Electric Circuits
Lecture 9 Second-Order Systems

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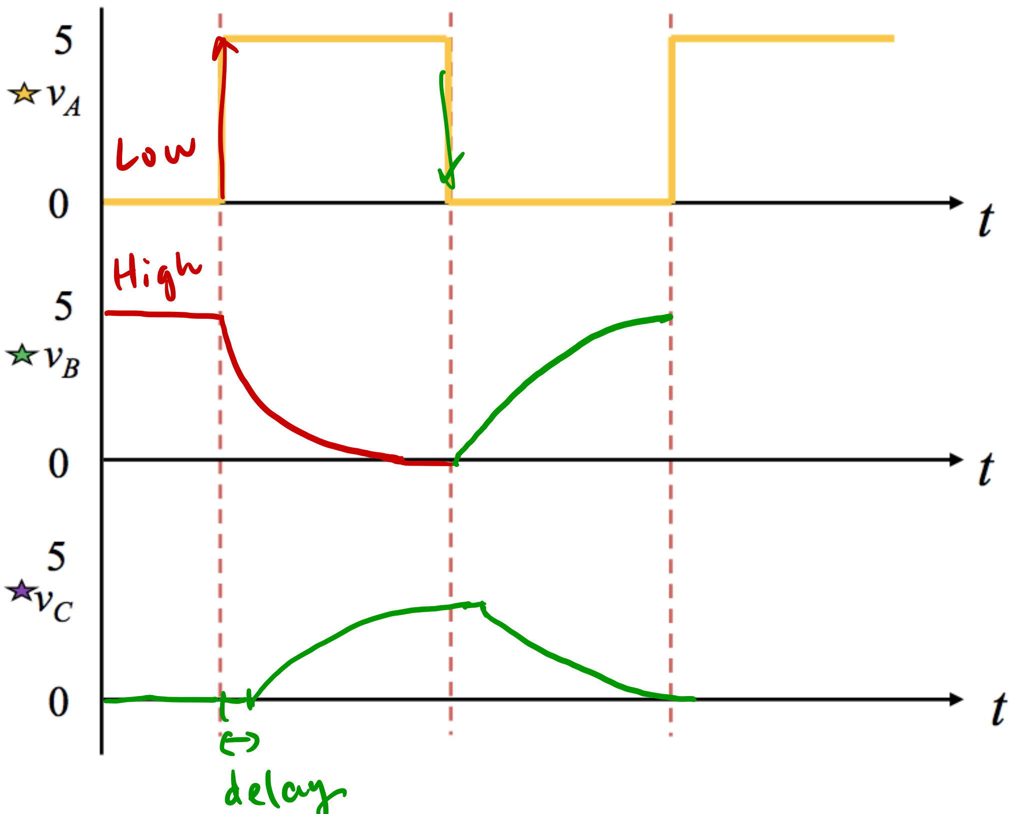
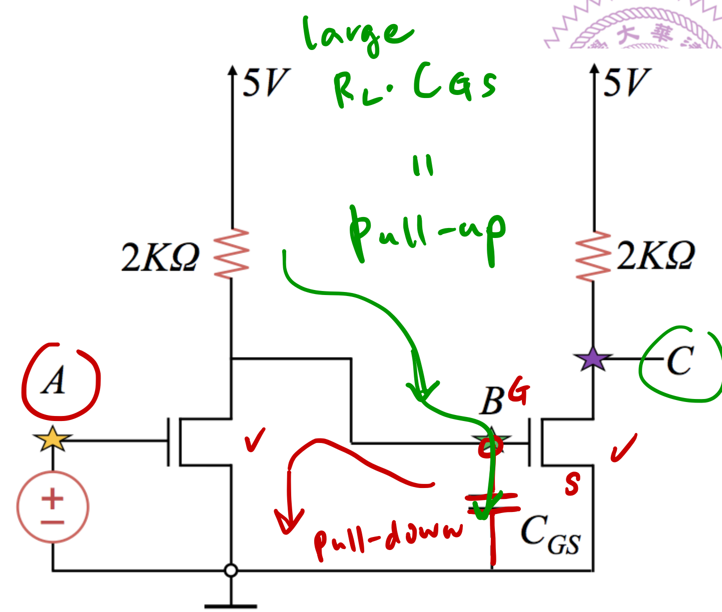
Lecture Outline

- Chapter 9 in the textbook

An Inverter Chain Example

- With C_{GS} of NMOS on node B.





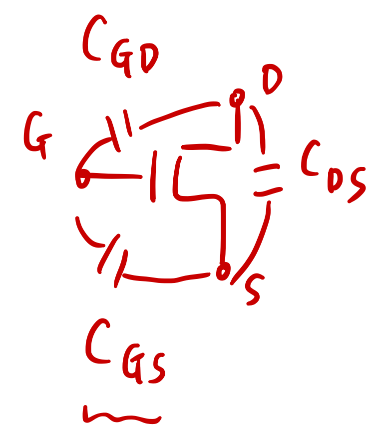
Pull-down

$(R_L \| R_{on}) \cdot C_{GS}$

$R_L >> R_{on}$

$\approx R_{on} \cdot C_{GS}$

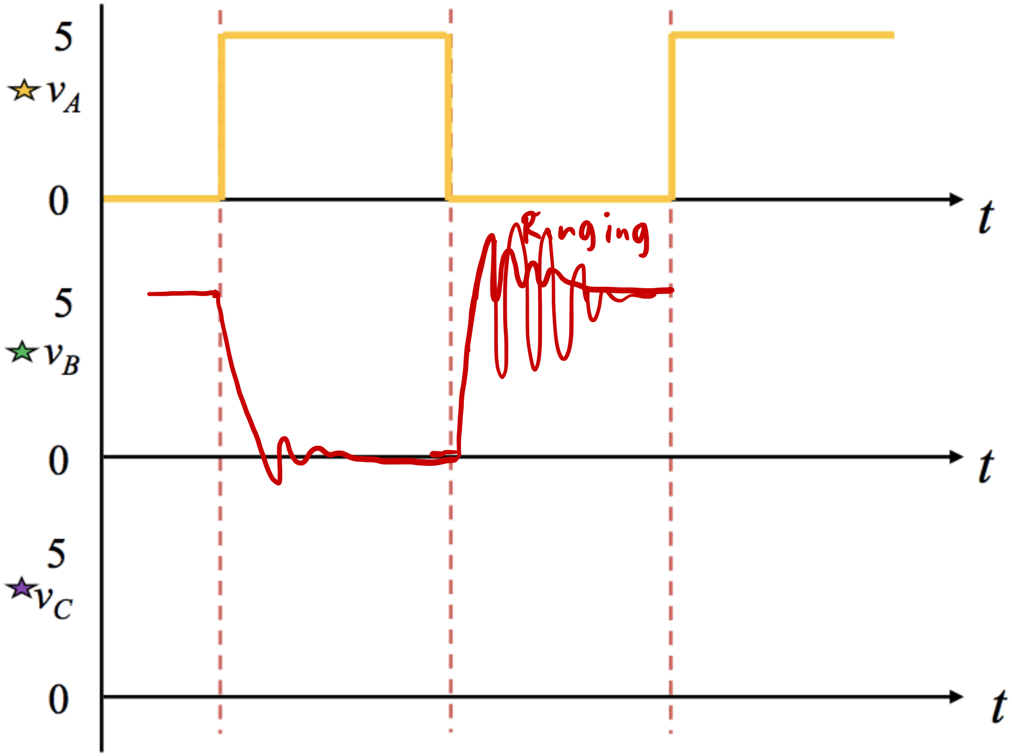
small

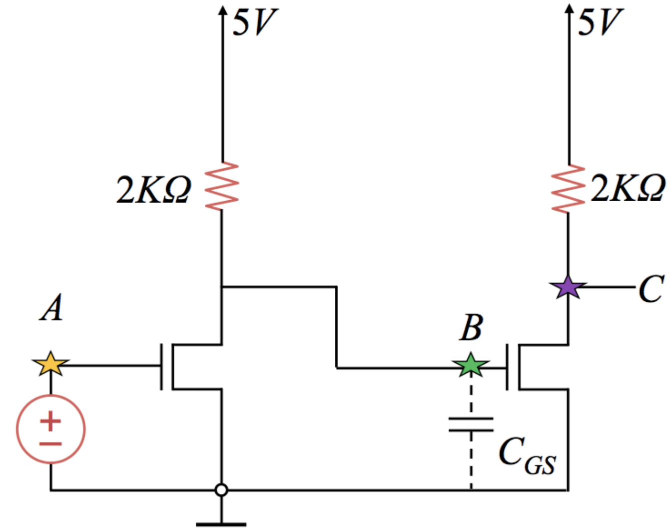


  - Slow rise time and delay in signal.
  - We can try to increase the speed with a smaller R_L (from 2 kΩ to 50 Ω).

Observed Output – Fast Case

- With C_{GS} of NMOS on node B.





  - Ringing behavior observed!!

Fast Case – What's Really Going On

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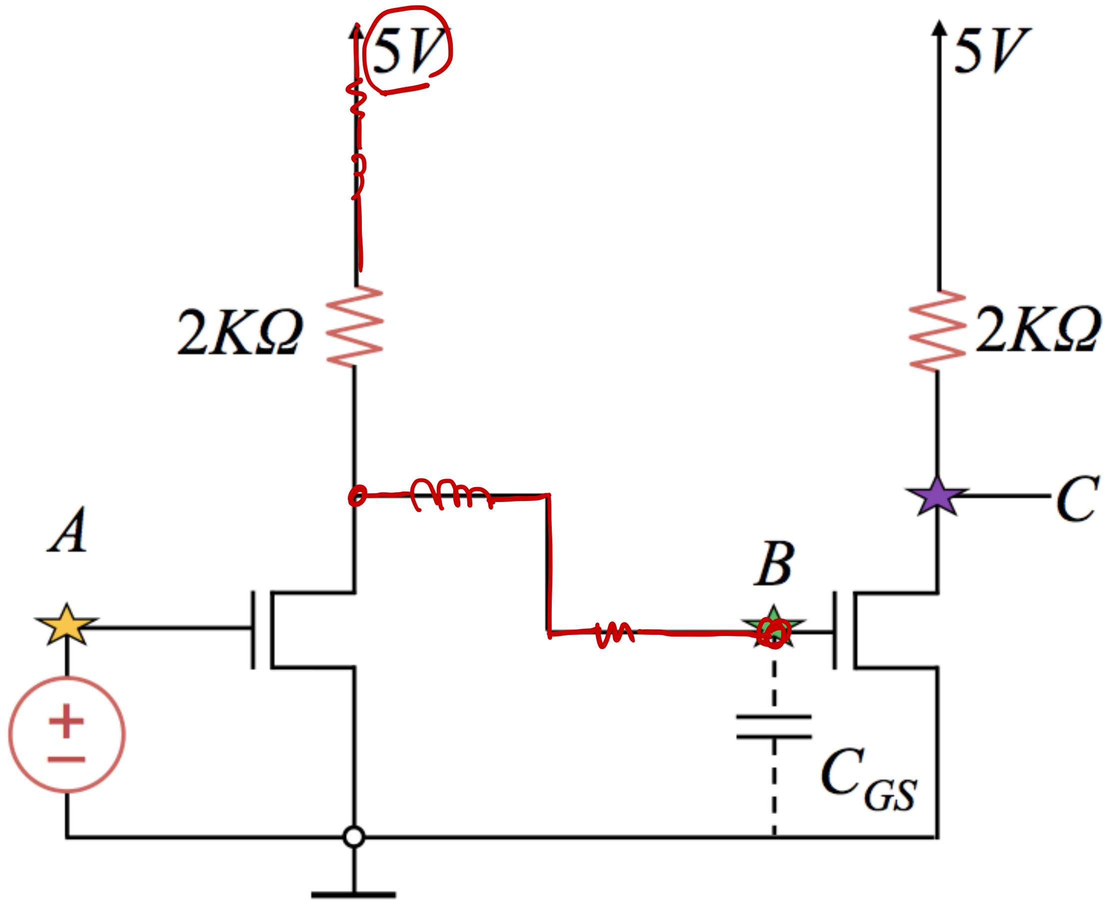
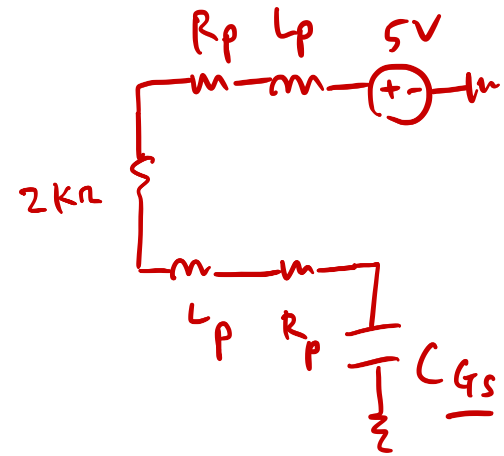
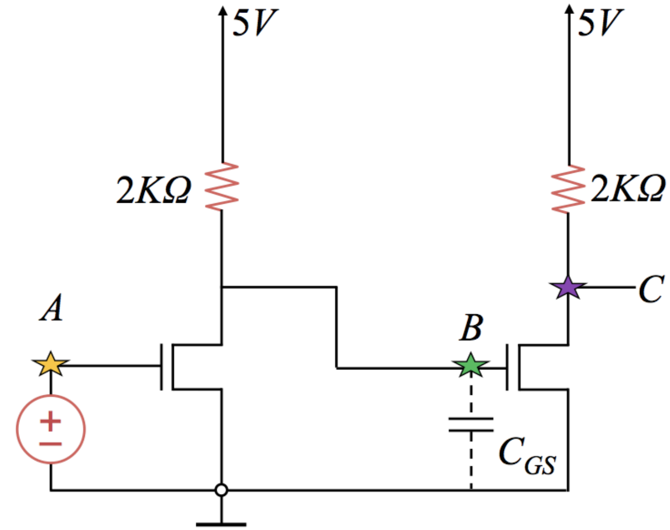
- The parasitic inductance of the wire is included.



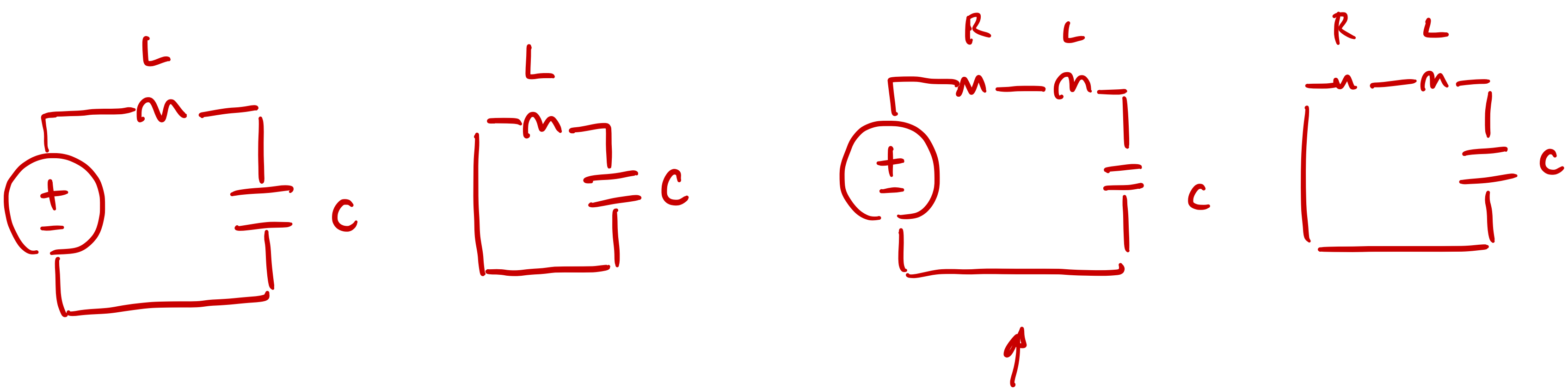


Second-Order Systems

- Involving R, L, C circuit elements.



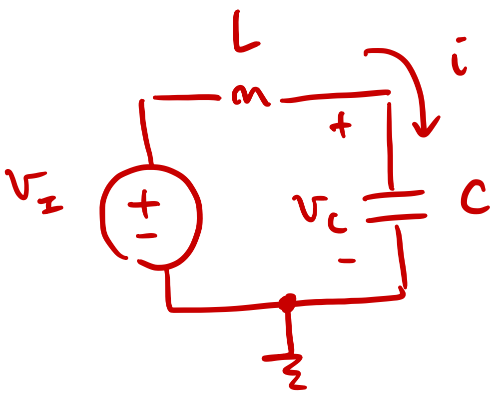
- Relevant circuit model



  - Two energy-storage elements → second-order system

First, Let's Analyze the LC Network

- We will introduce R into the circuit later.
- Node method



$$i = C\frac{dv_C}{dt} = \frac{1}{L}\int_{-\infty}^{t} (v_I - v_C)\,dt$$

$$\xrightarrow{\frac{d}{dt}} \quad C\frac{d^2 v_C}{dt^2} = \frac{1}{L}(v_I - v_C)$$

$$\Rightarrow \quad LC\frac{d^2 v_C}{dt^2} + v_C = v_I$$

Given $v_C(0)$, $v_C'(0)$

Method of Particular and Homogeneous Solutions

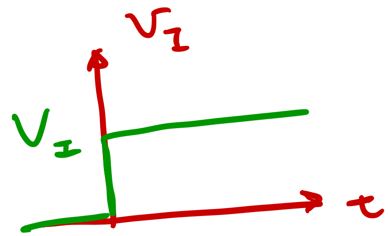
- Four-step procedure

✓ $$\boxed{LC\frac{\mathrm{d}^2 v}{\mathrm{d}t} + v = v_I}$$

1. Find the particular solution
2. Find the homogeneous solution
   - Four-step procedure
3. The total solution is the sum of the particular solution and homogeneous solution.
4. Use initial condition to solve for the remaining constraints.

Let's Solve

Consider $v_I(t) = V_I \cdot u(t)$



Given $v_c(0) = 0$

initial conditions $\left(\underbrace{i(0) = 0}_{} \right.$ $\left(i = C\frac{dv_c}{dt} \quad \frac{dv_c}{dt}(t=0) = 0 \right)$

Zero-state response, $v_c(t)$, $t > 0$

1. Particular Solution

$$LC\frac{d^2 v_c}{dt^2} + v_c = V_I$$

Guess that $v_{c,p} = K$

$$LC \cdot 0 + K = V_I$$

$$v_{c,p} = V_I$$

2. Homogeneous Solution

- Look for solution to $LC\dfrac{d^2 v_{CH}}{dt^2} + v_{CH} = 0$
- Four-step method:

1) Assume $v_{CH} = Ae^{st}$

2) $LC \cdot A \cdot s^2 \cdot e^{st} + Ae^{st} = 0$

$\Rightarrow (LCs^2 + 1)\underline{e^{st}} = 0$

$\Rightarrow LCs^2 + 1 = 0$ characteristic equation

3) $s^2 = \dfrac{-1}{LC}$ $\Rightarrow s = \pm j \cdot \sqrt{\dfrac{1}{LC}}$ $j^2 = -1$, $j = \sqrt{-1}$

$= \underbrace{\pm j \cdot \omega_0}$

$\omega_0 = \sqrt{\dfrac{1}{LC}}$: natural frequency of LC circuit (rad/s)

$\dfrac{\omega_0}{2\pi} = f_0$ = natural frequency (Hz)

4) $v_{CH} = A_1 e^{j\omega_0 t} + A_2 e^{-j\omega_0 t}$

3. Total Solution

$$v_C = v_{CH} + v_{CP} = A_1 e^{jw_0 t} + A_2 e^{-jw_0 t} + V_I \quad , \; w_0 = \sqrt{\frac{1}{LC}}$$

4. Find unknowns from initial conditions

$$v_C(0) = 0 = A_1 + A_2 + V_I \quad ①$$

$$i(0) = 0 = C \cdot \frac{dv_C}{dt}(t=0) = C \cdot A_1 \cdot jw_0 \cdot e^{jw_0 t} - C \cdot A_2 \cdot jw_0 \cdot e^{-jw_0 t} \Big|_{t=0}$$

$$= C\, jw_0 (A_1 - A_2) \quad ②$$

$$\Rightarrow \begin{cases} A_1 + A_2 = -V_I \\ A_1 - A_2 = 0 \end{cases} \Rightarrow A_1 = A_2 = -\frac{V_I}{2}$$

$$v_C = -\frac{V_I}{2} e^{jw_0 t} - \frac{V_I}{2} e^{-jw_0 t} + V_I \quad , \; w_0 = \sqrt{\frac{1}{LC}}$$

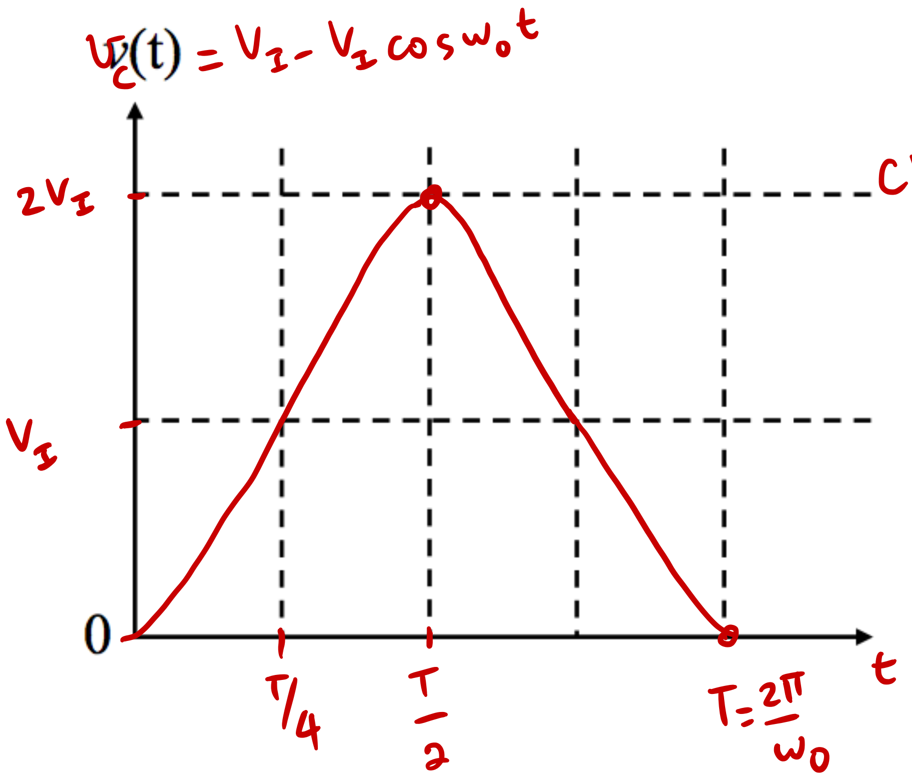
3. Total Solution

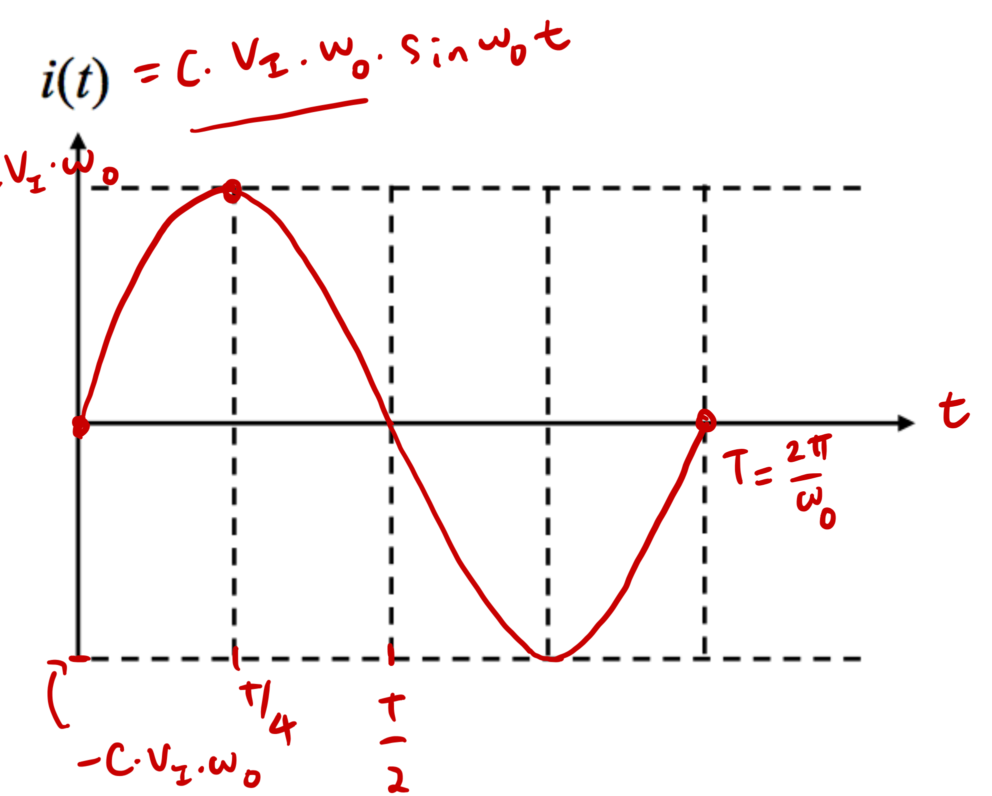
- Remember Euler relation: $e^{jx} = \cos x + j\sin x$

$v_C = V_I - V_I \cdot \cos(\omega_0 t)$

$i = C \cdot V_I \cdot \omega_0 \cdot \sin(\omega_0 t)$

Plotting the Total Solution





- $i(t)$ leads $v_c(t)$ by $T/4$

  $v_c(t)$ lags $i(t)$ by $T/4$

- Practice $v_L = ?$

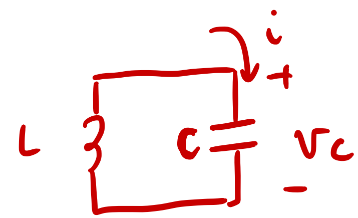
  v_L leads or lags i by ?

Summary of Method

1. Write DE for circuit by applying node method.
2. Find particular solution v_P by guessing and trial & error.
3. Find homogeneous solution v_{H}.
   A. Assume solution of the form. Ae^{st}
   B. Obtain characteristic equation. s^2
   C. Solve characteristic equation for roots s_i.
   D. Find v_H by summing $A_i \cdot e^{s_i t}$ terms.
4. Total solution is $v_p + v_H$, then solve for remaining constants using initial conditions.
   (A_i)

Example – Undriven LC Network Response

- What if we have:



- We can obtain the answer directly from the homogeneous solution (with $v_I = 0$).

$$LC\frac{d^2 v_c}{dt^2} + v_c = 0$$

Given $v_c(0) = V_1$, $i(0) = 0$

$$v_c(t) = A_1 e^{j\omega_0 t} + A_2 e^{-j\omega_0 t}$$

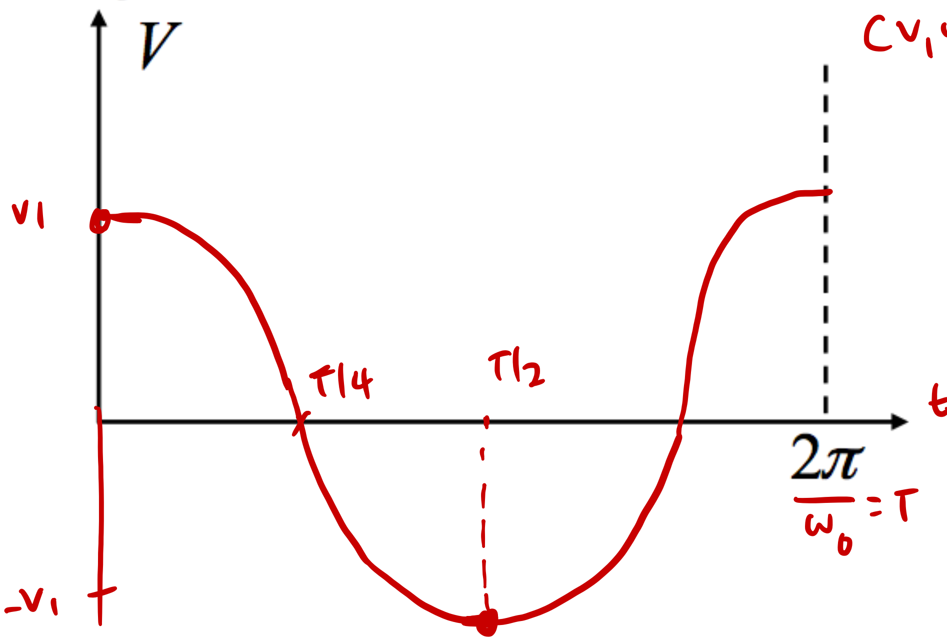
$$\begin{cases} A_1 + A_2 = V_1 \\ CA_1 j\omega_0 - CA_2 j\omega_0 = 0 \end{cases} \Rightarrow A_1 = A_2 = \frac{V_1}{2}$$

(oscillator)
振盪器

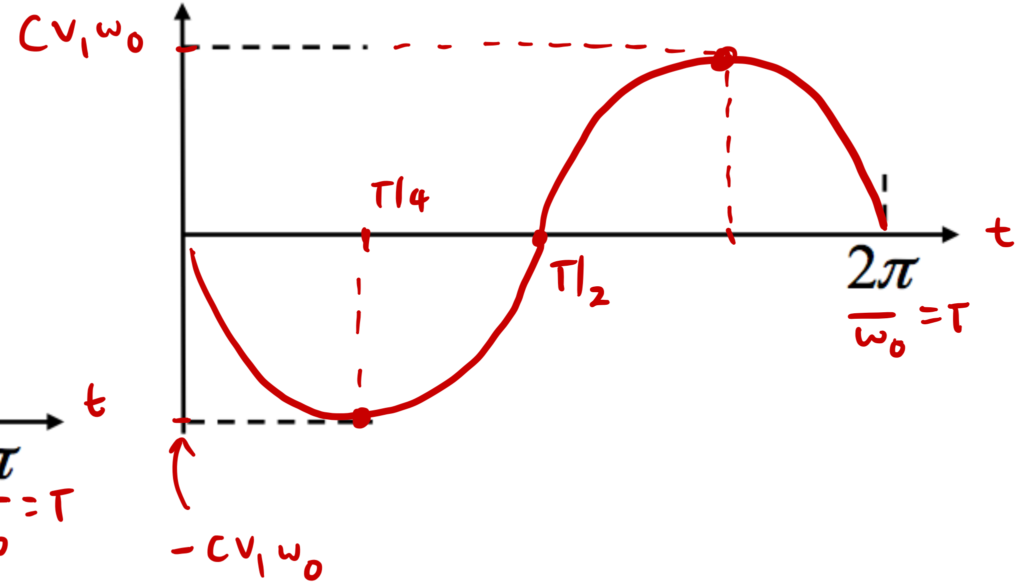
$$v_c(t) = \frac{V_1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) = \frac{V_1}{2} \cdot 2\cos\omega_0 t = V_1 \cdot \cos\omega_0 t$$

Undriven LC Network Response

$v_C = V_1 \cdot \cos\omega_0 t$



$i_C = -C \cdot V_1 \cdot \omega_0 \cdot \sin\omega_0 t$



- i_c leads v_c by $\underline{T/4}$

$$\frac{v_{peak}}{i_{peak}} = \sqrt{\frac{V_1}{CV_1\omega_0}} = \sqrt{\frac{1}{C\omega_0}} = \sqrt{\frac{L}{C}}$$

characteristic impedance of LC

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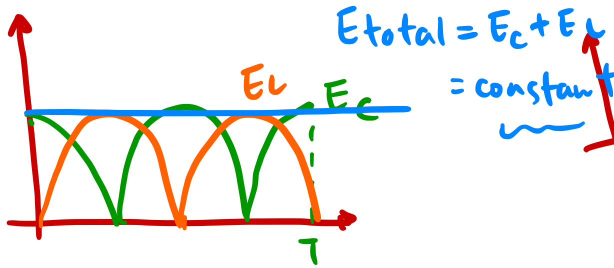
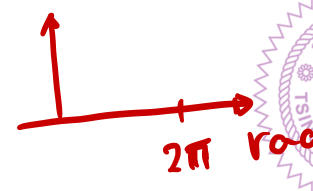
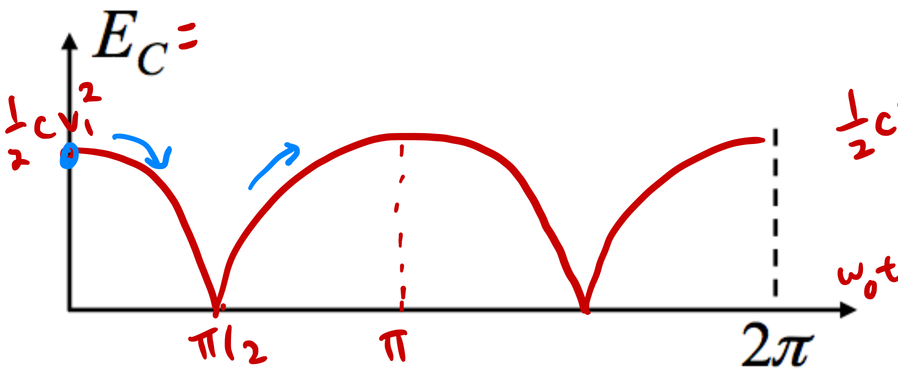
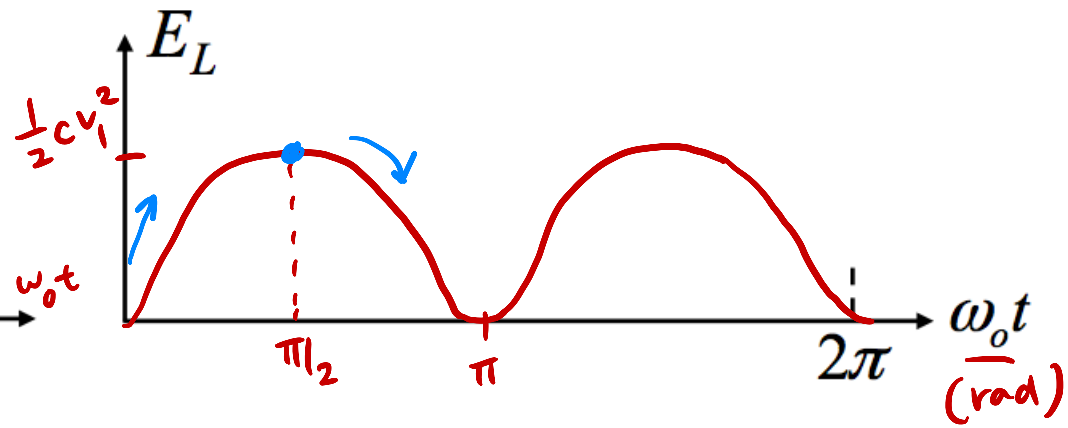
Energy









$$E_c = \frac{1}{2} C \cdot v_c^2 = \frac{1}{2} C \cdot (v_1 \cos \omega_0 t)^2$$

$$= \frac{1}{2} C v_1^2 \cos^2 \omega_0 t$$

$$E_{c,max} = \frac{1}{2} C v_1^2$$

$$E_L = \frac{1}{2} L \cdot i^2 = \frac{1}{2} L \cdot (-C v_1 \omega_0 \sin \omega_0 t)^2$$

$$= \frac{1}{2} L \cdot C^2 \cdot v_1^2 \cdot \omega_0^2 \cdot \sin^2 \omega_0 t$$

$$E_{L,max} = \frac{1}{2} L C^2 \cdot v_1^2 \cdot \frac{1}{LC} = \frac{1}{2} C v_1^2$$

$$E_c + E_L = C v_1^2 = \text{constant}$$

- Total energy in the system is a constant, but it goes back and forth between the capacitor and the inductor.