitted, but actually "absorbed" by the barrier!

7-9 The Java applet for this problem is identical to that for Problem 7.8 (http://info.brookscole.com/mp3e QMTools Simulations → Problem 7.8). The applet models an electron (mass 511 keV/c^2) in a potential $V(x)$ describing a square barrier of height 10.0 eV and width 1.00 Å on the interval [−5.0 Å, 5.0 Å]. The listing to the right of the graph includes a placeholder for one stationary state of the electron. Follow the applet instructions to display this state, and adjust its energy to 5.00 eV. The waveform to the right of the barrier is the transmitted wave at this energy; the waveform on the left is a mixture of incident and reflected waves. Follow the applet instructions to display the individual incident and reflected waves. The applet should resemble the screenshot below:



Activate the "Trace" feature to investigate the wave amplitudes at various locations. Observe that the incident wave amplitude A is unity (any discrepancy is due to the limitation inherent in representing decimal numbers in machine-readable, i.e., binary form). The amplitudes of the reflected and transmitted waves are $B = 0.815155$ and $F = 0.579242$, respectively, giving for the scattering coefficients at this energy $R(E) = |B/A|^2 = 0.66448$ and $T(E) = |F/A|^2 = 0.33552$, with $R(E) + T(E) = 1.00000$ in agreement with the sum rule to five figure accuracy. Also for this case,

$$\alpha L = \{2m[U-E]L^2\}^{1/2}/\hbar$$

$$= [2(511\times 10^3 \text{ eV}/c^2)(5.00 \text{ eV})(1 \text{ Å})^2]^{1/2}/(1973\times 10^2 \text{ eV}\cdot\text{Å}/c) = 1.146,$$

so the predicted value from Equation 7.9 is

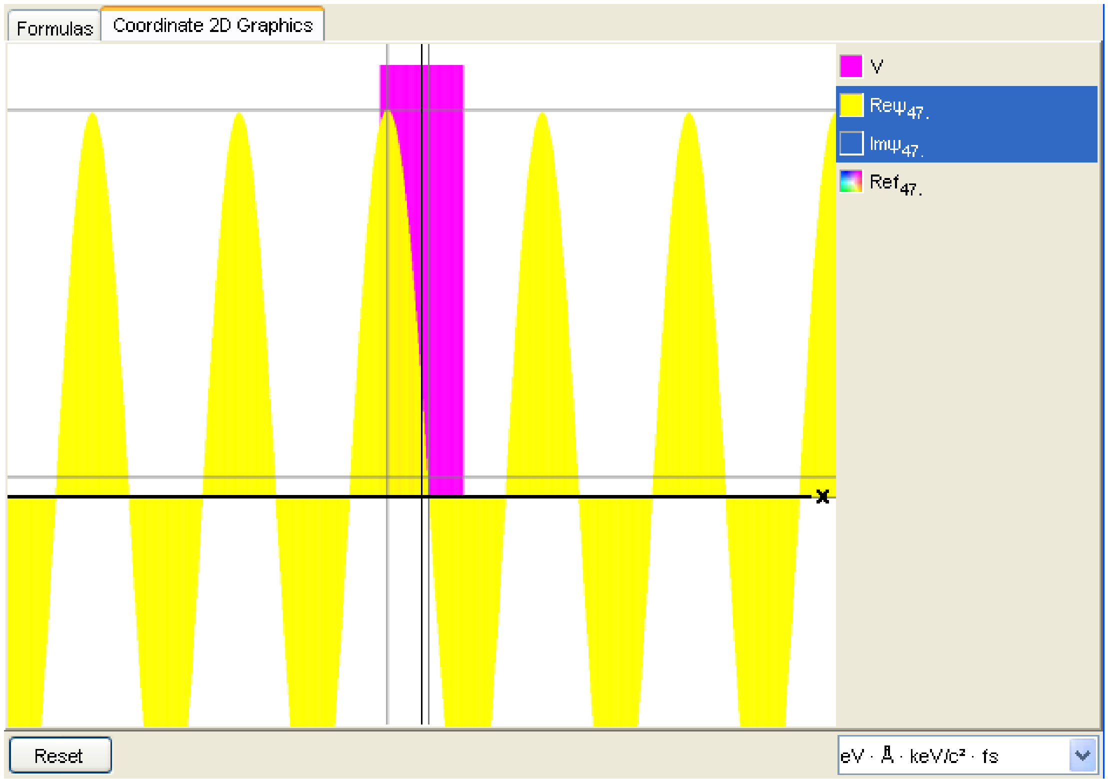
$$1/T(E) = 1 + \{U^2/[4E(U-E)]\}\sinh^2 \alpha L = 1 + [(10)^2/(4)(5)^2]\sinh^2(1.146) = 2.9977$$

or $T = 0.333596$ (exact solution), in good agreement with the numerical simulation. {Even better agreement is obtained by increasing the number of points at which the potential energy is sampled; the above results are reported for 1024 points.]

To describe protons, we need m = 938.38 MeV/c^2 (= 938380 keV/c^2), but to reproduce the same value for T, $\alpha L \propto \sqrt{m} \cdot L$ must remain the same. The required barrier width for proton transmission is, accordingly,

$$L \to L\sqrt{\frac{m_e}{m_p}} = (1\text{ Å})\sqrt{\frac{511}{938380}} = 0.02334\text{ Å}$$

7-10 The Java applet for this problem is available from our companion website (http://info.brookscole.com/mp3e QMTools Simulations → Problem 7.10). The applet models an electron (mass 511 keV/c^2) in a potential $V(x)$ describing a square barrier of height 10.0 eV and width 1.00 Å on the interval [−5.0 Å, 5.0 Å]. The square barrier potential appears on the tab labeled *Coordinate 2D Graphics* along with a stationary state (scattering) wave and its reflected component. The energy of these waves can be changed interactively from the stationary wave property panel (see applet instructions). Try any value $E > U = 10.0$ eV and vary E until the reflected wave amplitude vanishes. To "fine tune" the energy, enter the trial value as an argument to the reflection function ***Reflection* (…)** located on the *Spectrum* pane of the *Formulas* tab, and search for the zeroes of this function using trial and error. In this way we find that the two lowest energies giving rise to perfect transmission (zero reflection) occur at $E_1 = 47.83$ eV and $E_2 = 161.3$ eV, respectively. With the "Trace" feature activated and only the real part of the waveform displayed, we obtain (after zooming in to the limit!) the corresponding electron wavelengths in the barrier region from $(1/4)\lambda_1 = 0.4953$ Å (peak to node) and $(1/2)\lambda_2 = 0.5018$ Å (node to node). In this way we deduce the values $\lambda_1 = 1.981$ Å $(\approx 2L)$ for energy E_1 and $\lambda_2 = 1.004$ Å $(\approx L)$ for energy E_2. The screen snapshot below illustrates the peak-to-node measurement for determining λ_1 (approximate locations are marked by the light gray vertical lines):



Transmission resonances arise from the interference of the electron waves reflected from the leading and trailing edges of the barrier. If these reflected waves interfere destructively, there will be no reflection ($R = 0$) and thus perfect transmission. The wave reflected from the rear of the barrier must travel the extra distance $2L$ before recombining with the wave reflected at the front, but this wave also suffers an intrinsic phase shift of π radians, as discussed in Example 7.3. Thus, the condition for destructive interference becomes simply $2L = n\lambda$, where $n = 1, 2, \ldots$

7-11 (a) The matter wave reflected from the trailing edge of the well $(x = L)$ must travel the extra distance $2L$ before combining with the wave reflected from the leading edge $(x = 0)$. For $\lambda_2 = 2L$, these two waves interfere destructively since the latter suffers a phase shift of 180° upon reflection, as discussed in Example 7.3.

(b) The wave functions in all three regions are free particle plane waves. In regions 1 and 3 where $U(x) = U$ we have

$$\Psi(x, t) = Ae^{i(k'x-\omega t)} + Be^{i(-k'x-\omega t)} \qquad x < 0$$
$$\Psi(x, t) = Fe^{i(k'x-\omega t)} + Ge^{i(-k'x-\omega t)} \qquad x < 0$$

with $k' = \dfrac{[2m(E-U)]^{1/2}}{\hbar}$. In this case $G = 0$ since the particle is incident from the left. In region 2 where $U(x) = 0$ we have

$$\Psi(x,\ t) = Ce^{i(-kx-\omega t)} + De^{i(kx-\omega t)} \qquad 0 < x < L$$

with $k = \frac{(2mE)^{1/2}}{\hbar} = \frac{2\pi}{\lambda_2} = \frac{\pi}{L}$ for the case of interest. The wave function and its slope are continuous everywhere, and in particular at the well edges $x = 0$ and $x = L$. Thus, we must require

$$A + B = C + D \qquad [\text{continuity of } \Psi \text{ at } x = 0]$$

$$k'A - k'B = kD - kC \qquad \left[\text{continuity of } \frac{\partial \Psi}{\partial x} \text{ at } x = 0\right]$$

$$Ce^{-ikL} + De^{ikL} = Fe^{ik'L} \qquad [\text{continuity of } \Psi \text{ at } x = L]$$

$$kDe^{ikL} - kCe^{-ikL} = k'Fe^{ik'L} \qquad \left[\text{continuity of } \frac{\partial \Psi}{\partial x} \text{ at } x = L\right]$$

For $kL = \pi$, $e^{\pm ikL} = -1$ and the last two requirements can be combined to give $kD - kC = k'C + k'D$. Substituting this into the second requirement implies $A - B = C + D$, which is consistent with the first requirement only if $B = 0$, i.e., no reflected wave in region 1.

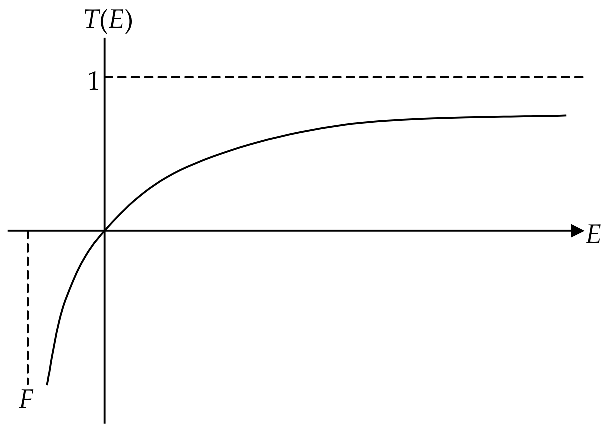
7-12 (a) For $E > 0$ solutions to the wave equation on either side of the origin are free particle plane waves with wavenumber $k = \left(\frac{2mE}{\hbar^2}\right)^{1/2}$:

$$\psi(x) = Ae^{+ikx} + Be^{-ikx} \qquad \text{for } x < 0$$

$$\psi(x) = Fe^{+ikx} + Ge^{-ikx} \qquad \text{for } x > 0$$

We take $G = 0$ (no reflected wave in the region to the right of the well) for particles incident on the delta well from the left.

(b) Some fraction of these are transmitted, as given by $T = \left|\frac{F}{A}\right|^2$. To find *T* we impose the slope condition on the waveform to get $ik(A - B) - ikF = -\left(\frac{2mS}{\hbar^2}\right)F$ and demand continuity of the wave at $x = 0$: $A + B = F$. Dividing the first equation through by *ik* and adding the result to the second gives $A = F\left[1 + i\left(\frac{mS}{\hbar^2 k}\right)\right] = F\left[1 + \frac{i}{-E_0/E}\right]$. In the second step we have written $E = \frac{\hbar^2 k^2}{2m}$ and $E_0 = \frac{-mS^2}{2\hbar^2}$ as convenient parameterizations for the scattering problem. The transmission coefficient is $T(E) = \left|\frac{A}{F}\right|^{-2} = \left[1 + \frac{-E_0}{E}\right]^{-1}$. The transmission coefficient for the delta well is sketched in the Figure below. $T(E)$ increases with *E*, approaching 1 (perfect transmission) only asymptotically as *E* becomes large (since $E_0 < 0$).



(c) Although the interpretation of T as a transmission factor is sensible only for nonnegative particle energies, it is interesting that the right-hand side becomes infinite if we take $E = E_0 = -\dfrac{mS^2}{2\hbar^2}$.

(d) For $E = |E_0|$, we find $T = \dfrac{1}{2}$, so that exactly half of the particles incident on the well with this energy are transmitted, and the other half reflected.

7-13 As in Problem 7-12, waveform continuity and the slope condition at the site of the delta well demand $A + B = F$ and $ik(A - B) - ikF = -\left(\dfrac{2mS}{\hbar^2}\right)F$. Dividing the second of these equations by ik and subtracting from the first gives $2B + F = F + \dfrac{\left(2mS/\hbar^2\right)F}{ik}$, or $B = -i\left(\dfrac{mS}{\hbar^2 k}\right)F = -iF\left(\dfrac{-E_0}{E}\right)^{1/2}$. Thus, the reflection coefficient R is $R(E) = \left|\dfrac{B}{A}\right|^2 = \left|\dfrac{B}{F}\right|^2\left|\dfrac{F}{A}\right|^2 = \left(\dfrac{-E_0}{E}\right)\left[1 + \left(\dfrac{-E_0}{E}\right)\right]^{-1}$. Then, with $T(E)$ from Problem 7-12, $T(E) = \left[1 + \left(\dfrac{-E_0}{E}\right)\right]^{-1}$, we find $R(E) + T(E) = \left(1 - \dfrac{E_0}{E}\right)\left[1 + \left(\dfrac{-E_0}{E}\right)\right]^{-1} = 1$.

7-14 Suppose the marble has mass 20 g. Suppose the wall of the box is 12 cm high and 2 mm thick. While it is inside the wall, $U = mgy = (0.02\text{ kg})\left(9.8\text{ m/s}^2\right)(0.12\text{ m}) = 0.023\,5\text{ J}$ and $E = K = \dfrac{1}{2}mv^2 = \dfrac{1}{2}(0.02\text{ kg})(0.8\text{ m/s})^2 = 0.006\,4\text{ J}$. Then

$$\alpha = \frac{\sqrt{2m(U - E)}}{\hbar} = \frac{\sqrt{2(0.02\text{ kg})(0.017\,1\text{ J})}}{1.055 \times 10^{-34}\text{ J}\cdot\text{s}} = 2.5 \times 10^{32}\text{ m}^{-1}$$

and the transmission coefficient is

$$e^{-2\alpha L} = e^{-2\left(2.5\times10^{32}\right)\left(2\times10^{-3}\right)} = e^{-10\times10^{29}} = e^{-2.30\left(4.3\times10^{29}\right)} = 10^{-4.3\times10^{29}} \sim 10^{-10^{30}}.$$

7-15 Divide the barrier region into N subintervals of length $\Delta x = x_{i+1} - x_i$. For the barrier in the i^{th} subinterval, denote by A_i and F_i the incident and transmitted wave amplitudes, respectively. The transmission coefficient for this interval is then $T_i = \left|\frac{F_i}{A_i}\right|^2$, and that for the entire barrier is $T(E) = \left|\frac{F_N}{A_1}\right|^2$. Now consider the product $\Pi T_i = T_1 T_2 T_3 \ldots T_N = \left(\frac{|F_1|^2}{|A_1|^2}\right)\left(\frac{|F_2|^2}{|A_2|^2}\right)\left(\frac{|F_3|^2}{|A_3|^2}\right)\ldots\left(\frac{|F_N|^2}{|A_N|^2}\right)$. Assuming the transmitted wave intensity for one barrier becomes the incident wave intensity for the next, we have $|F_1|^2 = |A_2|^2$, $|F_2|^2 = |A_3|^2$ etc., so that $T(E) = \left|\frac{F_N}{A_1}\right|^2 = T_1 T_2 T_3 \ldots T_N$. Next, we assume that Δx is sufficiently small and that $U(x)$ is sensibly constant over each interval (so that the square barrier result can be used for T_i), yet large enough to approximate $\sinh \alpha_i \Delta x$ with $\frac{1}{2} e^{\alpha_i \Delta x}$, where α_i, is the value taken by α in the i^{th} subinterval: $\alpha_i = \frac{[2m(U_i - E)]^{1/2}}{\hbar}$. Then, $\frac{1}{T_i} = 1 + \left[\frac{U_i^2}{4E(U_i - E)}\right]\sinh^2(\alpha_i \Delta x) \approx \left[\frac{U_i^2}{16E(U_i - E)}\right] e^{2\alpha_i \Delta x}$ and the transmission coefficient for the entire barrier becomes $T(E) \approx \Pi\left\{\left[\frac{16E(U_i - E)}{U_i^2}\right] e^{-2\alpha_i \Delta x}\right\} \approx \left[\frac{\Pi 16E(U_i - E)}{U_i^2}\right] e^{-\Sigma 2\alpha_i \Delta x}$. To recover Equation 7.10, we approximate the sum in the exponential by an integral, and note that the product in square brackets is a term of order 1: $T(E) \sim e^{\Sigma 2\alpha_i \Delta x} \approx e^{-\int 2\alpha(x)dx}$ where now $\alpha(x) = \frac{2m[U(x) - E]^{1/2}}{\hbar}$.

7-16 Since the alpha particle has the combined mass of 2 protons and 2 neutrons, or about $3\,755.8\ \text{MeV}/c^2$, the first approximation to the decay length δ is

$$\delta \approx \frac{\hbar}{(2mU)^{1/2}} = \frac{197.3\ \text{MeV fm}/c}{\left[2\left(3\,755.8\ \text{MeV}/c^2\right)(30\ \text{MeV})\right]^{1/2}} = 0.415\,6\ \text{fm}.$$

This gives an effective width for the (infinite) well of $R + \delta = 9.415\,6\ \text{fm}$, and a ground state energy $E_1 = \frac{\pi^2 (197.3\ \text{MeV fm}/c)^2}{2\left(3\,755.8\ \text{MeV}/c^2\right)(9.415\,6\ \text{fm})^2} = 0.577\ \text{MeV}$. From this E we calculate $U - E = 29.42\ \text{MeV}$ and a new decay length

$$\delta = \frac{197.3\ \text{MeV fm}/c}{\left[2\left(3\,755.8\ \text{MeV}/c^2\right)(29.42\ \text{MeV})\right]^{1/2}} = 0.419\,7\ \text{fm}.$$

This, in turn, increases the effective well width to 9.419 7 fm and lowers the ground state energy to $E_1 = 0.576$ MeV . Since our estimate for E has changed by only 0.001 MeV, we may be content with this value. With a kinetic energy of E_1 , the alpha particle in the ground state has speed $v_1 = \left(\frac{2E_1}{m}\right)^{1/2} = \left[\frac{2(0.576\text{ MeV})}{(3\,755.8\text{ MeV}/c^2)}\right]^{1/2} = 0.017\,5c$. In order to be ejected with a kinetic energy of 4.05 MeV, the alpha particle must have been preformed in an excited state of the nuclear well, not the ground state.

7-17 The collision frequency f is the reciprocal of the transit time for the alpha particle crossing the nucleus, or $f = \frac{v}{2R}$, where v is the speed of the alpha. Now v is found from the kinetic energy which, inside the nucleus, is not the total energy E but the difference $E-U$ between the total energy and the potential energy representing the bottom of the nuclear well. At the nuclear radius $R = 9$ fm , the Coulomb energy is

$$\frac{k(Ze)(2e)}{R} = 2Z\left(\frac{ke^2}{a_0}\right)\left(\frac{a_0}{R}\right) = 2(88)(27.2\text{ eV})\left(\frac{5.29\times10^4\text{ fm}}{9\text{ fm}}\right) = 28.14\text{ MeV}\,.$$

From this we conclude that $U = -1.86$ MeV to give a nuclear barrier of 30 MeV overall. Thus an alpha with $E = 4.05$ MeV has kinetic energy $4.05 + 1.86 = 5.91$ MeV inside the nucleus. Since the alpha particle has the combined mass of 2 protons and 2 neutrons, or about $3\,755.8\text{ MeV}/c^2$ this kinetic energy represents a speed

$$v = \left(\frac{2E_k}{m}\right)^{1/2} = \left[\frac{2(5.91)}{3\,755.8\text{ MeV}/c^2}\right]^{1/2} = 0.056c\,.$$

Thus, we find for the collision frequency $f = \frac{v}{2R} = \frac{0.056c}{2(9\text{ fm})} = 9.35\times10^{20}\text{ Hz}$.

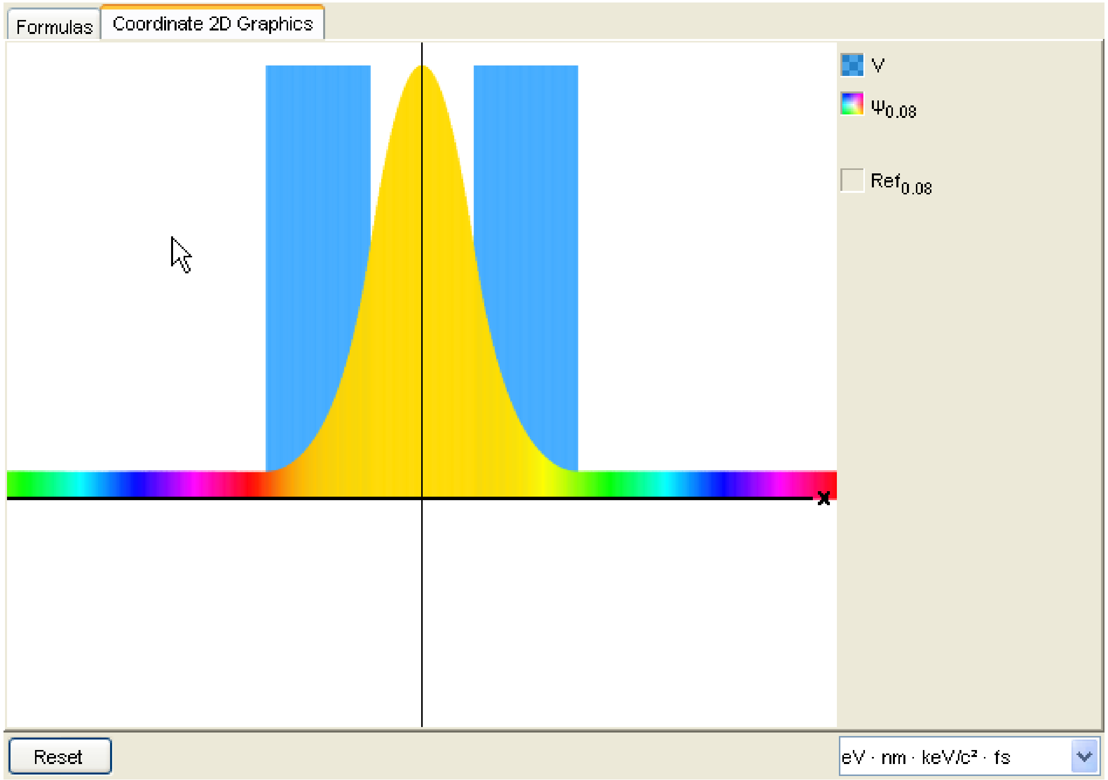
7-18 Any one conduction electron of the metal is virtually free to move about with a speed v fixed by its kinetic energy $E_k = \frac{1}{2}mv^2$, but the average energy per electron available for motion in any specific direction (say, normal to the surface) is reduced from this by the factor 1/3 to account for the random directions of travel:

$$\langle E_k\rangle = \frac{1}{2}m\left\{\langle v_x^2\rangle + \langle v_y^2\rangle + \langle v_z^2\rangle\right\} = \frac{3}{2}m\langle v_x^2\rangle,\text{ or }\frac{1}{2}m\langle v_x^2\rangle = \frac{1}{3}\langle E_k\rangle\,.$$

For a sample with dimension L normal to the surface, the time elapsed between collisions with this surface is $\frac{2L}{|v_x|}$, for any one electron. The reciprocal of this time is the collision frequency. For two electrons, collisions occur twice as often, and so forth, so that the collision frequency for N electrons is $\frac{N|v_x|}{2L}$. Making the identification $|v_x|^2 = \langle v_x^2 \rangle$ allows us to write the collision frequency f in terms of electron energy as $f = \frac{N}{2L}\left(\frac{2E_k}{3m}\right)^{1/2}$. The density of copper is $8.96\ \text{g/cm}^3$, so one cubic centimeter represents an amount of copper equal to 8.96 g, or the equivalent of $\frac{8.96}{63.54} = 0.141$ moles (the atomic weight of copper is 63.54). Since each mole contains a number of atoms equal to Avogadro's number $N_A = 6.02\times10^{23}$, the number of copper atoms in our sample is $0.141N_A$ or about 8.49×10^{22}, which is also the number N of conduction electrons.

The most energetic electrons in copper have kinetic energies of about 7 eV. Using this for E_k, $L = 1$ cm, and $N = 8.49\times10^{22}$ gives for the collision frequency $f = 3.85\times10^{30}$ Hz.

7-19 The Java applet for this problem is available from our companion website (http://info.brookscole.com/mp3e QMTools Simulations → Problem 7.19). The applet models an electron in gallium arsenide with potential energy *V(x)* representing a double barrier. Note that the effective electron mass in these materials is $m^* = 0.067\ m_e = 34.237\ \text{keV/c}^2$. The double barrier potential appears on the *Coordinate 2D Graphics* tab along with a stationary state (scattering) wave and its reflected component. The energy of these waves can be changed interactively from the stationary wave property panel (see applet instructions). Gradually increase the energy until the reflected wave amplitude vanishes. [*Caution*: at very small energies a reflected wave cannot be computed; if you encounter a warning to that effect, just continue raising the energy until the reflected wave reappears.] For $E = 0.08293$ eV we find $T(E) = 0.999964$, or nearly perfect transmission. The screen snapshot below shows the stationary state waveform at the resonance energy:



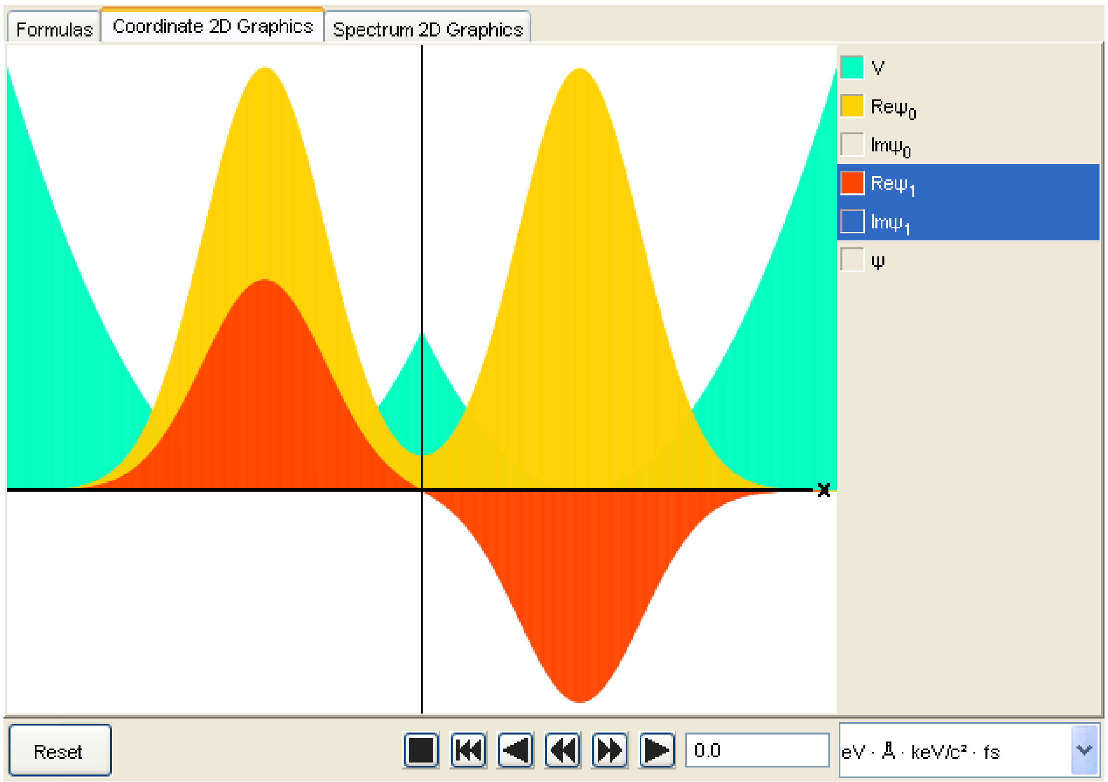
The transmission probability drops to about 50% at $E = 0.08254$ eV $(T(E) = 0.4991)$, and again at $E = 0.08332$ eV $(T(E) = 0.5027)$. [*Note*: *T(E)* can be read directly from the *Formulas* tab by entering the desired energy as an argument to the transmission function ***Transmission* (…)** located on the *Spectrum* pane.] Thus, the transmission resonance at $E = 0.08293$ eV has width

$$\Delta E = 0.08332 - 0.08254 = 0.00078 \text{ eV}.$$

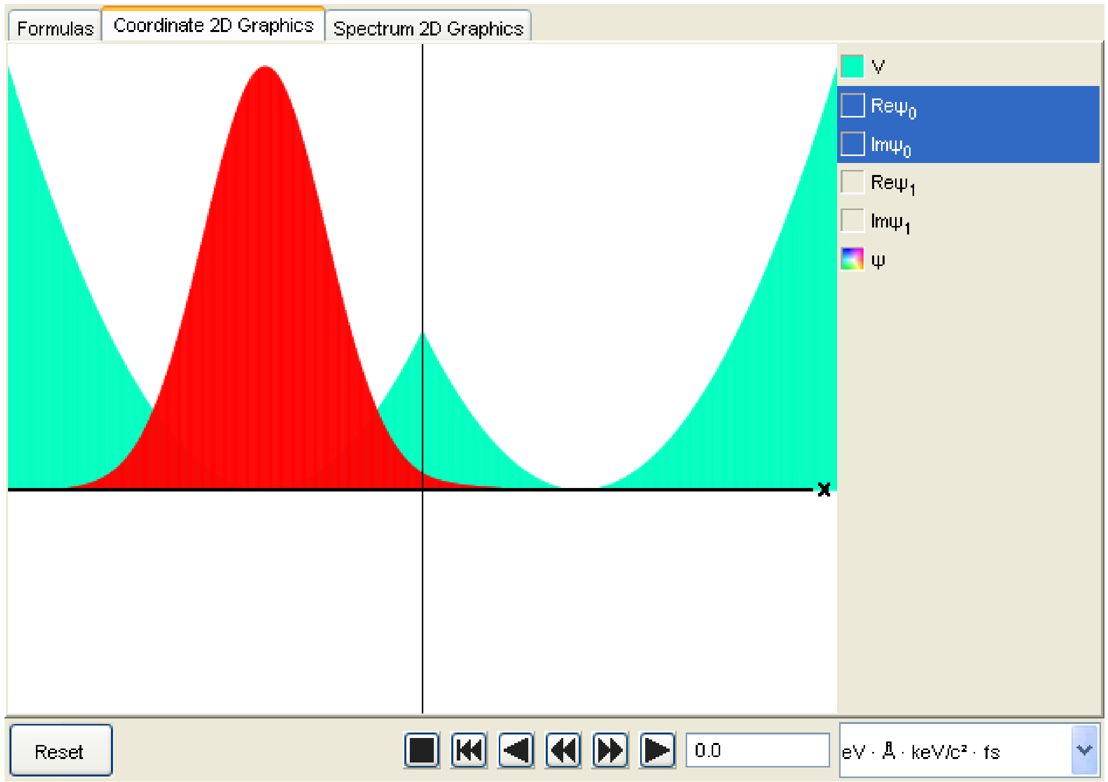
This resonance is very sharp, amounting to less than 1% of the resonance energy.

7-20 The Java applet for this problem is available from our companion website (http://info.brookscole.com/mp3e QMTools Simulations → Problem 7.20). The applet shows the double oscillator potential of Equation 7.15 with parameters to model the nitrogen atom in the ammonia molecule, as discussed in the text. These values appear on the *Formulas* tab of the applet, along with the reduced mass of the nitrogen-hydrogen group, $2.47 \text{ u} = 2300805 \text{ keV/c}^2$. On the *Coordinate 2D Graphics* tab the double oscillator potential is plotted for 1024 points over the interval [−1 Å, +1 Å].

(a) The listing to the right of the graph includes placeholders for two stationary states of the nitrogen atom. Follow the applet instructions to display each state in turn and adjust its energy until no discernible mismatch results in the wavefunction. For this potential well, we find the ground and first excited state energies at $E_0 = 0.038633$ eV and $E_1 = 0.038936$ eV (waveforms with zero and one node, respectively). The ground state is symmetric and the first excited state antisymmetric about the midpoint of the double oscillator well. These two states differ in energy by only $\Delta E = 3.03 \times 10^{-4}$ eV ! The screen snapshot below shows the two stationary states as they appear in the applet:

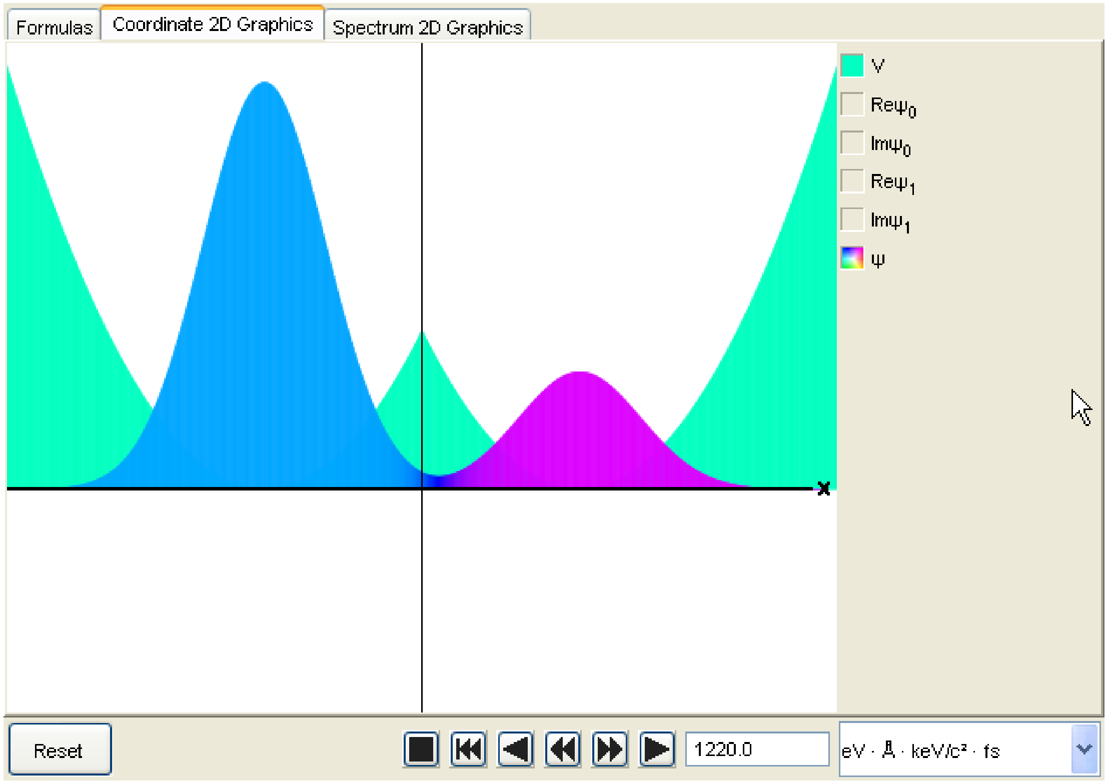


(b) Located on the *Spectrum* pane of the applet *Formulas* tab, the envelope function $Psi0(E) = 1$ specifies that all stationary states in the input range will be added with unit amplitude. Plot this function for 1024 points over the energy range from 0.0385 eV to 0.0390 eV. Only stationary states with energies in this range are actually added, and these are just the two lowest-lying states already found in (a). Refer to the applet instructions to display the Schrödinger wavefunction that results from this addition. The Schrödinger wavefunction has a Gaussian-like appearance and is confined to the left-side oscillator well, as shown in the screen snapshot below:



Evidently the nitrogen atom described by this waveform is initially localized to one side of the basal plane in the ammonia molecule.

(c) Start the clock to animate the display. With the passage of time, the waveform in the right-side well grows steadily in amplitude. The following screen snapshot shows the [non-stationary state] wavefunction for the nitrogen atom as it appears at 1220 clock "ticks":



After about 6850 "ticks", we can assert with confidence that the atom has moved over completely to the right-side well; with another 6850 "ticks", the atom takes up its original position on the left. Since each "tick" is 1 fs = 10^{-15} s, the recurrence time is $13700\times10^{-15}\ \text{s} = 1.37\times10^{-11}\ \text{s}$. The associated frequency is

$$f = \frac{1}{1.37\times10^{-11}\ \text{s}} = 7.30\times10^{10}\ \text{Hz}$$

The frequency f defines for this process a characteristic energy

$$E = hf = (4.136\times10^{-15}\ \text{eV}\cdot\text{s})(7.30\times10^{10}\ \text{Hz}) = 3.02\times10^{-4}\ \text{eV}$$

that is [nearly] identical to the energy separation $\Delta E = 3.03\times10^{-4}$ eV between the two stationary states used to construct the initial wavefunction!