Electric Circuits
Lecture 12 Impedance and Frequency Response

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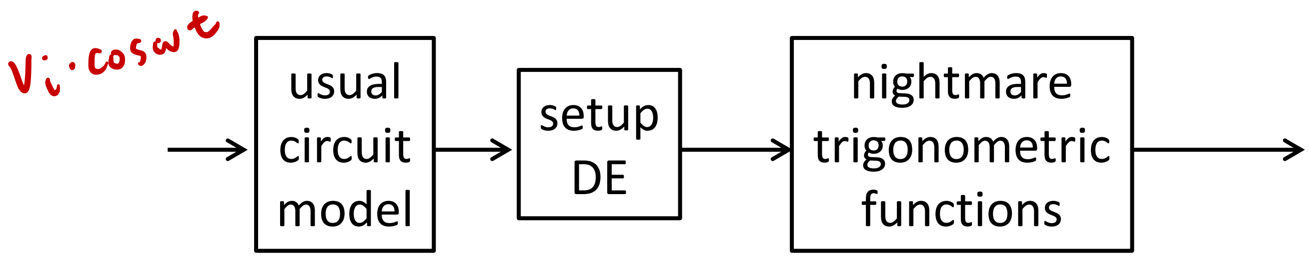
Jenny Yi-Chun Liu

jennyliu@gapp.nthu.edu.tw

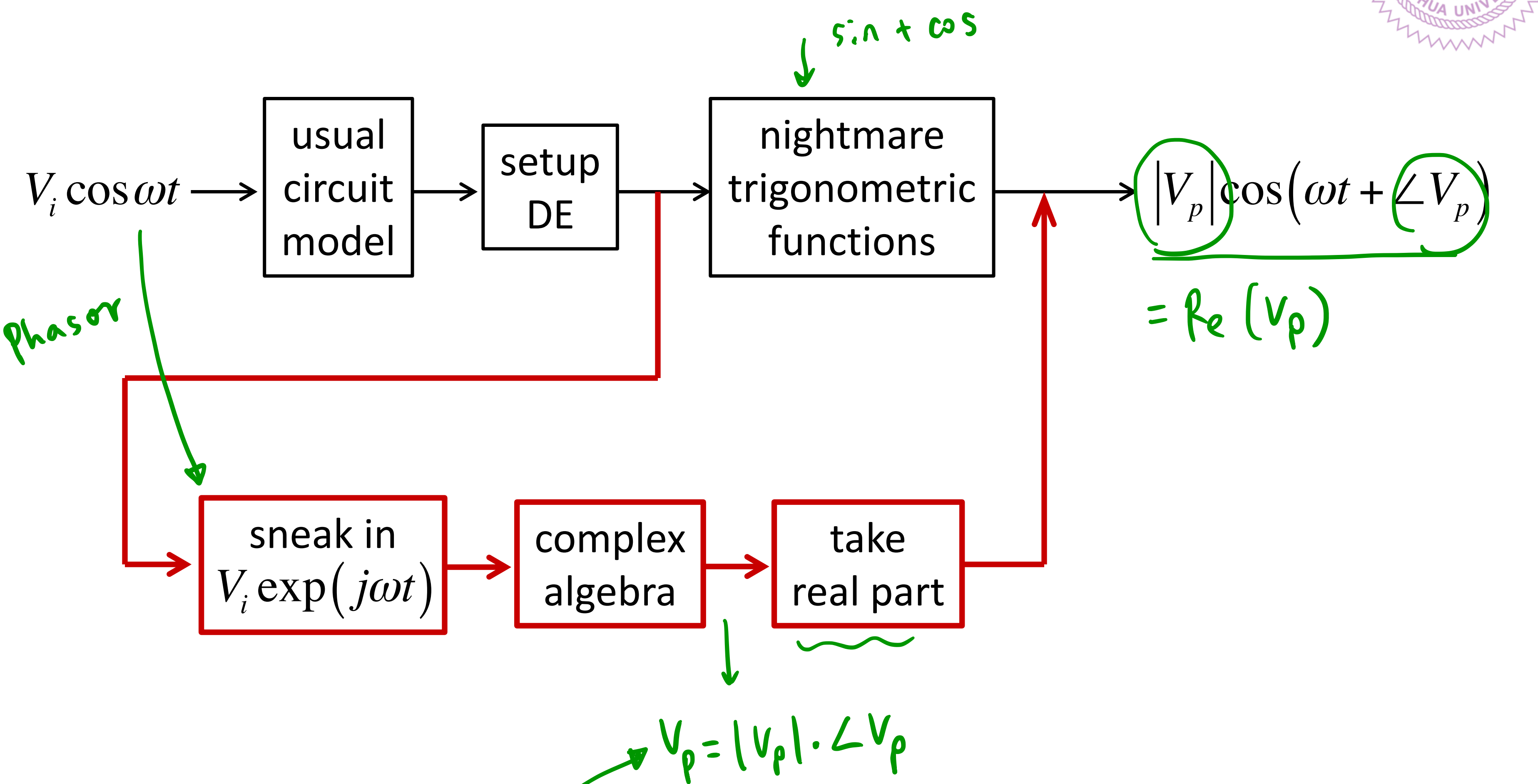
Lecture Outline

- Review
- Chapter 10 in the textbook

Review: Sinusoidal Steady State Analysis Approach



Review: Sinusoidal Steady State Analysis Approach



- V_p contains all the information we need.
  - Complex amplitude gives the amplitude and phase of the output cosine.

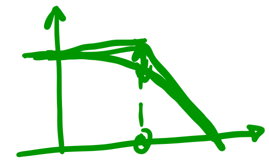
V_p

Review

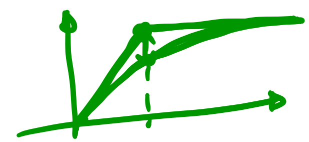
- Frequency response. Transfer function, $H = \dfrac{V_o}{V_i} = |H| \cdot \angle H$

1 Magnitude plot, $|H|$

① sketch low frequency asympotote, $\omega \to 0$



② high freq, $\omega \to \infty$



③ Two asympototes intersect at corner freq, ω_c

2. Phase plot, $\angle H$

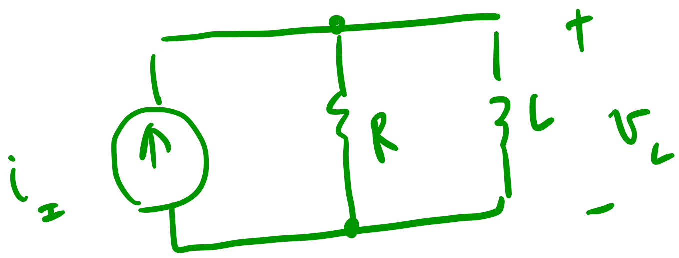
① low freq, $\omega < \frac{1}{10}\omega_c$

② high freq, $\omega > 10\omega_c$

③ At ω_c, phase $= \pm 45^\circ$

Review

- Practice



$i_I(t) = I_0 \cos\omega t$. Find SSS of $v_L(t)$.

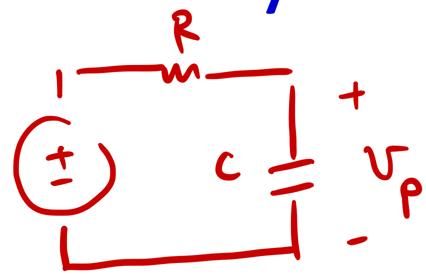
Review

Is There an Even Simpler Way to Get V_P?

complex amplitude

$$V_p = \frac{V_i}{1+j\omega RC}$$

$v_i = V_i \cdot \cos \omega t \rightarrow V_i$

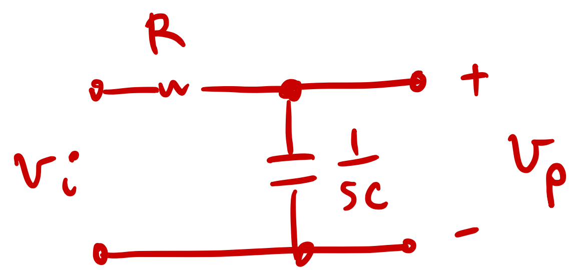


- with $s = j\omega$

- Divide numerator and denominator by sC

$$V_p = V_i \cdot \frac{\frac{1}{sC}}{\frac{1}{sC} + R}$$

voltage divider



- Let’s explore this further…

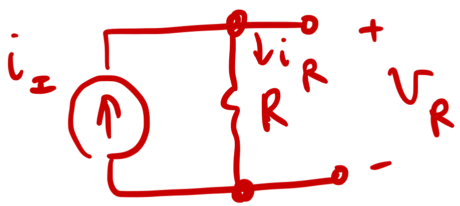
阻抗 complex

The Impedance Model

(phasor)

exponential representation

- Consider resistor:



$i_I = I_0 \cos \omega t$

$i_R = I_0 \cdot \cos \omega t$

$v_R = i_R \cdot R$

$\Rightarrow$

$\tilde{i}_I = I_0 \cdot e^{st}$

$\tilde{i}_R = I_0 \cdot e^{st} = I_R \cdot e^{st}$

$\tilde{v}_R = V_R \cdot e^{st} = \tilde{i}_R \cdot R$

$V_R = I_0 \cdot R = I_R \cdot R$

- We know from previous sequence that response to $e^{j\omega t}$ is also of the form of $e^{j\omega t}$

complex amplitude follows Ohm's law.

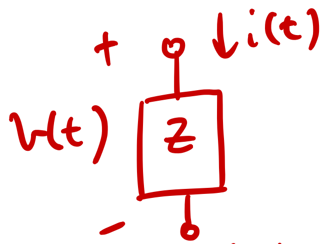
(sinusoidal)

- In other words, if $v_R = R \cdot i_R$ $\Rightarrow$ $V_R = I_R \cdot R$

Impedance $= R$

Impedance

Phasor



$$v(t) = \underline{V_m} \cdot \cos(\omega t + \underline{\theta}) \longrightarrow V = V_m \cdot \angle\theta = V_m \cdot e^{j\omega t} \cdot e^{j\theta}$$

$$i(t) = \underline{I_m} \cdot \cos(\omega t + \underline{\phi}) \longrightarrow I = I_m \cdot \angle\phi$$

$$\text{Impedance} = \frac{V}{I} = \frac{V_m \cdot \angle\theta}{I_m \cdot \angle\phi} = \frac{V_m}{I_m} \angle(\theta - \phi) = R + j\underline{X}$$

R : resistance

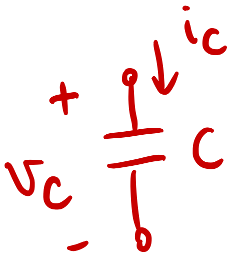
X : reactance

$X > 0$, inductive

$X < 0$, capacitive

The Impedance Model

- Consider capacitor:



$$i_c = C \cdot \frac{dv_c}{dt}$$

$v_c \longrightarrow V_c \cdot e^{st}$, V_c : complex amplitude of v_c

$i_c \longrightarrow I_c \cdot e^{st}$ I_c : " i_c

$$I_c \cdot e^{st} = C \cdot \frac{d V_c e^{st}}{dt} = C \cdot V_c \cdot s \cdot e^{st}$$

$$\frac{V_c}{I_c} = \frac{1}{sC} = \underbrace{\frac{1}{j\omega C}}_{= -j \cdot \frac{1}{\omega C}} = Z_c$$

$\omega = 0$, $Z_c \to \infty$, open

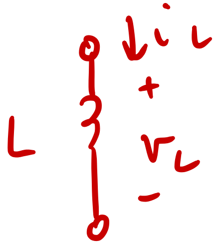
$\omega = \infty$, $Z_c \to 0$, short

- In other words, the complex amplitudes are related as

$$\frac{V_c}{I_c} = \frac{1}{sC}$$

The Impedance Model

- Consider inductor:



$$v_L = L \cdot \frac{di_L}{dt}$$

$$V_L \cdot e^{st} = L \cdot \frac{d(I_L \cdot e^{st})}{dt}$$

$$= L \cdot I_L \cdot s\, e^{st}$$

$$\Rightarrow \frac{V_L}{I_L} = sL = Z_L$$

$$= j\omega L$$

$V_L \cdot \cos\omega t$

Phasor

$v_L \rightarrow V_L \cdot e^{st}$ $(s = j\omega)$

$i_L \rightarrow I_L \cdot e^{st}$

V_L, I_L: complex amplitude of v_L and i_L

The Impedance Model – Summary

Resistor

$Z_R = R$

$\angle Z_R = 0^\circ$

$|Z_R| = R$

Capacitor

$Z_C = \frac{1}{j\omega C}$

$\angle Z_C = -90^\circ$

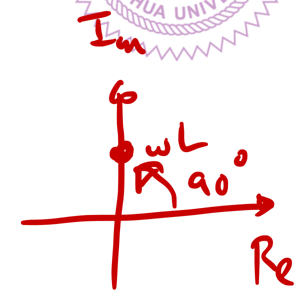
$|Z_C| = \frac{1}{\omega C}$

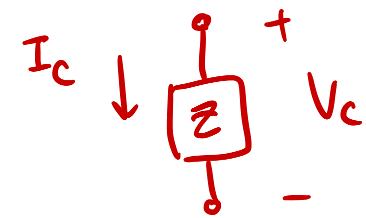
Inductor

$Z_L = j\omega L$

$\angle Z_L = 90^\circ$

$|Z_L| = \omega L$



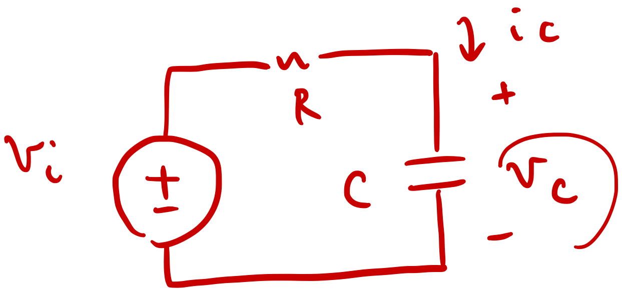


- For a drive of the form $V_c \cdot e^{j\omega t}$, the complex amplitude V_c is related to the complex amplitude I_c algebraically, by a generalization of Ohm's Law.

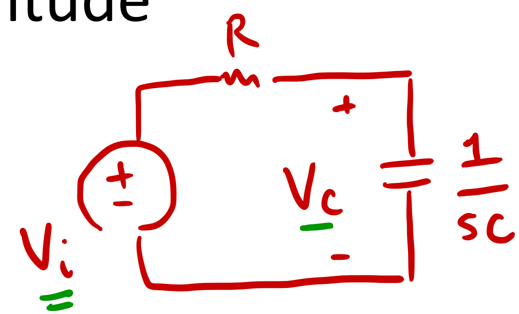
$$\frac{V_c}{I_c} = Z$$

Back to RC Network

- Find v_c
- Impedance circuit model $C \to \frac{1}{sC}$ $\quad L \to sL$
- Work on complex amplitude



$\Rightarrow$



KCL

$$\frac{V_i - V_c}{R} = \frac{V_c}{\frac{1}{sC}}$$

$$\Rightarrow V_c = V_i \cdot \frac{1}{1 + sRC}$$

$$= V_i \cdot \frac{\frac{1}{sC}}{\frac{1}{sC} + R}$$

- All our old friends apply! KVL, KCL, superposition, Norton, Thevenin, …
- The denominator: charcteristic equation

Signal Notation

$V_I = 10V$

$I_I = 1A$

- Uppercase symbols with uppercase subscript
  - DC or operating-point variables

    constant

$v_I = 10 + 2\cos\omega t$

$i_I = 1 + \sin(\omega t + \theta)$

- Lowercase symbols with uppercase subscript
  - Total instantaneous variables   dc + ac (signal)

$v_i = 2\cos\omega t$

$i_i = \exp(t)$

- Lowercase symbols with lower subscript
  - Incremental instantaneous variables

    signal (ac)

V_i

I_i

- Upper symbols with lower subscript
  - Complex amplitude or real amplitude of sinusoids (either total variables or incremental variables)

There You Have It!

1. Replace the (sinusoidal) source by their complex (or real) amplitude. $V_i \cos(\omega t+\theta) \longrightarrow$ phasor $V_i \cdot e^{j\omega t} e^{j\theta} = |V_i| \angle \theta$

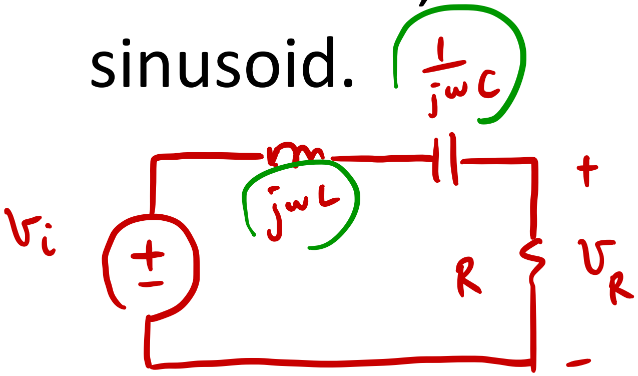
   $C \to \frac{1}{sC}$, $L \to sL$

2. Replace circuit elements by their impedances. The resulting diagram is called the impedance model of the network.
3. Determine the complex amplitude of voltages and currents in the circuit using any standard circuit analysis method.
4. Obtain the time variables from the complex amplitudes. For example, for a voltage signal with complex amplitude of $\underline{V_o}$:

$$v_o(t) = |V_o| \cdot \cos(\omega t + \angle V_o)$$

Another Example – Recall Series RLC Network

- Remember, we only want the steady-state response to sinusoid.



Assume $v_i = V_i \cdot \cos\omega t$, Find $v_R(t)$

① $v_i \longrightarrow V_i \cdot e^{j\omega t}$

② Impedance model

③ $V_R = \dfrac{R}{j\omega L + \frac{1}{j\omega C} + R} \cdot V_i \xrightarrow{\text{Real}}$

④ $v_R(t) = |V_R| \cdot \cos(\omega t + \angle V_R)$

$$H = \frac{V_R}{V_i} = \frac{R}{j\omega L + \frac{1}{j\omega C} + R}$$

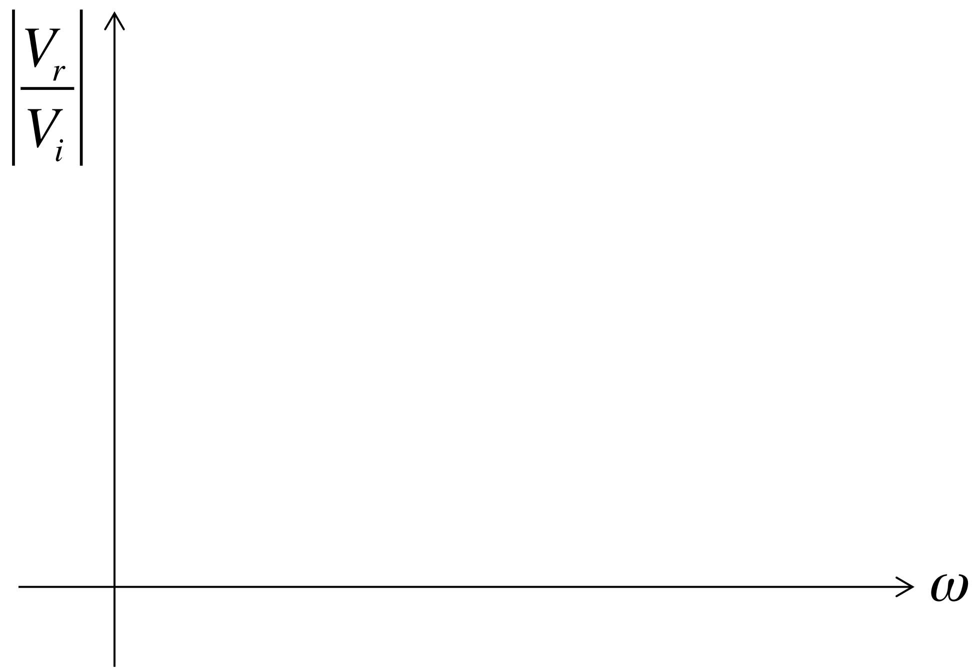
Series RLC Network

- The transfer function $\frac{V_r}{V_i} = \frac{\frac{sR}{L}}{s^2 + \frac{sR}{L} + \frac{1}{LC}} = \frac{\frac{j\omega R}{L}}{-\omega^2 + \frac{j\omega R}{L} + \frac{1}{LC}}$

Graphically

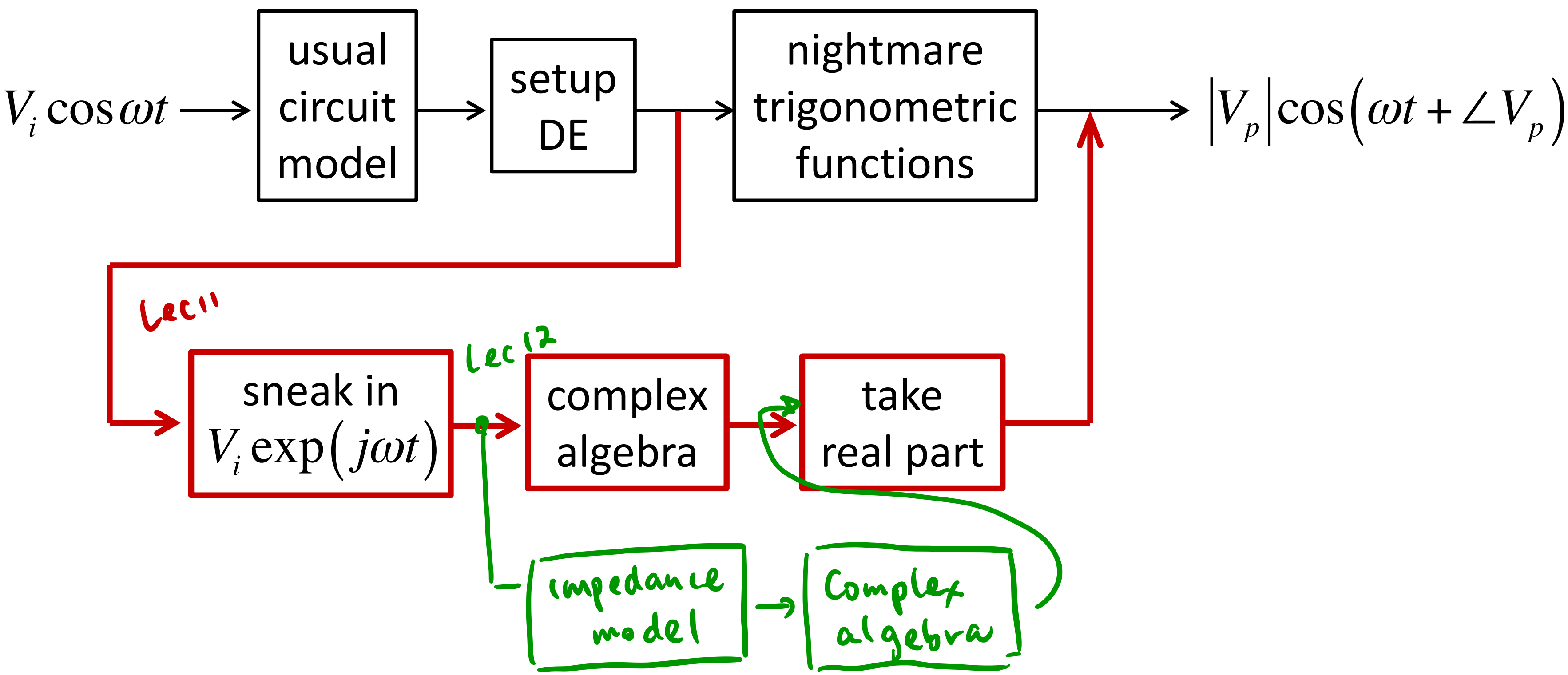
skip

- Magnitude plot $\left|\dfrac{V_r}{V_i}\right| = \dfrac{\omega RC}{\sqrt{\left(1-\omega^2 LC\right)^2 + \left(\omega RC\right)^2}}$



- Passes signals of frequencies in a middle band.

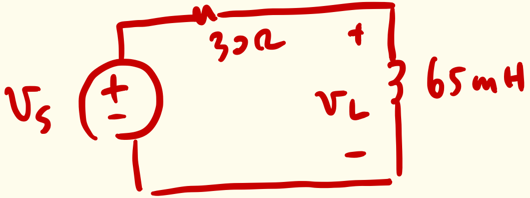
The Big Picture Again



- No differential equations!! No trigonometry!!

- Example 1.

$v_s = 12 \cdot \cos(1000t + 15^\circ)$, Find $v_L(t)$

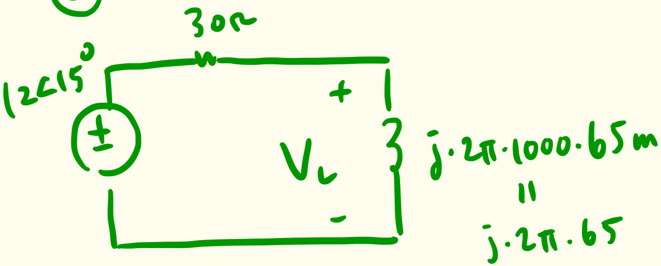


- Practice using method Lec 11.

① $v_s \longrightarrow 12 \cdot e^{j2\pi \cdot 1000t} \cdot e^{j15^\circ}$

$= 12 \cdot \angle 15^\circ$

②

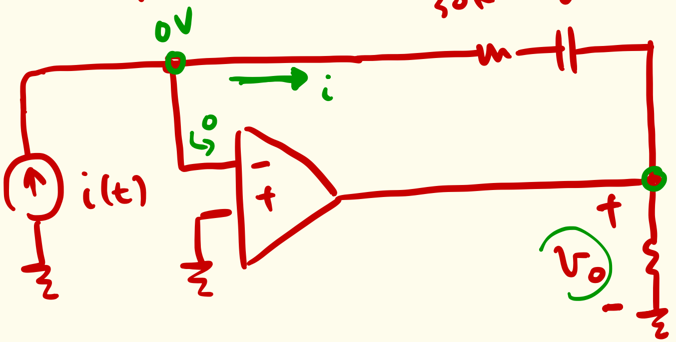


③ $V_L = V_i \cdot \dfrac{j2\pi \cdot 65}{30 + j2\pi \cdot 65}$

④ $Re(V_L) = v_L(t)$

Example 2.

30kΩ 8nF → $\frac{1}{j\omega C}$ Ideal operational amp.



$\omega = 4000$ rad/s

$i(t) = 120\mu \cdot \cos \omega t$,

① $i(t) \xrightarrow{\text{phasor}} I = 120\mu \cdot e^{j\omega t}$

$= 120\mu \cdot \angle 0^\circ$

$$Z_C = \frac{1}{j\omega C} = \frac{1}{j \cdot \omega \cdot 8n}$$

$$I = \frac{0 - V_o}{30k + \frac{1}{j4000 \cdot 8n}} = 120\mu \cdot \angle 0^\circ$$

$$\Rightarrow V_o = 120\mu \cdot \left(30k - j\frac{10^9}{4000 \cdot 8}\right) = 3.6 - j$$

$$V_o(t) = |V_o| \cdot \cos\left(4000t + \angle V_o\right)$$

$$H = \frac{V_o}{I} = \frac{3.6 - j}{120\mu} \; (V/A)$$